MACROECONOMIC THEORY **T. J. KEHOE**
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ANSWERS TO MIDTERM EXAMINATION

1. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \le \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$
$$c_t^i \ge 0.$$

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$.

(b) With sequential market markets structure, there are markets for goods and bonds open every period. Consumers trade goods and bonds among themselves.

A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, and bond holdings $\hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots;$ $\hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$ such that

- Given $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, the consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots;\ \hat{b}_0^i, \hat{b}_1^i, \hat{b}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_t^i + b_{t+1}^i \le w_t^i + (1+\hat{r}_t) b_t^i,\ t = 0,1,\ldots$$
$$c_t^i \ge 0,\ b_t^i \ge -B$$
$$b_0^i = 0.$$

Here $b_t^i \ge -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$

- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0,1,\ldots$

(c) **Proposition 1:** Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots;$ $\hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1$$
$$\hat{b}_{t+1}^i = w_t^i + (1+\hat{r}_t)\hat{b}_t^i - \hat{c}_t^i,\ t = 0, 1, \ldots.$$

Proposition 2: Suppose that $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots;$ $\hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium where

$$\hat{p}_0 = 1$$
$$\hat{p}_t = \prod_{s=1}^{t} \frac{1}{(1+\hat{r}_s)},\ t = 1, 2, \ldots.$$

(d) Using the two consumers' first order conditions

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t,\ i = 1, 2,$$

we can write

$$\frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1}.$$

In even periods,

$$c_t^1 + c_t^2 = 3$$

$$c_t^1 + \frac{\lambda^1}{\lambda^2} c_t^1 = 3$$

$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 3.$$

Similarly, in odd periods,

$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 6.$$

Normalizing $p_0 = 1$, we can use the first order condition to write

$$p_t = \begin{cases} \beta^t & \text{if } t \text{ is even} \\ \frac{1}{2}\beta^t & \text{if } t \text{ is odd} \end{cases},$$

which implies that

$$p_t c_t^1 = \beta^t \frac{3\lambda^2}{\lambda^1 + \lambda^2}.$$

Consequently,

$$\sum_{t=0}^{\infty} p_t c_t^1 = \frac{3\lambda^2}{\lambda^1 + \lambda^2} \sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = \sum_{t=0}^{\infty} p_t w_t^1$$

$$\frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = 2\sum_{t=0}^{\infty} p_{2t} + 2\sum_{t=0}^{\infty} p_{2t+1}$$

$$\frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = 2\sum_{t=0}^{\infty} \beta^{2t} + \beta\sum_{t=0}^{\infty} \beta^{2t}$$

$$\frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = \frac{2+\beta}{1-\beta^2}$$

$$\frac{\lambda^2}{\lambda^1 + \lambda^2} = \frac{\frac{2}{3} + \frac{1}{3}\beta}{1+\beta} = \frac{2+\beta}{3(1+\beta)},$$

which implies that

$$\frac{\lambda^1}{\lambda^1 + \lambda^2} = \frac{1+2\beta}{3(1+\beta)}.$$

$$c_t^1 = \begin{cases} \dfrac{2+\beta}{1+\beta} & \text{if } t \text{ is even} \\ \dfrac{4+2\beta}{1+\beta} & \text{if } t \text{ is odd} \end{cases}$$

$$c_t^2 = \begin{cases} \dfrac{1+2\beta}{1+\beta} & \text{if } t \text{ is even} \\ \dfrac{2+4\beta}{1+\beta} & \text{if } t \text{ is odd} \end{cases}.$$

(We can even work out λ^1 and λ^2, although the question does not require this and it would be a waste of precious time to do so during the exam.

$$\lambda^1 = \frac{1}{c_0^1} = \frac{1+\beta}{2+\beta}$$

$$\lambda^2 = \frac{1}{c_0^2} = \frac{1+\beta}{1+2\beta}.$$

Check:

$$\frac{\lambda^1}{\lambda^1+\lambda^2} = \frac{\frac{1+\beta}{2+\beta}}{\frac{1+\beta}{2+\beta}+\frac{1+\beta}{1+2\beta}} = \frac{\frac{1}{2+\beta}}{\frac{1}{2+\beta}+\frac{1}{1+2\beta}}$$

$$\frac{\lambda^1}{\lambda^1+\lambda^2} = \frac{1+2\beta}{1+2\beta+2+\beta} = \frac{1+2\beta}{3(1+\beta)}.)$$

To calculate the sequential markets equilibrium, we just use the formulas from proposition 1 in part c:

$$r_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1 = \begin{cases} \dfrac{1}{2\beta} - 1 & \text{if } t \text{ is even} \\ \dfrac{2}{\beta} - 1 & \text{if } t \text{ is odd} \end{cases}.$$

$$\hat{b}_{t+1}^i = w_t^i + (1+\hat{r}_t)\hat{b}_t^i - \hat{c}_t^i$$

$$\hat{b}_1^1 = w_0^1 - \hat{c}_0^1 = 2 - \frac{1+\beta}{1+\beta} = \frac{\beta}{1+\beta}.$$

In even periods, consumer 1 lends $\frac{\beta}{1+\beta}$ to consumer 2. Consumer 2 pays back $\frac{2}{1+\beta}$ to consumer 1 in odd periods.

