MACROECONOMIC THEORY **T. J. KEHOE**
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ANSWERS TO MIDTERM EXAMINATION

1. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \dots$ and consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \dots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \dots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \dots$, consumer i, $i = 1, 2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \dots$ to solve

$$\max \ \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$
$$c_t^i \geq 0.$$

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0, 1, \dots$.

(b) With sequential market markets structure, there are markets for goods and bonds open every period. Consumers trade goods and bonds among themselves.

A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_1, \hat{r}_2, \hat{r}_3, \dots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \dots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \dots$, and asset holdings $\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \dots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \dots$ such that

- Given $\hat{r}_1, \hat{r}_2, \hat{r}_3, \dots$, the consumer i, $i = 1, 2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \dots; \hat{b}_1^i, \hat{b}_2^i, \hat{b}_3^i, \dots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_0^i + b_1^i \leq w_0^i$$
$$c_t^i + b_{t+1}^i \leq w_t^i + \left(1 + \hat{r}_t\right) b_t^i,\ t = 1, 2, \dots$$
$$b_t^i \geq -B$$
$$c_t^i \geq 0.$$

Here $b_t^i \geq -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0, 1, \dots$

- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0,1,\ldots$

(c) **Proposition 1:** Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1$$
$$\hat{b}_1^i = w_0^i - \hat{c}_0^i$$
$$\hat{b}_{t+1}^i = w_t^i + (1+\hat{r}_t)\hat{b}_t^i - \hat{c}_t^i,\ t = 1,2,\ldots.$$

Proposition 2: Suppose that $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium where

$$\hat{p}_0 = 1$$
$$\hat{p}_t = \prod_{s=1}^{t} \frac{1}{(1+\hat{r}_s)},\ t = 1,2,\ldots.$$

(d) Using the two consumers' first order conditions

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t,$$

we can write

$$\frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1}.$$

In even periods,

$$c_t^1 + c_t^2 = 3$$
$$c_t^1 + \frac{\lambda^1}{\lambda^2} c_t^1 = 3$$
$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 3.$$

Similarly, in odd periods,

$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 5 .$$

Normalizing $p_0 = 1$, we can use the first order condition to write

$$p_t = \begin{cases} \beta^t & \text{if } t \text{ is even} \\ \frac{3}{5}\beta^t & \text{if } t \text{ is odd} \end{cases} ,$$

which implies that

$$p_t c_t^1 = \beta^t \frac{3\lambda^2}{\lambda^1 + \lambda^2} .$$

Consequently,

$$\sum_{t=0}^{\infty} p_t c_t^1 = \frac{3\lambda^2}{\lambda^1 + \lambda^2} \sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = \sum_{t=0}^{\infty} p_t w_t^1$$

$$\frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = 2\sum_{t=0}^{\infty} p_{2t} + \sum_{t=0}^{\infty} p_{2t+1}$$

$$\frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = 2\sum_{t=0}^{\infty} \beta^{2t} + \frac{3}{5}\beta \sum_{t=0}^{\infty} \beta^{2t}$$

$$\frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = \frac{2 + \frac{3}{5}\beta}{1-\beta^2}$$

$$\frac{\lambda^2}{\lambda^1 + \lambda^2} = \frac{\frac{2}{3} + \frac{1}{5}\beta}{1+\beta} = \frac{10+3\beta}{15(1+\beta)} ,$$

which implies that

$$\frac{\lambda^1}{\lambda^1 + \lambda^2} = \frac{5+12\beta}{15(1+\beta)} .$$

$$c_t^1 = \begin{cases} \frac{10+3\beta}{5(1+\beta)} & \text{if } t \text{ is even} \\ \frac{10+3\beta}{3(1+\beta)} & \text{if } t \text{ is odd} \end{cases}$$

$$c_t^2 = \begin{cases} \dfrac{5+12\beta}{5(1+\beta)} & \text{if } t \text{ is even} \\ \dfrac{5+12\beta}{3(1+\beta)} & \text{if } t \text{ is odd} \end{cases}.$$

(We can even work out λ^1 and λ^2, although the question does not require this and it would be a waste of precious time to do so during the exam.

$$\lambda^1 = \frac{1}{c_0^1} = \frac{5(1+\beta)}{10+3\beta}$$
$$\lambda^2 = \frac{1}{c_0^2} = \frac{5(1+\beta)}{5+12\beta}.$$

Check:

$$\frac{\lambda^1}{\lambda^1+\lambda^2} = \frac{\dfrac{5(1+\beta)}{10+3\beta}}{\dfrac{5(1+\beta)}{10+3\beta}+\dfrac{5(1+\beta)}{5+12\beta}} = \frac{\dfrac{1}{10+3\beta}}{\dfrac{1}{10+3\beta}+\dfrac{1}{5+12\beta}} = \frac{5+12\beta}{5+12\beta+10+3\beta} = \frac{5+12\beta}{15(1+\beta)}.)$$

