MACROECONOMIC THEORY **T. J. KEHOE**
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ANSWERS TO MIDTERM EXAMINATION

1. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \ \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$
$$c_t^i \geq 0 .$$

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$.

(b) With sequential market markets structure, there are markets for goods and bonds open every period. Consumers trade goods and bonds among themselves.

A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, and asset holdings $\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ such that

- Given $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, the consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots; \hat{b}_1^i, \hat{b}_2^i, \hat{b}_3^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_0^i + b_1^i \leq w_0^i$$
$$c_t^i + b_{t+1}^i \leq w_t^i + \left(1 + \hat{r}_t\right) b_t^i, \ t = 1,2,\ldots$$
$$b_t^i \geq -B, \ c_t^i \geq 0$$
$$b_0^i = 0 .$$

Here $b_t^i \geq -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$
- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0,1,\ldots$

(c) **Proposition 1:** Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1$$

$$\hat{b}_1^i = w_0^i - \hat{c}_0^i$$

$$\hat{b}_{t+1}^i = w_t^i + (1+\hat{r}_t)\hat{b}_t^i - \hat{c}_t^i,\ t = 1, 2, \ldots.$$

Proposition 2: Suppose that $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium where

$$\hat{p}_0 = 1$$

$$\hat{p}_t = \prod_{\tau=1}^{t} \frac{1}{(1+\hat{r}_\tau)},\ t = 1, 2, \ldots.$$

(d) Using the two consumers' first order conditions

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t,$$

we can write

$$\frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1}.$$

In every period,

$$c_t^1 + c_t^2 = 3$$

$$c_t^1 + \frac{\lambda^1}{\lambda^2} c_t^1 = 3$$

$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 3.$$

Since this implies that c_t^1 is constant, we can normalize $p_0 = 1$ and use the first order condition to write

$$p_t = \beta^t,$$

which implies that

$$p_t c_t^1 = \beta^t \frac{3\lambda^2}{\lambda^1 + \lambda^2}.$$

Consequently,

$$\sum_{t=0}^{\infty} p_t c_t^1 = \frac{3\lambda^2}{\lambda^1+\lambda^2}\sum_{t=0}^{\infty}\beta^t = \frac{1}{1-\beta}\frac{3\lambda^2}{\lambda^1+\lambda^2} = \sum_{t=0}^{\infty} p_t w_t^1$$

$$\frac{1}{1-\beta}\frac{3\lambda^2}{\lambda^1+\lambda^2} = 2\sum_{t=0}^{\infty} p_{2t} + \sum_{t=0}^{\infty} p_{2t+1}$$

$$\frac{1}{1-\beta}\frac{3\lambda^2}{\lambda^1+\lambda^2} = 2\sum_{t=0}^{\infty} \beta^{2t} + \beta\sum_{t=0}^{\infty} \beta^{2t}$$

$$\frac{1}{1-\beta}\frac{3\lambda^2}{\lambda^1+\lambda^2} = \frac{2+1\beta}{1-\beta^2}$$

$$\frac{\lambda^2}{\lambda^1+\lambda^2} = \frac{2+\beta}{3(1+\beta)},$$

which implies that

$$\frac{\lambda^1}{\lambda^1+\lambda^2} = \frac{1+2\beta}{3(1+\beta)}.$$

$$c_t^1 = \frac{2+\beta}{1+\beta}$$

$$c_t^2 = \frac{1+2\beta}{1+\beta}.$$

(We can even work out λ^1 and λ^2, although the question does not require this and it would be a waste of precious time to do so during the exam.

$$\lambda^1 = \frac{1}{c_0^1} = \frac{1+\beta}{2+\beta}$$

$$\lambda^2 = \frac{1}{c_0^2} = \frac{1+\beta}{1+2\beta}.$$

Check:

$$\frac{\lambda^1}{\lambda^1+\lambda^2} = \frac{\frac{1+\beta}{2+\beta}}{\frac{1+\beta}{2+\beta}+\frac{1+\beta}{1+2\beta}} = \frac{\frac{1}{2+\beta}}{\frac{1}{2+\beta}+\frac{1}{1+2\beta}} = \frac{1+2\beta}{1+2\beta+2+\beta} = \frac{1+2\beta}{3(1+\beta)}.)$$