(e) A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots$; $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$; $\hat{c}_0^3, \hat{c}_1^3, \hat{c}_2^3, \ldots$ and bond holdings $\hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots$; $\hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$; $\hat{b}_0^3, \hat{b}_1^3, \hat{b}_2^3, \ldots$ such that

- Given $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, the consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$; $\hat{b}_0^i, \hat{b}_1^i, \hat{b}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_t^i + b_{t+1}^i \le w_t^i + (1+\hat{r}_t) b_t^i,\ t = 0,1,\ldots$$
$$c_t^i \ge 0,\ b_t^i \ge -B$$
$$b_0^i = 0.$$

Here $b_t^i \geq -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{c}_t^3 = w_t^1 + w_t^2 + w_t^3$, $t = 0,1,\ldots$

- $\hat{b}_t^1 + \hat{b}_t^2 + \hat{b}_t^3 = 0$, $t = 0,1,\ldots$

2. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 1. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max \ \log c_1^0 \\ &\text{s.t } \ \hat{p}_1 c_1^0 \leq \hat{p}_1 w_2 + m \\ &c_1^0 \geq 0 . \end{aligned}$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer t, $t = 1,2,\ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ to solve

$$\begin{aligned} &\max \ c_t^t + \log c_{t+1}^t \\ &\text{s.t. } \ \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \leq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 \\ &c_t^t, c_{t+1}^t \geq 0 . \end{aligned}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t-1$ and t trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and asset holdings $\hat{s}_2^1, \hat{s}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max \ \log c_1^0 \\ &\text{s.t } \ c_1^0 \leq w_2 + m \\ &c_1^0 \geq 0 . \end{aligned}$$

- Given $\hat{r}_{t+1}$, consumer t, $t = 1,2,\ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{s}_{t+1}^t$ to solve

$$\begin{aligned} &\max\ c_t^t + \log c_{t+1}^t \\ &\text{s.t. } c_t^t + s_{t+1}^t \le w_1 \\ &c_{t+1}^t \le w_2 + (1+\hat{r}_{t+1}) s_{t+1}^t \\ &c_t^t, c_{t+1}^t \ge 0 . \end{aligned}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1,2,\ldots$.

- $\hat{s}_2^1 = m$, $\hat{s}_{t+1}^t = \left[\prod_{\tau=2}^{t} (1+\hat{r}_\tau)\right] m$, $t = 2,3,\ldots$.

(c) Since there is no fiat money, there is only one good per period, there is only one consumer type in each generation, and consumers live for only two periods, the equilibrium allocation is autarky:

$$\hat{c}_1^0 = w_2$$
$$(\hat{c}_t^t, \hat{c}_{t+1}^t) = (w_1, w_2) .$$

The first order conditions from the consumers' problems in the Arrow-Debreu equilibrium imply that

$$\hat{c}_{t+1}^t = \frac{\hat{p}_t}{\hat{p}_{t+1}} .$$

Normalizing $\hat{p}_1 = 1$, we obtain $\hat{p}_t = w_2^{1-t}$. Similarly, the first order conditions from the consumers' problems in the sequential markets equilibrium, imply that

$$1 + \hat{r}_{t+1} = \hat{c}_{t+1}^t = w_2$$

or $\hat{r}_t = w_2 - 1$. Since the equilibrium allocation is autarky, $\hat{s}_{t+1}^t = 0$.

(d) An allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is **feasible** if

$$\hat{c}_t^{t-1} + \hat{c}_t^t \le w_2 + w_1,\ t = 1,\ 2,\ \ldots .$$

An allocation is **Pareto efficient** if it is feasible and there exists no other allocation $\overline{c}_1^0, (\overline{c}_1^1, \overline{c}_2^1), (\overline{c}_2^2, \overline{c}_3^2), \ldots$ that is also feasible and satisfies

$$\log \overline{c}_1^0 \ge \log \hat{c}_1^0$$
$$\overline{c}_t^t + \log \overline{c}_{t+1}^t \ge \hat{c}_t^t + \log \hat{c}_{t+1}^t,\ t = 1,2,\ldots,$$

with at least one inequality strict.

If $w_2 > 1$, the equilibrium allocation is Pareto efficient. Suppose not. Then there exists a feasible allocation that is Pareto superior. If

$$\bar{c}_t^t + \log \bar{c}_{t+1}^t > \hat{c}_t^t + \log \hat{c}_{t+1}^t,$$

then

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t > \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Otherwise, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ would not solve the maximization problem of generation t. Similarly, $\log \bar{c}_1^0 > \log \hat{c}_1^0$ implies $\hat{p}_1 \bar{c}_1^0 > \hat{p}_1 w_2$.

Suppose that

$$\bar{c}_t^t + \log \bar{c}_{t+1}^t \geq \hat{c}_t^t + \log \hat{c}_{t+1}^t$$

but that

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t < \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Then let

$$\tilde{c}_t^t = \bar{c}_t^t + \frac{\hat{p}_t w_1 + \hat{p}_{t+1} w_2 - \hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t}{\hat{p}_t} > \bar{c}_t^t$$

and $\tilde{c}_{t+1}^t = \bar{c}_{t+1}^t$. Then

$$\tilde{c}_t^t + \log \tilde{c}_{t+1}^t > \hat{c}_t^t + \log \hat{c}_{t+1}^t$$

but

$$\hat{p}_t \tilde{c}_t^t + \hat{p}_{t+1} \tilde{c}_{t+1}^t = \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Once again, this would imply that $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ would not solve the maximization problem of generation t, which is impossible. Consequently,