To calculate the sequential markets equilibrium, we just use the formulas from proposition 1 in part c. For example,

$$r_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1 = \begin{cases} \dfrac{5}{3\beta} - 1 & \text{if } t \text{ is odd} \\ \dfrac{3}{5\beta} - 1 & \text{if } t \text{ is even} \end{cases}.$$

Notice that, in $t=0$,

$$\hat{b}_1^1 = 2 - \frac{10+3\beta}{5(1+\beta)} = \frac{2\beta}{5(1+\beta)}.$$

That is, in even periods, consumer 1 lends $\dfrac{2\beta}{5(1+\beta)}$ to consumer 2, who pays back $\dfrac{2}{3(1+\beta)}$ in odd periods.

(e) A **sequential markets equilibrium** is sequences of rental rates on capital $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$; wages $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$; interest rates on bonds $\hat{r}_0^b, \hat{r}_1^b, \hat{r}_2^b, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots$; $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$; capital holdings $\hat{k}_0^1, \hat{k}_1^1, \hat{k}_2^1, \ldots$; $\hat{k}_0^2, \hat{k}_1^2, \hat{k}_2^2, \ldots$; bond holdings $\hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots$; $\hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$; and production plans $(\hat{y}_0, \hat{k}_0, \hat{\ell}_0), (\hat{y}_1, \hat{k}_1, \hat{\ell}_1), (\hat{y}_2, \hat{k}_2, \hat{\ell}_2), \ldots$ such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$, $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \hat{r}_2^b, \ldots$, consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots, \hat{k}_0^i, \hat{k}_1^i, \hat{k}_2^i, \ldots$, $\hat{b}_0^i, \hat{b}_1^i, \hat{b}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_t^i + k_{t+1}^i + b_{t+1}^i \le \hat{w}_t^i \bar{\ell}_t^i + (1 + \hat{r}_t^k - \delta) k_t^i + (1 + \hat{r}_t^b) b_t^i,\ t = 0,1,\ldots$$
$$c_t^i \ge 0,\ k_t^i \ge 0\ b_t^i \ge -B$$
$$k_0^i = \bar{k}_0^i,\ b_0^i = 0.$$

- Given $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$, $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$, firms choose $(\hat{y}_0, \hat{k}_0, \hat{\ell}_0), (\hat{y}_1, \hat{k}_1, \hat{\ell}_1), (\hat{y}_2, \hat{k}_2, \hat{\ell}_2), \ldots$ to minimize costs, and $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$, $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$ are such that firms earn 0 profits:

$$\hat{r}_t^k = \alpha\theta \hat{k}_t^{\alpha-1} \hat{\ell}_t^{1-\alpha}$$
$$\hat{w}_t = (1-\alpha)\theta \hat{k}_t^{\alpha} \hat{\ell}_t^{-\alpha}$$

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{k}_{t+1} - (1-\delta)\hat{k}_t = \hat{y}_t = \theta \hat{k}_t^{\alpha} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$
- $\hat{k}_t^1 + \hat{k}_t^2 = \hat{k}_t$, $t = 0,1,\ldots$
- $\bar{\ell}_t^1 + \bar{\ell}_t^2 = \hat{\ell}_t$, $t = 0,1,\ldots$
- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0,1,\ldots$

2. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 1. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\max\ \log c_1^0$$
$$\text{s.t}\ \ \hat{p}_1 c_1^0 \le \hat{p}_1 w_2 + m$$
$$c_1^0 \ge 0.$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer t, $t = 1,2,\ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ to solve

$$\max\ c_t^t + \log c_{t+1}^t$$

$$\text{s.t. } \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \leq \hat{p}_t w_1 + \hat{p}_{t+1} w_2$$
$$c_t^t, c_{t+1}^t \geq 0 .$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1,\ 2,\ \ldots$.

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t-1$ and t trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and asset holdings $\hat{b}_2^1, \hat{b}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\max \ \log c_1^0$$
$$\text{s.t } c_1^0 \leq w_2 + m$$
$$c_1^0 \geq 0 .$$

- Given $\hat{r}_{t+1}$, consumer *t*, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{b}_{t+1}^t$ to solve

$$\max \ c_t^t + \log c_{t+1}^t$$
$$\text{s.t. } c_t^t + b_{t+1}^t \leq w_1$$
$$c_{t+1}^t \leq w_2 + (1 + \hat{r}_{t+1}) b_{t+1}^t$$
$$c_t^t, c_{t+1}^t \geq 0 .$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.
- $\hat{b}_2^1 = m$, $\hat{b}_{t+1}^t = \left[\prod_{\tau=2}^t (1 + \hat{r}_\tau)\right] m$, $t = 2, 3, \ldots$.