To calculate the sequential markets equilibrium, we just use the formulas from proposition 1 in part c. For example,

$$r_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1 = \frac{1}{\beta} - 1.$$

Notice that, in $t=0$,

$$\hat{b}_1^1 = 2 - \frac{2+1\beta}{1+\beta} = \frac{\beta}{1+\beta}.$$

$$\hat{b}_1^2 = -\hat{b}_1^1 = -\frac{\beta}{1+\beta}$$

Consequently, consumer 1 lends $\beta/(1+\beta)$ in even periods, and consumer 2 pays back

$$(1+r_1)\hat{b}_1^1 = \left(1+\frac{1}{\beta}-1\right)\frac{\beta}{1+\beta} = \frac{1}{1+\beta}$$

in odd periods. That is,

$$(\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \hat{b}_4^1, ...) = \left(\frac{\beta}{1+\beta}, 0, \frac{\beta}{1+\beta}, 0, ...\right)$$

$$(\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \hat{b}_4^2, ...) = \left(-\frac{\beta}{1+\beta}, 0, -\frac{\beta}{1+\beta}, 0, ...\right).$$

(e) A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots$; $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, asset holdings $\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, ...$; $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, ...$, and storage levels $\hat{x}_0^1, \hat{x}_1^1, \hat{x}_2^1, ...$; $\hat{x}_0^2, \hat{x}_1^2, \hat{x}_2^2, ...$,such that

- Given $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, the consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$; $\hat{b}_1^i, \hat{b}_2^i, \hat{b}_3^i, ...$; $\hat{x}_0^i, \hat{x}_1^i, \hat{x}_2^i, ...$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_0^i + b_1^i + x_0^i \le w_0^i$$
$$c_t^i + b_{t+1}^i + x_t^i \le w_t^i + \left(1+\hat{r}_t\right) b_t^i + x_{t-1}^i,\ t = 1,2,\ldots$$
$$b_t^i \ge -B,\ c_t^i \ge 0,\ x_t^i \ge 0.$$

Here $b_t^i \ge -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_0^1 + \hat{c}_0^2 + x_0^1 + x_0^2 = w_0^1 + w_0^2$
  $\hat{c}_t^1 + \hat{c}_t^2 + x_t^1 + x_t^2 = w_t^1 + w_t^2 + x_{t-1}^1 + x_{t-1}^2$, $t = 1,2,\ldots$
- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0,1,\ldots$

The storage technology will not be used in the equilibrium because saving in the good, which has a gross return of 1, is dominated by saving in assets, which has a gross return of $1 + r_{t+1} > 1$. To see this, take the first-order condition of the consumer's problem with respect to b_{t+1}^i,

$$-\lambda_t^i + \lambda_{t+1}^i(1+r_{t+1}) = 0$$

and the first order condition with respect to x_t^i :

$$-\lambda_t^i + \lambda_{t+1}^i \leq 0, \; = 0 \text{ if } x_t^i > 0.$$

Since in the equilibrium of the model without storage, $1 + r_{t+1} = 1/\beta > 1$,

$$-\lambda_t^i + \lambda_{t+1}^i = -\lambda_t^i + \frac{\lambda_t^i}{1 + r_{t+1}} = -\lambda_t^i + \lambda_t^i \beta = -(1-\beta)\lambda_t^i < 0,$$

which implies that $x_t^i = 0$.