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t \geq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Similarly, $\log \bar{c}_1^0 \geq \log \hat{c}_1^0$ implies $\hat{p}_1 \bar{c}_1^0 \geq \hat{p}_1 w_2$. Therefore

$$\hat{p}_1 \bar{c}_1^0 \geq \hat{p}_1 w_2$$
$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t \geq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 ,\ t = 1, 2, \ldots,$$

with at least one inequality strict. Adding these inequalities up, we obtain

$$\sum_{t=1}^{\infty} \hat{p}_t (\bar{c}_t^{t-1} + \bar{c}_t^t) > \sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) .$$

It is here that $\hat{p}_t = w_2^{1-t}$ plays its role in ensuring that these series converge.

$$\sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) = \sum_{t=1}^{\infty} w_2^{1-t} (w_1 + w_2) = \frac{w_1 + w_2}{1 - w_2^{-1}} < \infty$$

Multiplying the feasibility condition in period t by $\hat{p}_t > 0$ and adding up yields

$$\sum_{t=1}^{\infty} \hat{p}_t (\overline{c}_t^{t-1} + \overline{c}_t^t) \leq \sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) < \infty,$$

which is a contradiction.

(e) A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_1, \hat{r}_2, \dots$, an allocation $\hat{c}_1^{10}, \hat{c}_1^{20}, (\hat{c}_1^{11}, \hat{c}_2^{11}), (\hat{c}_1^{21}, \hat{c}_2^{21}), (\hat{c}_2^{12}, \hat{c}_3^{12}), (\hat{c}_2^{22}, \hat{c}_3^{22})$ …, and asset holdings $\hat{s}_1^{11}, \hat{s}_1^{21}, \hat{s}_2^{12}, \hat{s}_2^{22} \dots$ such

- Consumer $i0$ chooses $\hat{c}_1^{i0}$, $i = 1, 2$, to solve

$$\begin{gathered} \max \ u_{i0}(c_1^{i0}) \\ \text{s.t } c_1^{i0} \leq w_2^i + m^i \\ c_1^{i0} \geq 0. \end{gathered}$$

- Given $\hat{r}_t$, consumer it, $i = 1, 2$, $t = 1, 2, \dots$, chooses $(\hat{c}_t^{it}, \hat{c}_{t+1}^{it})$ and $\hat{s}_t^{it}$ to solve

$$\begin{gathered} \max \ u_i(c_t^{it}, c_{t+1}^{it}) \\ \text{s.t. } c_t^{it} + s_t^{it} \leq w_1^i \\ c_{t+1}^{it} \leq w_2^i + (1 + \hat{r}_t) s_t^{it} \\ c_t^{it}, c_{t+1}^{it} \geq 0. \end{gathered}$$

- $\hat{c}_t^{1t-1} + \hat{c}_t^{2t-1} + \hat{c}_t^{1t} + \hat{c}_t^{2t} = w_2^1 + w_2^2 + w_1^1 + w_1^2$, $t = 1, 2, \dots$.

- $\hat{s}_1^1 + \hat{s}_1^2 = m^1 + m^2$
  $\hat{s}_t^{1t} + \hat{s}_t^{2t} = \left[\prod_{\tau=1}^{t-1} (1 + \hat{r}_\tau) \right] (m^1 + m^2)$, $t = 2, 3, \dots$.

3. (a) A **sequential markets equilibrium** is wages $\hat{w}_1, \hat{w}_2, \hat{w}_3, \dots$, interest rates on bonds $\hat{r}_2^b, \hat{r}_3^b, \dots$, rental rates on capital, $\hat{r}_1^k, \hat{r}_2^k, \hat{r}_3^k, \dots$, consumption levels $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \dots$, bond holdings $\hat{b}_2^1, \hat{b}_3^2, \dots$, and capital holdings $\hat{k}_2^1, \hat{k}_3^2, \dots$ such that

- Given $\hat{w}_1$ and $\hat{r}_1^k$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{gathered} \max \ \log c_1^0 \\ \text{s.t } c_1^0 \leq \hat{w}_1 \overline{\ell}_2 + (1 + \hat{r}_1^k - \delta) \overline{k}_1^0 + m \\ c_1^0 \geq 0. \end{gathered}$$

- Given $\hat{w}_t$, $\hat{w}_{t+1}$, $\hat{r}_{t+1}^b$, $\hat{r}_{t+1}^k$, consumer *t*, $t = 1,2,\ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$, $\hat{b}_{t+1}^t$, $\hat{k}_{t+1}^t$, to solve

$$\begin{gathered}\max\ \ \log c_t^t + \log c_{t+1}^t \\ \text{s.t.}\ \ c_t^t + k_{t+1}^t + b_{t+1}^t \le \hat{w}_t \overline{\ell}_1 \\ c_{t+1}^t \le \hat{w}_{t+1}\overline{\ell}_2 + (1+\hat{r}_{t+1}^k - \delta)k_{t+1}^t + (1+\hat{r}_{t+1}^b)b_{t+1}^t \\ c_t^t, c_{t+1}^t, k_{t+1}^t \ge 0\,.\end{gathered}$$

- Firms minimize costs and earn 0 profits. In particular,

  $\hat{r}_t^k = \alpha\theta(\hat{k}_t^{t-1})^{\alpha-1}(\overline{\ell}_2 + \overline{\ell}_1)^{1-\alpha}$, $t = 0,1,\ldots$

  $\hat{w}_t = (1-\alpha)\theta(\hat{k}_t^{t-1})^{\alpha}(\overline{\ell}_2 + \overline{\ell}_1)^{-\alpha}$, $t = 0,1,\ldots$.

- $\hat{c}_t^{t-1} + \hat{c}_t^t + \hat{k}_{t+1}^t - (1-\delta)\hat{k}_t^{t-1} = \theta(\hat{k}_t^{t-1})^{\alpha}(\overline{\ell}_2 + \overline{\ell}_1)^{1-\alpha}$, $t = 1,2,\ldots$.

- $\hat{b}_2^1 = m$, $\hat{b}_{t+1}^t = \left[\prod_{\tau=2}^{t}(1+\hat{r}_\tau)\right]m$, $t = 2,3,\ldots$.

(b) An **Arrow-Debreu equilibrium** is prices $\hat{p}_1, \hat{p}_2, \hat{p}_3, \ldots$, wages $\hat{w}_1, \hat{w}_2, \hat{w}_3, \ldots$, rental rates on capital, $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, consumption levels $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and capital holdings $\hat{k}_2^1, \hat{k}_3^2, \ldots$ such that

- Given $\hat{p}_1$, $\hat{w}_1$, $\hat{r}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{gathered}\max\ \ \log c_1^0 \\ \text{s.t}\ \ \hat{p}_1 c_1^0 \le \hat{w}_1\overline{\ell}_2 + (\hat{p}_1(1-\delta) + \hat{r}_1)\overline{k}_1^0 + m \\ c_1^0 \ge 0\,.\end{gathered}$$

- Given $\hat{p}_t$, $\hat{p}_{t+1}$, $\hat{w}_t$, $\hat{w}_{t+1}$, $\hat{r}_{t+1}$, consumer *t*, $t = 1,2,\ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$, $\hat{k}_{t+1}^t$, to solve

$$\begin{gathered}\max\ \ \log c_t^t + \log c_{t+1}^t \\ \text{s.t.}\ \ \hat{p}_t(c_t^t + k_{t+1}^t) + \hat{p}_{t+1}c_{t+1}^t \le \hat{w}_t\overline{\ell}_1 + \hat{w}_{t+1}\overline{\ell}_2 + (\hat{p}_{t+1}(1-\delta) + \hat{r}_{t+1})k_{t+1}^t \\ c_t^t, c_{t+1}^t, k_{t+1}^t \ge 0\,.\end{gathered}$$

- Firms minimize costs and earn 0 profits. In particular,

  $\hat{r}_t^k = \hat{p}_t\alpha\theta(\hat{k}_t^{t-1})^{\alpha-1}(\overline{\ell}_2 + \overline{\ell}_1)^{1-\alpha}$, $t = 0,1,\ldots$

  $\hat{w}_t = \hat{p}_t(1-\alpha)\theta(\hat{k}_t^{t-1})^{\alpha}(\overline{\ell}_2 + \overline{\ell}_1)^{-\alpha}$, $t = 0,1,\ldots$.

- $\hat{c}_t^{t-1} + \hat{c}_t^t + \hat{k}_{t+1}^t - (1-\delta)\hat{k}_t^{t-1} = \theta(\hat{k}_t^{t-1})^{\alpha}(\overline{\ell}_2 + \overline{\ell}_1)^{1-\alpha}$, $t = 1,2,\ldots$.

(b) **Proposition 1:** Suppose that $\hat{p}_1, \hat{p}_2, \hat{p}_3, \ldots,\ \hat{w}_1, \hat{w}_2, \hat{w}_3, \ldots,\ \hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots,$ $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots,\ \hat{k}_2^1, \hat{k}_3^2, \ldots$ is an Arrow Debreu equilibrium with fiat money m. Then $\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \ldots,\ \hat{r}_2^b, \hat{r}_3^b, \ldots,\ \hat{r}_1^k, \hat{r}_2^k, \hat{r}_3^k, \ldots,\ \hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots,\ \hat{b}_2^1, \hat{b}_3^2, \ldots,\ \hat{k}_2^1, \hat{k}_3^2, \ldots$ is a sequential markets equilibrium in a model with fiat money $\tilde{m}$ where

$$\tilde{m} = \frac{m}{\hat{p}_1}$$

$$\tilde{w}_t = \frac{\hat{w}_t}{\hat{p}_t}$$

$$\hat{r}_t^b = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1$$

$$\hat{r}_t^k = \frac{\hat{r}_t}{\hat{p}_t}$$

$$\hat{b}_{t+1}^t = \hat{w}_t \overline{\ell}_1 - \hat{c}_t^t - \hat{k}_{t+1}^t,\ t = 1, 2, \ldots.$$

Proof: An Arrow-Debreu equilibrium is characterized by first order conditions for the consumers' problems,

(AD.1) $$\frac{1}{\hat{c}_t^t} = \lambda^t \hat{p}_t$$

(AD.2) $$\frac{1}{\hat{c}_{t+1}^t} = \lambda^t \hat{p}_{t+1}$$

(AD.3) $$-\hat{p}_t + \hat{p}_{t+1}(1-\delta) + \hat{r}_{t+1} = 0,$$,

budget constraints,

(AD.4) $$\hat{p}_1 \hat{c}_1^0 = \hat{w}_1 \overline{\ell}_2 + (\hat{p}_1(1-\delta) + \hat{r}_1)\overline{k}_1^0 + m$$