2. (a) An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{gathered} \max \; c_1^0 \\ \text{s.t} \;\; \hat{p}_1 c_1^0 \leq \hat{p}_1 w_2 + m \\ c_1^0 \geq 0. \end{gathered}$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer *t*, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ to solve

$$\begin{gathered} \max 2\log c_t^t + c_{t+1}^t \\ \text{s.t.} \;\; \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \leq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 \\ c_t^t, c_{t+1}^t \geq 0. \end{gathered}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t-1$ and t trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and asset holdings $\hat{b}_2^1, \hat{b}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{gathered} \max \; c_1^0 \\ \text{s.t} \;\; c_1^0 \leq w_2 + m \\ c_1^0 \geq 0. \end{gathered}$$

- Given $\hat{r}_{t+1}$, consumer t, $t = 1,2,\ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{b}_{t+1}^t$ to solve

$$\begin{gathered} \max 2\log c_t^t + c_{t+1}^t \\ \text{s.t. } c_t^t + b_{t+1}^t \leq w_1 \\ c_{t+1}^t \leq w_2 + (1+\hat{r}_{t+1})b_{t+1}^t \\ c_t^t, c_{t+1}^t \geq 0 . \end{gathered}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1,2,\ldots$.
- $\hat{b}_2^1 = m$, $\hat{b}_{t+1}^t = \left[\prod_{\tau=2}^{t}(1+\hat{r}_\tau)\right]m$, $t = 2,3,\ldots$.

(c) Since there is no fiat money, there is only one good per period, there is only one consumer type in each generation, and consumers live for only two periods, the equilibrium allocation is autarky:

$$\begin{gathered} \hat{c}_1^0 = w_2 \\ (\hat{c}_t^t, \hat{c}_{t+1}^t) = (w_1, w_2) \end{gathered}$$

The first order conditions from the consumers' problems in the Arrow-Debreu equilibrium are

$$\begin{gathered} \frac{2}{c_t^t} - \lambda^t p_t = 0 \\ 1 - \lambda^t p_{t+1} \leq 0 \,,\ = 0 \text{ if } c_{t+1}^t > 0 \ . \end{gathered}$$

Since $c_{t+1}^t = w_2 > 0$, the second condition holds with equality. Consequently,

$$\begin{gathered} \lambda^t = \frac{1}{p_{t+1}} \\ \frac{2}{\lambda^t p_t} = \frac{2p_{t+1}}{p_t} = c_t^t = w_1 \\ \frac{p_{t+1}}{p_t} = \frac{w_1}{2} . \end{gathered}$$

Normalizing $\hat{p}_1 = 1$, we obtain $\hat{p}_t = (w_1/2)^{t-1}$. Similarly, the first order conditions from the consumers' problems in the sequential markets equilibrium, imply that

$$1 + \hat{r}_{t+1} = \frac{2}{\hat{c}_t^t} = \frac{2}{w_1}$$

or $\hat{r}_t = 2/w_1 - 1$. Since the equilibrium allocation is autarky, $\hat{b}_{t+1}^t = 0$.

(d) An allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is **feasible** if

$$\hat{c}_t^{t-1} + \hat{c}_t^t \le w_2 + w_1 \,,\ t = 1,\ 2,\ \ldots .$$

An allocation is **Pareto efficient** if it is feasible and there exists no other allocation $\bar{c}_1^0, (\bar{c}_1^1, \bar{c}_2^1), (\bar{c}_2^2, \bar{c}_3^2), \ldots$ that is also feasible and satisfies

$$\bar{c}_1^0 \ge \hat{c}_1^0$$
$$2\log \bar{c}_t^t + \bar{c}_{t+1}^t \ge 2\log \hat{c}_t^t + \hat{c}_{t+1}^t \,,\ t = 1, 2, \ldots ,$$

with at least one inequality strict.