(AD.5) $$\hat{p}_t(\hat{c}_t^t + \hat{k}_{t+1}^t) + \hat{p}_{t+1}\hat{c}_{t+1}^t = \hat{w}_t \overline{\ell}_1 + \hat{w}_{t+1}\overline{\ell}_2 + (\hat{p}_{t+1}(1-\delta) + \hat{r}_{t+1})\hat{k}_{t+1}^t,$$

profit maximization conditions,

(AD.6) $$\hat{r}_t = \hat{p}_t \alpha\theta(\hat{k}_t^{t-1})^{\alpha-1}(\overline{\ell}_2 + \overline{\ell}_1)^{1-\alpha}$$

(AD.7) $$\hat{w}_t = \hat{p}_t(1-\alpha)\theta(\hat{k}_t^{t-1})^{\alpha}(\overline{\ell}_2 + \overline{\ell}_1)^{-\alpha},$$

and goods market clearing conditions,

(AD.8) $$\hat{c}_t^{t-1} + \hat{c}_t^t + \hat{k}_{t+1}^t - (1-\delta)\hat{k}_t^{t-1} = \theta(\hat{k}_t^{t-1})^{\alpha}(\overline{\ell}_2 + \overline{\ell}_1)^{1-\alpha}.$$

A sequential markets equilibrium is characterized by first order conditions for the consumers' problems,

(SM.1) $$\frac{1}{\hat{c}_t^t} = \mu_t^t$$

(SM.2) $$\frac{1}{\hat{c}_{t+1}^{t}} = \mu_{t+1}^{t}$$

(SM.3) $$-\mu_t^t + (1+\hat{r}_{t+1}^k - \delta)\mu_{t+1}^t = 0$$

(SM.4) $$-\mu_t^t + (1+\hat{r}_{t+1}^b)\mu_{t+1}^t = 0,$$

budget constraints,

(SM.5) $$\hat{c}_1^0 = \hat{w}_1\overline{\ell}_2 + (1+\hat{r}_1^k - \delta)\overline{k}_1^0 + m$$

(SM.6) $$\hat{c}_t^t + \hat{k}_{t+1}^t + \hat{b}_{t+1}^t = \hat{w}_t\overline{\ell}_1$$

(SM.7) $$\hat{c}_{t+1}^t = \hat{w}_{t+1}\overline{\ell}_2 + (1+\hat{r}_{t+1}^k - \delta)\hat{k}_{t+1}^t + (1+\hat{r}_{t+1}^b)\hat{b}_{t+1}^t,$$

profit maximization conditions,

(SM.8) $$\hat{r}_t^k = \alpha\theta(\hat{k}_t^{t-1})^{\alpha-1}(\overline{\ell}_2 + \overline{\ell}_1)^{1-\alpha}$$

(SM.9) $$\tilde{w}_t = (1-\alpha)\theta(\hat{k}_t^{t-1})^{\alpha}(\overline{\ell}_2 + \overline{\ell}_1)^{-\alpha}$$

goods market clearing conditions,

(SM.10) $$\hat{c}_t^{t-1} + \hat{c}_t^t + \hat{k}_{t+1}^t - (1-\delta)\hat{k}_t^{t-1} = \theta(\hat{k}_t^{t-1})^{\alpha}(\overline{\ell}_2 + \overline{\ell}_1)^{1-\alpha},$$

and bonds market clearing conditions,

(SM.11) $$\hat{b}_2^1 = \tilde{m}$$

(SM.12) $$\hat{b}_{t+1}^t = \left[\prod\nolimits_{s=2}^{t}(1+\hat{r}_s^b)\right]\tilde{m}.$$

Let

$$\mu_t^t = \lambda^t \hat{p}_t$$
$$\mu_{t+1}^t = \lambda^t \hat{p}_{t+1}.$$

Notice that

$$\begin{aligned}
-\mu_t^t + (1+\hat{r}_{t+1}^k - \delta)\mu_{t+1}^t &= -\lambda^t \hat{p}_t + (1+\hat{r}_{t+1}^k - \delta)\lambda^t \hat{p}_{t+1} \\
&= \lambda^t\left[-\hat{p}_t + \hat{p}_{t+1}(1-\delta) + \hat{p}_{t+1}\frac{\hat{r}_{t+1}}{\hat{p}_{t+1}}\right] \\
&= \lambda^t\left[-\hat{p}_t + \hat{p}_{t+1}(1-\delta) + \hat{r}_{t+1}\right] \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
-\mu_t^t + (1+\hat{r}_{t+1}^b)\mu_{t+1}^t &= -\lambda^t \hat{p}_t + (1+\hat{r}_{t+1}^b)\lambda^t \hat{p}_{t+1} \\
&= \lambda^t\left[-\hat{p}_t + \hat{p}_{t+1}(1+\frac{\hat{p}_t}{\hat{p}_{t+1}} - 1)\right] \\
&= 0.
\end{aligned}$$

Consequently, if the first order conditions for the Arrow-Debreu equilibrium (AD.1)-(AD.3) are satisfied, so are those for the sequential markets equilibrium (SM.1)-(SM.4).

If we divide the budget constraint for the initial old in the Arrow-Debreu equilibrium (AD.4) through by $\hat{p}_1$, it becomes the budget constraint for the initial old in the sequential markets equilibrium (SM.5). The definition of $\hat{b}^t_{t+1}$ implies that the budget constraint of consumer t in period t (SM.6) is satisfied. If we multiply

$$\hat{b}^t_{t+1} = \hat{w}_t \bar{\ell}_1 - \hat{c}^t_t - \hat{k}^t_{t+1}$$

by $\hat{p}_t$ and subtract it from the budget constraint of consumer t in the Arrow-Debreu equilibrium (AD.5), we obtain

$$\hat{p}_{t+1}\hat{c}^t_{t+1} = \hat{w}_{t+1}\bar{\ell}_2 + \hat{p}_t\hat{b}_{t+1} + (\hat{p}_{t+1}(1-\delta) + \hat{r}_{t+1})\hat{k}^t_{t+1}.$$

Dividing this equation through by $\hat{p}_{t+1}$, we obtain the budget constraint of consumer t in period $t+1$ in the sequential markets equilibrium (SM.7).

Dividing the profit maximization conditions in the Arrow-Debreu equilibrium (AD.6)-(AD.7) by $\hat{p}_t$, we obtain the profit maximization conditions in the sequential markets equilibrium (SM.8)-(SM.9).

The goods market clearing condition in the Arrow-Debreu equilibrium (AD.8) is the same as that in the sequential markets equilibrium (SM.10).

All that remains is to show the bonds market clearing conditions (SM.11)-(SM.12) hold. Adding the budget constraints in period t of consumer $t-1$ (SM.6) and that of consumer t (SM.7) — which we have already shown to hold — we obtain

$$\begin{gathered}
\hat{c}^0_1 + \hat{c}^1_1 + \hat{k}^1_2 + \hat{b}^1_2 = \hat{w}_1\bar{\ell}_1 + \hat{w}_1\bar{\ell}_2 + (1+\hat{r}^k_1 - \delta)\bar{k}^0_1 + \tilde{m} \\
\hat{c}^0_1 + \hat{c}^1_1 + \hat{k}^1_2 - (1-\delta)\bar{k}^0_1 + \hat{b}^1_2 = \left[\hat{w}_1\bar{\ell}_1 + \hat{w}_1\bar{\ell}_2 + \hat{r}^k_1\bar{k}^0_1\right] + \tilde{m} \\
\hat{b}^1_2 = \tilde{m}
\end{gathered}$$

in period 1 and

$$\begin{gathered}
\hat{c}^{t-1}_t + \hat{c}^t_t + \hat{k}^t_{t+1} + \hat{b}^t_{t+1} = \hat{w}_t\bar{\ell}_1 + \hat{w}_t\bar{\ell}_2 + (1+\hat{r}^k_t - \delta)\hat{k}^t_t + (1+\hat{r}^b_{t+1})\hat{b}^t_{t+1} \\
\hat{c}^{t-1}_t + \hat{c}^t_t + \hat{k}^t_{t+1} - (1-\delta)\hat{k}^t_t + \hat{b}^t_{t+1} = \left[\hat{w}_t\bar{\ell}_1 + \hat{w}_t\bar{\ell}_2 + \hat{r}^k_t\hat{k}^t_t\right] + (1+\hat{r}^b_t)\hat{b}^t_t \\
\hat{b}^t_{t+1} = (1+\hat{r}^b_t)\hat{b}^t_t
\end{gathered}$$

in periods $t = 2, 3, \ldots$, which implies that the bonds market clearing conditions (SM.11)-(SM.12) hold. □

Proposition 2: Suppose that $\hat{w}_1, \hat{w}_2, \hat{w}_3, \ldots,\ \hat{r}_2^b, \hat{r}_3^b, \ldots,\ \hat{r}_1^k, \hat{r}_2^k, \hat{r}_3^k, \ldots,\ \hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots,$ $\hat{b}_2^1, \hat{b}_3^2, \ldots,\ \hat{k}_2^1, \hat{k}_3^2, \ldots$ is a sequential markets equilibrium in a model with fiat money m. Then $\hat{p}_1, \hat{p}_2, \hat{p}_3, \ldots,\ \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \ldots,\ \hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots,\ \hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots,\ \hat{k}_2^1, \hat{k}_3^2, \ldots$ is an Arrow Debreu equilibrium with fiat money m where

$$\hat{p}_1 = 1$$

$$\hat{p}_t = \prod_{s=2}^{t} \frac{1}{1+\hat{r}_s^b}\ ,\ s = 2, 3, \ldots$$

$$\tilde{w}_t = \hat{p}_t \hat{w}_t$$

$$\hat{r}_t = \hat{p}_t \hat{r}_t^k\ .$$

Proof: Let

$$\lambda^t = \frac{\mu_t^t}{\hat{p}_t}\ .$$

Then the first order condition for consumer t in period t in the sequential markets equilibrium (SM.1) implies the corresponding constraint in the Arrow-Debreu equilibrium (AD.1). Our definition of $\hat{p}_t$ implies that

$$\hat{p}_{t+1} = \frac{1}{1+\hat{r}_{t+1}^b} \hat{p}_t\ .$$

Consequently, the first order condition for consumer t in period $t+1$ (SM.2) and the first order condition for bonds in the sequential markets equilibrium (SM.4) imply the first order condition for consumer t in period $t+1$ in the Arrow-Debreu equilibrium (AD.2). Furthermore the first order condition for capital in the sequential markets equilibrium (SM.3) implies the corresponding condition in the Arrow-Debreu equilibrium (AD.3).