If $(w_1, w_2) = (1,1)$, the equilibrium allocation is Pareto efficient. We prove this by contradiction. If the equilibrium allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is not Pareto efficient, then there is another feasible allocation that $\bar{c}_1^0, (\bar{c}_1^1, \bar{c}_2^1), (\bar{c}_2^2, \bar{c}_3^2), \ldots$ is Pareto superior to $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ If

$$2\log \bar{c}_t^t + \bar{c}_{t+1}^t > 2\log \hat{c}_t^t + \hat{c}_{t+1}^t ,$$

then

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t > \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Otherwise, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ would not solve the maximization problem of generation t. Similarly, $\bar{c}_1^0 > \hat{c}_1^0$ implies $\hat{p}_1 \bar{c}_1^0 > \hat{p}_1 w_2$.

Suppose that

$$2\log \bar{c}_t^t + \bar{c}_{t+1}^t \ge 2\log \hat{c}_t^t + \hat{c}_{t+1}^t$$

but that

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t < \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Then let

$$\tilde{c}_t^t = \bar{c}_t^t + \frac{\hat{p}_t w_1 + \hat{p}_{t+1} w_2 - \hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t}{\hat{p}_t} > \bar{c}_t^t$$

and $\tilde{c}_{t+1}^t = \bar{c}_{t+1}^t$. Then

$$2\log \tilde{c}_t^t + \tilde{c}_{t+1}^t > 2\log \hat{c}_t^t + \hat{c}_{t+1}^t$$

but

$$\hat{p}_t \tilde{c}_t^t + \hat{p}_{t+1} \tilde{c}_{t+1}^t = \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Once again, this would imply that $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ would not solve the maximization problem of generation t, which is impossible. Consequently,

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t \ge \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Similarly, $\bar{c}_1^0 \geq \hat{c}_1^0$ implies $\hat{p}_1 \bar{c}_1^0 \geq \hat{p}_1 w_2$. Therefore

$$\hat{p}_1 \bar{c}_1^0 \geq \hat{p}_1 w_2$$

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t \geq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 \, , \; t = 1, 2, \ldots ,$$

with at least one inequality strict. Adding these inequalities up, we obtain

$$\sum\nolimits_{t=1}^{\infty} \hat{p}_t (\bar{c}_t^{t-1} + \bar{c}_t^t) > \sum\nolimits_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) \, .$$

It is here that $\hat{p}_t = (w_1 / 2)^{t-1}$, where $w_1 / 2 < 1$ plays its role in ensuring that these series converge.

$$\sum\nolimits_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) = \sum\nolimits_{t=1}^{\infty} (w_1 / 2)^{t-1} (w_1 + w_2) = \frac{w_1 + w_2}{1 - \frac{w_1}{2}} = \frac{1+1}{1 - \frac{1}{2}} = 4 < \infty$$

Multiplying the feasibility condition in period t by $\hat{p}_t > 0$ and adding up yields

$$\sum\nolimits_{t=1}^{\infty} \hat{p}_t (\bar{c}_t^{t-1} + \bar{c}_t^t) \leq \sum\nolimits_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) = 4 < \infty \, ,$$

which is a contradiction.

(e) A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_1, \hat{r}_2, \ldots$, an allocation $\hat{c}_1^{10}, \hat{c}_1^{20}, (\hat{c}_1^{11}, \hat{c}_2^{11}), (\hat{c}_1^{21}, \hat{c}_2^{21}), (\hat{c}_2^{12}, \hat{c}_3^{12}), (\hat{c}_2^{22}, \hat{c}_3^{22})$ …, and asset holdings $\hat{b}_2^{11}, \hat{b}_2^{21}, \hat{b}_3^{12}, \hat{b}_3^{22} \ldots$ such

- Consumer $i0$ chooses $\hat{c}_1^{i0}$, $i = 1, 2$, to solve

$$\begin{gathered} \max \; u_{i0}(c_1^{i0}) \\ \text{s.t } \; c_1^{i0} \leq w_2^i + m^i \\ c_1^{i0} \geq 0 \, . \end{gathered}$$

- Given $\hat{r}_t$, consumer it, $i = 1, 2$, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^{it}, \hat{c}_{t+1}^{it})$ and $\hat{b}_{t+1}^{it}$ to solve

$$\begin{gathered} \max \; u_i(c_t^{it}, c_{t+1}^{it}) \\ \text{s.t. } \; c_t^{it} + b_{t+1}^{it} \leq w_1^i \\ c_{t+1}^{it} \leq w_2^i + (1 + \hat{r}_{t+1}) b_{t+1}^{it} \\ c_t^{it}, c_{t+1}^{it} \geq 0 \, . \end{gathered}$$

- $\hat{c}_t^{1t-1} + \hat{c}_t^{2t-1} + \hat{c}_t^{1t} + \hat{c}_t^{2t} = w_2^1 + w_2^2 + w_1^1 + w_1^2$, $t = 1, 2, \ldots$.

- $\hat{b}_2^1 + \hat{b}_2^2 = m^1 + m^2$
  $\hat{b}_{t+1}^{1t} + \hat{b}_{t+1}^{2t} = \left[\prod\nolimits_{\tau=1}^{t-1} (1 + \hat{r}_{\tau+1}) \right] (m^1 + m^2)$, $t = 2, 3, \ldots$.

3. (a) An **Arrow-Debreu equilibrium** is sequences of prices of goods $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, rental rates $\hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \ldots$, and capital stocks $\hat{k}_0^1, \hat{k}_1^1, \ldots; \hat{k}_0^2, \hat{k}_1^2, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, $\hat{w}_0, \hat{w}_1, \ldots$, and $\hat{r}_0$, consumer i chooses $\hat{c}_0^i, \hat{c}_1^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{w}_t \overline{\ell}_t^i + (\hat{p}_0(1-\delta) + \hat{r}_0)\overline{k}_0$$
$$c_t \geq 0.$$

(Here we have the consumers sell their initial capital to firms and have firms make capital accumulation decisions. If we have consumers make capital accumulation decisions, then consumers choose $\hat{k}_0, \hat{k}_1, \ldots$ and the budget constraint is $\sum_{t=0}^{\infty} \hat{p}_t \left(c_t^i + k_{t+1}^i - (1-\delta)k_t^i\right) \leq \sum_{t=0}^{\infty} \left(\hat{w}_t \overline{\ell}_t^i + \hat{r}_t k_t^i\right)$.)

- $\hat{r}_t = \hat{p}_t \alpha\theta(\hat{k}_t^1 + \hat{k}_t^1)^{\alpha-1}(\overline{\ell}_t^1 + \overline{\ell}_t^2)^{1-\alpha} = \hat{p}_t \alpha\theta(\hat{k}_t^1 + \hat{k}_t^1)^{\alpha-1} 3^{1-\alpha}$, $t = 0, 1, \ldots$

- $\hat{w}_t = \hat{p}_t(1-\alpha)\theta(\hat{k}_t^1 + \hat{k}_t^2)^{\alpha} 3^{-\alpha}$, $t = 0, 1, \ldots$

- $(\hat{p}_{t+1}(1-\delta) + \hat{r}_{t+1}) - \hat{p}_t \leq 0$, $= 0$ if $\hat{k}_{t+1} > 0$, $t = 0, 1, \ldots$

(A good answer would explain that these are the profit maximization conditions for constant returns. Notice that, if we have consumers make capital accumulation decisions, then the zero profit condition on accumulating capital is a first order condition for utility maximization and does not need to be included as a separate equilibrium condition.)