Since $\hat{p}_1 = 1$, the budget constraint for the initial old in the sequential markets equilibrium (SM.5) is the same as the budget constraint for the initial old in the Arrow-Debreu equilibrium (AD.4). If we multiply the budget constraint of consumer t in period t in the sequential markets equilibrium (SM.6) by $\hat{p}_t$ and add it to the budget constraint of consumer t in period $t+1$ (SM.7) multiplied by $\hat{p}_{t+1}$, we use

$$\hat{b}_{t+1}^t = (1+\hat{r}_t^b)\hat{b}_t^t = \frac{\hat{p}_t}{\hat{p}_{t+1}} \hat{b}_t^t$$

to obtain the budget constraint of consumer t in the Arrow-Debreu equilibrium (SM.7).

Multiplying the profit maximization conditions in the sequential markets equilibrium (SM.8)-(SM.9) by $\hat{p}_t$, we obtain the profit maximization conditions in the Arrow-Debreu equilibrium (AD.6)-(AD.7).

The goods market clearing condition in the sequential markets equilibrium (SM.10) is the same as that in the Arrow-Debreu equilibrium (AD.8). □

(c) A **sequential markets equilibrium** is wages $\hat{w}_1, \hat{w}_2, \hat{w}_3, \ldots$, interest rates on bonds $\hat{r}_2^b, \hat{r}_3^b, \ldots$, rental rates on capital, $\hat{r}_1^k, \hat{r}_2^k, \hat{r}_3^k, \ldots$, consumption levels $\hat{c}_1^{-1}, (\hat{c}_1^0, \hat{c}_2^0), (\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1), (\hat{c}_2^2, \hat{c}_3^2, \hat{c}_4^2), \ldots$, bond holdings $\hat{b}_2^0, (\hat{b}_2^1, \hat{b}_3^1), (\hat{b}_3^2, \hat{b}_4^2), \ldots$, and capital holdings $\hat{k}_2^0, (\hat{k}_2^1, \hat{k}_3^1), (\hat{k}_3^2, \hat{k}_4^2), \ldots$ such that

- Given $\hat{w}_1$ and $\hat{r}_1^k$, consumer -1 chooses $\hat{c}_1^{-1}$ to solve

$$\begin{gathered}\max \ \log c_1^{-1} \\ \text{s.t } \ c_1^{-1} \le \hat{w}_1 \overline{\ell}_2 + (1 + \hat{r}_1^k - \delta)\overline{k}_1^{-1} + m^{-1} \\ c_1^{-1} \ge 0 .\end{gathered}$$

- Given $\hat{w}_1, \hat{w}_2, \hat{r}_2^b, \hat{r}_1^k, \hat{r}_2^k$, and m^0, consumer 0 chooses $(\hat{c}_1^0, \hat{c}_2^0), \hat{b}_2^0, \hat{k}_2^0$, to solve

$$\begin{gathered}\max \ \log c_1^0 + \log c_2^0 \\ \text{s.t. } \ c_1^0 + k_2^0 + b_2^0 \le \hat{w}_1 \overline{\ell}_2 + (1 + \hat{r}_1^k - \delta)\overline{k}_1^0 + m^0 \\ c_2^0 \le \hat{w}_2 \overline{\ell}_3 + (1 + \hat{r}_2^k - \delta)k_2^0 + (1 + \hat{r}_2^b)b_2^0 \\ c_t^t, c_{t+1}^t, k_{t+1}^t \ge 0 .\end{gathered}$$

- Given $\hat{w}_t, \hat{w}_{t+1}, \hat{w}_{t+2}, \hat{r}_{t+1}^b, \hat{r}_{t+2}^b$, $\hat{r}_{t+1}^k, \hat{r}_{t+2}^k$, consumer t, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{c}_{t+2}^t), (\hat{b}_{t+1}^t, \hat{b}_{t+2}^t), (\hat{k}_{t+1}^t, \hat{k}_{t+2}^t)$, to solve

$$\begin{gathered}\max \ \log c_t^t + \log c_{t+1}^t + \log c_{t+2}^t \\ \text{s.t. } \ c_t^t + k_{t+1}^t + b_{t+1}^t \le \hat{w}_t \overline{\ell}_1 \\ c_{t+1}^t + k_{t+2}^t + b_{t+2}^t \le \hat{w}_{t+1} \overline{\ell}_2 + (1 + \hat{r}_{t+1}^k - \delta)k_{t+1}^t + (1 + \hat{r}_{t+1}^b)b_{t+1}^t \\ c_{t+2}^t \le \hat{w}_{t+2} \overline{\ell}_3 + (1 + \hat{r}_{t+2}^k - \delta)k_{t+2}^t + (1 + \hat{r}_{t+2}^b)b_{t+2}^t \\ c_t^t, c_{t+1}^t, c_{t+2}^t, k_{t+1}^t, k_{t+2}^t \ge 0 .\end{gathered}$$

- Firms minimize costs and earn 0 profits. In particular,

  $\hat{r}_t^k = \alpha\theta(\hat{k}_t^{t-2} + \hat{k}_t^{t-1})^{\alpha-1}(\overline{\ell}_3 + \overline{\ell}_2 + \overline{\ell}_1)^{1-\alpha}$, $t = 0, 1, \ldots$

  $\hat{w}_t = (1-\alpha)\theta(\hat{k}_t^{t-2} + \hat{k}_t^{t-1})^{\alpha}(\overline{\ell}_3 + \overline{\ell}_2 + \overline{\ell}_1)^{-\alpha}$, $t = 0, 1, \ldots$.