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{k}_{t+1}^1 + \hat{k}_{t+1}^2 - (1-\delta)(\hat{k}_t^1 + \hat{k}_t^2) = \theta(\hat{k}_t^1 + \hat{k}_t^1)^{\alpha}(\overline{\ell}_t^1 + \overline{\ell}_t^2)^{1-\alpha} = \theta(\hat{k}_t^1 + \hat{k}_t^1)^{\alpha} 3^{1-\alpha}$, $t = 0, 1, \ldots$

(b) A **sequential markets equilibrium** is sequences of wages $\hat{w}_0, \hat{w}_1, \ldots$, rental rates $\hat{r}_0^k, \hat{r}_1^k, \ldots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \ldots$, capital stocks $\hat{k}_0^1, \hat{k}_1^1, \ldots; \hat{k}_0^2, \hat{k}_1^2, \ldots$ and bond holdings $\hat{b}_0^1, \hat{b}_1^1, \ldots; \hat{b}_0^2, \hat{b}_1^2, \ldots$, such that

- Given $\hat{w}_0, \hat{w}_1, \ldots$, $\hat{r}_0^k, \hat{r}_1^k, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \ldots$, consumer i, $i = 1, 2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \ldots$, $\hat{k}_0^i, \hat{k}_1^i, \ldots$, and $\hat{b}_0^i, \hat{b}_1^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_t^i + k_{t+1}^i + b_{t+1}^i \leq \hat{w}_t \overline{\ell}_t^i + (1 + \hat{r}_t^k - \delta)k_t^i + (1 + \hat{r}_t^b)b_t^i, \; t = 0, 1, \ldots$$

$$c_t^i, k_t^i \geq 0, \; b_t^i \geq -B$$
$$k_0^i = \bar{k}_0^i, \; b_0^i = 0.$$

- $\hat{w}_t = (1-\alpha)\theta(\hat{k}_t^1 + \hat{k}_t^2)^\alpha 3^{-\alpha}$, $t = 0,1,\ldots$
- $\hat{r}_t^k = \alpha\theta(\hat{k}_t^1 + \hat{k}_t^2)^{\alpha-1} 3^{1-\alpha}$, $t = 0,1,\ldots$
- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{k}_{t+1}^1 + \hat{k}_{t+1}^2 - (1-\delta)(\hat{k}_t^1 + \hat{k}_t^2) = \theta(\hat{k}_t^1 + \hat{k}_t^2)^\alpha 3^{1-\alpha}$, $t = 0,1,\ldots$
- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 1,2,\ldots$

(c) **Proposition 1:** Suppose that $\hat{p}_0, \hat{p}_1, \ldots; \hat{w}_0, \hat{w}_1, \ldots; \hat{r}_0, \hat{r}_1, \ldots; \hat{c}_0^1, \hat{c}_1^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \ldots;$ $\hat{k}_0^1, \hat{k}_1^1, \ldots; \hat{k}_0^2, \hat{k}_1^2, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_0^k, \hat{r}_1^k, \ldots; \hat{r}_0^b, \hat{r}_1^b, \ldots; \tilde{w}_0, \tilde{w}_1, \ldots;$ $\hat{c}_0^1, \hat{c}_1^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \ldots; \hat{k}_0^1, \hat{k}_1^1, \ldots; \hat{k}_0^2, \hat{k}_1^2, \ldots; \hat{b}_0^1, \hat{b}_1^1, \ldots; \hat{b}_0^2, \hat{b}_1^2, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t^k = \frac{\hat{r}_t}{\hat{p}_t}$$
$$\hat{r}_t^b = \hat{r}_t^k - \delta$$
$$\tilde{w}_t = \frac{\hat{w}_t}{\hat{p}_t}$$
$$\hat{b}_0^i = 0$$
$$\hat{b}_{t+1}^i = \hat{w}_t \bar{\ell}_t^i + (1 + \hat{r}_t^k - \delta)\hat{k}_t^i + (1 + \hat{r}_t^b)\hat{b}_t^i - \hat{c}_t^i - \hat{k}_{t+1}^i, \; t = 0,1,\ldots.$$