- $\hat{c}_t^{t-2} + \hat{c}_t^{t-1} + \hat{c}_t^t + \hat{k}_{t+1}^{t-1} + \hat{k}_{t+1}^t - (1-\delta)(\hat{k}_t^{t-2} + \hat{k}_t^{t-1}) = \theta(\hat{k}_t^{t-2} + \hat{k}_t^{t-1})^{\alpha}(\overline{\ell}_3 + \overline{\ell}_2 + \overline{\ell}_1)^{1-\alpha}$, $t = 1, 2, \ldots$.

- $\hat{b}_2^0 + \hat{b}_2^1 = m^{-1} + m^0$, $\hat{b}_{t+1}^{t-1} + \hat{b}_{t+1}^t = \left[\prod_{\tau=2}^{t}(1 + \hat{r}_\tau)\right](m^{-1} + m^0)$, $t = 2, 3, \ldots$.

(d) An **Arrow-Debreu equilibrium** is prices $\hat{p}_1, \hat{p}_2, \hat{p}_3, ...$, wages $\hat{w}_1, \hat{w}_2, \hat{w}_3, ...$, rental rates on capital, $\hat{r}_1, \hat{r}_2, \hat{r}_3, ...$, consumption levels $\hat{c}_1^{-1}, (\hat{c}_1^0, \hat{c}_2^0), (\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1), (\hat{c}_2^2, \hat{c}_3^2, \hat{c}_4^2), \ldots$, and capital holdings $\hat{k}_2^0, (\hat{k}_2^1, \hat{k}_3^1), (\hat{k}_3^2, \hat{k}_4^2), \ldots$ such that

- Given $\hat{p}_1$, $\hat{w}_1$, $\hat{r}_1$, consumer 0 chooses $\hat{c}_1^{-1}$ to solve

$$\max \ \log c_1^{-1}$$
$$\text{s.t} \ \ \hat{p}_1 c_1^{-1} \le \hat{w}_1 \bar{\ell}_2 + (\hat{p}_1(1-\delta) + \hat{r}_1)\bar{k}_1^{-1} + m^{-1}$$
$$c_1^{-1} \ge 0 .$$

- Given $\hat{p}_1$, $\hat{p}_2$, $\hat{w}_1$, $\hat{w}_1$, $\hat{r}_1$, $\hat{r}_2$, consumer 0 chooses $(\hat{c}_1^0, \hat{c}_2^0)$, $\hat{k}_2^0$, to solve

$$\max \ \log c_1^0 + \log c_2^0$$
$$\text{s.t.} \ \ \hat{p}_1(c_1^0 + k_2^0) + \hat{p}_2 c_2^0 \le \hat{w}_1 \bar{\ell}_2 + \hat{w}_2 \bar{\ell}_3 + (\hat{p}_1(1-\delta) + \hat{r}_1)\bar{k}_1^0 + (\hat{p}_2(1-\delta) + \hat{r}_2)k_2^0 + m^0$$
$$c_1^0, c_2^0, k_2^0 \ge 0 .$$

- Given $\hat{p}_t$, $\hat{p}_{t+1}$, $\hat{p}_{t+2}$, $\hat{w}_t$, $\hat{w}_{t+1}$, $\hat{w}_{t+2}$, $\hat{r}_{t+1}$, consumer t, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{c}_{t+2}^t)$, $(\hat{k}_{t+1}^t, \hat{k}_{t+2}^t)$, to solve

$$\max \ \log c_t^t + \log c_{t+1}^t + \log c_{t+2}^t$$
$$\text{s.t.} \ \ \hat{p}_t(c_t^t + k_{t+1}^t) + \hat{p}_{t+1}(c_{t+1}^t + k_{t+2}^t) + \hat{p}_{t+2} c_{t+2}^t$$
$$\le \hat{w}_t \bar{\ell}_1 + \hat{w}_{t+1} \bar{\ell}_2 + \hat{w}_{t+2} \bar{\ell}_3 + (\hat{p}_{t+1}(1-\delta) + \hat{r}_{t+1})k_{t+1}^t + (\hat{p}_{t+2}(1-\delta) + \hat{r}_{t+2})k_{t+2}^t$$
$$c_t^t, c_{t+1}^t, k_{t+1}^t \ge 0 .$$

- Firms minimize costs and earn 0 profits. In particular,

  $\hat{r}_t = \hat{p}_t \alpha \theta (\hat{k}_t^{t-2} + \hat{k}_t^{t-1})^{\alpha-1} (\bar{\ell}_3 + \bar{\ell}_2 + \bar{\ell}_1)^{1-\alpha}$, $t = 0, 1, \ldots$
  $\hat{w}_t = \hat{p}_t (1-\alpha)\theta (\hat{k}_t^{t-2} + \hat{k}_t^{t-1})^{\alpha} (\bar{\ell}_3 + \bar{\ell}_2 + \bar{\ell}_1)^{-\alpha}$, $t = 0, 1, \ldots$.

- $\hat{c}_t^{t-2} + \hat{c}_t^{t-1} + \hat{c}_t^t + \hat{k}_{t+1}^{t-1} + \hat{k}_{t+1}^t - (1-\delta)(\hat{k}_t^{t-2} + \hat{k}_t^{t-1}) = \theta(\hat{k}_t^{t-2} + \hat{k}_t^{t-1})^{\alpha} (\bar{\ell}_3 + \bar{\ell}_2 + \bar{\ell}_1)^{1-\alpha}$, $t = 1, 2, \ldots$.