Proposition 2: Suppose that $\hat{r}_0^k, \hat{r}_1^k, \ldots; \hat{r}_0^b, \hat{r}_1^b, \ldots; \tilde{w}_0, \tilde{w}_1, \ldots; \hat{c}_0^1, \hat{c}_1^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \ldots;$ $\hat{k}_0^1, \hat{k}_1^1, \ldots; \hat{k}_0^2, \hat{k}_1^2, \ldots; \hat{b}_0^1, \hat{b}_1^1, \ldots; \hat{b}_0^2, \hat{b}_1^2, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_0, \hat{p}_1, \ldots; \hat{w}_0, \hat{w}_1, \ldots; \hat{r}_0, \hat{r}_1, \ldots; \hat{c}_0^1, \hat{c}_1^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \ldots;$ $\hat{k}_0^1, \hat{k}_1^1, \ldots; \hat{k}_0^2, \hat{k}_1^2, \ldots$ is an Arrow-Debreu equilibrium where

$$\hat{p}_0 = 1$$
$$\hat{p}_t = \prod_{\tau=1}^{t} \frac{1}{1 + \hat{r}_\tau^k - \delta}, \; t = 1,2,\ldots$$
$$\hat{r} = \hat{p}_t \hat{r}_t^k$$
$$\tilde{w}_t = \hat{p}_t \hat{w}_t.$$

(Notice that, according to the way in which we have done things, we need to use separate symbols for the Arrow-Debreu wage, the price of labor services in period t in terms of period 0 goods, and the sequential markets wage, of labor services in period t in terms of period t goods.)

(d) A **sequential markets equilibrium** is sequences of rental rates $\hat{r}_0^k, \hat{r}_1^k, \ldots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \ldots$, labor levels $\hat{\ell}_0^1, \hat{\ell}_1^1, \ldots; \hat{\ell}_0^2, \hat{\ell}_1^2$, capital stocks $\hat{k}_0^1, \hat{k}_1^1, \ldots; \hat{k}_0^2, \hat{k}_1^2, \ldots$ and bond holdings $\hat{b}_0^1, \hat{b}_1^1, \ldots; \hat{b}_0^2, \hat{b}_1^2, \ldots$ such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \ldots$, and $\hat{w}_0, \hat{w}_1, \ldots$, the consumer chooses $\hat{c}_0^i, \hat{c}_1^i, \ldots; \hat{\ell}_0^i, \hat{\ell}_1^i, \ldots;$ $\hat{k}_0^i, \hat{k}_1^i, \ldots$;and $\hat{b}_0^i, \hat{b}_1^i, \ldots$ to solve

$$\sum_{t=0}^{\infty} \beta^t \left(\gamma \log c_t^i + (1-\gamma) \log(\bar{\ell}_t^i - \ell_t^i) \right)$$
$$\text{s.t. } c_t^i + k_{t+1}^i + b_{t+1}^i \leq \hat{w}_t \ell_t^i + (1 + \hat{r}_t^k - \delta) k_t^i + (1 + \hat{r}_t^b) b_t^i, \ t = 0, 1, \ldots$$
$$c_t^i, k_t^i \geq 0, \ \bar{\ell}_t^i \geq \ell_t^i \geq 0, \ b_t^i \geq -B$$
$$k_0^i = \bar{k}_0, \ b_0^i = 0.$$

- $\hat{r}_t^k = \alpha\theta(\hat{k}_t^1 + \hat{k}_t^2)^{\alpha-1}(\hat{\ell}_t^1 + \hat{\ell}_t^2)^{1-\alpha}$, $t = 0, 1, \ldots$

- $\hat{w}_t = (1-\alpha)\theta(\hat{k}_t^1 + \hat{k}_t^2)^{\alpha}(\hat{\ell}_t^1 + \hat{\ell}_t^2)^{-\alpha}$, $t = 0, 1, \ldots$

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{k}_{t+1}^1 + \hat{k}_{t+1}^2 - (1-\delta)(\hat{k}_t^1 + \hat{k}_t^2) = \theta(\hat{k}_t^1 + \hat{k}_t^2)^{\alpha}(\hat{\ell}_t^1 + \hat{\ell}_t^2)^{1-\alpha}$, $t = 0, 1, \ldots$

- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 1, 2, \ldots$