(c) Since there is no fiat money, there is only one good per period, there is only one consumer type in each generation, and consumers live for only two periods, the equilibrium allocation is autarky:

$$\hat{c}_1^0 = w_2$$
$$(\hat{c}_t^t, \hat{c}_{t+1}^t) = (w_1, w_2)$$

The first order conditions from the consumers' problems in the Arrow-Debreu equilibrium are

$$1 - \lambda^t p_t \leq 0 ,\ = 0 \text{ if } c_t^t > 0 .$$
$$\frac{1}{c_{t+1}^t} - \lambda^t p_{t+1} = 0 ,$$

which imply

$$\frac{p_{t+1}}{p_t} = \frac{1}{c_{t+1}^t} = \frac{1}{w_2}$$

Normalizing $\hat{p}_1 = 1$, we obtain $\hat{p}_t = (1/w_2)^{t-1} = w_2^{1-t}$. Similarly, the first order conditions from the consumers' problems in the sequential markets equilibrium, imply that

$$1 + \hat{r}_{t+1} = \hat{c}_{t+1}^t = w_2$$

or $\hat{r}_t = w_2 - 1$. Since the equilibrium allocation is autarky, $\hat{b}_{t+1}^t = 0$.

(d) An allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is **feasible** if

$$\hat{c}_t^{t-1} + \hat{c}_t^t \leq w_2 + w_1, \ t = 1,\ 2,\ \ldots.$$

An allocation is **Pareto efficient** if it is feasible and there exists no other allocation $\overline{c}_1^0, (\overline{c}_1^1, \overline{c}_2^1), (\overline{c}_2^2, \overline{c}_3^2), \ldots$ that is also feasible and satisfies

$$\log \overline{c}_1^0 \geq \log \hat{c}_1^0$$
$$\overline{c}_t^t + \log \overline{c}_{t+1}^t \geq \hat{c}_t^t + \log \hat{c}_{t+1}^t, \ t = 1, 2, \ldots,$$

with at least one inequality strict.

If $w_2 > 1$, the equilibrium allocation is Pareto efficient. Suppose not. Then there exists a feasible allocation that is Pareto superior. If

$$\overline{c}_t^t + \log \overline{c}_{t+1}^t > \hat{c}_t^t + \log \hat{c}_{t+1}^t,$$

then

$$\hat{p}_t \overline{c}_t^t + \hat{p}_{t+1} \overline{c}_{t+1}^t > \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Otherwise, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ would not solve the maximization problem of generation t. Similarly, $\log \overline{c}_1^0 > \log \hat{c}_1^0$ implies $\hat{p}_1 \overline{c}_1^0 > \hat{p}_1 w_2$.

Suppose that

$$\overline{c}_t^t + \log \overline{c}_{t+1}^t \geq \hat{c}_t^t + \log \hat{c}_{t+1}^t$$

but that

$$\hat{p}_t \overline{c}_t^t + \hat{p}_{t+1} \overline{c}_{t+1}^t < \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Then let

$$\overline{\overline{c}}_t^t = \overline{c}_t^t + \frac{\hat{p}_t w_1 + \hat{p}_{t+1} w_2 - \hat{p}_t \overline{c}_t^t + \hat{p}_{t+1} \overline{c}_{t+1}^t}{\hat{p}_t} > \overline{c}_t^t$$

and $\overline{\overline{c}}_{t+1}^t = \overline{c}_{t+1}^t$. Then

$$\bar{\bar{c}}_t^t + \log \bar{\bar{c}}_{t+1}^t > \hat{c}_t^t + \log \hat{c}_{t+1}^t$$

but

$$\hat{p}_t \bar{\bar{c}}_t^t + \hat{p}_{t+1} \bar{\bar{c}}_{t+1}^t = \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Once again, this would imply that $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ would not solve the maximization problem of generation t, which is impossible. Consequently,

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t \geq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Similarly, $\log \bar{c}_1^0 \geq \log \hat{c}_1^0$ implies $\hat{p}_1 \bar{c}_1^0 \geq \hat{p}_1 w_2$.
Therefore

$$\hat{p}_1 \bar{c}_1^0 \geq \hat{p}_1 w_2$$
$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t \geq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 , \ t = 1, 2, \ldots ,$$

with at least one inequality strict. Adding these inequalities up, we obtain

$$\sum_{t=1}^{\infty} \hat{p}_t (\bar{c}_t^{t-1} + \bar{c}_t^t) > \sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) .$$

It is here that $\hat{p}_t = w_2^{1-t}$ plays its role in ensuring that these series converge.

$$\sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) = \sum_{t=1}^{\infty} w_2^{1-t} (w_1 + w_2) = \frac{w_1 + w_2}{1 - w_2^{-1}} < \infty$$

Multiplying the feasibility condition in period t by $\hat{p}_t > 0$ and adding up yields

$$\sum_{t=1}^{\infty} \hat{p}_t (\bar{c}_t^{t-1} + \bar{c}_t^t) \leq \sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) < \infty ,$$

which is a contradiction.

(e) An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$, and an allocation $\hat{c}_1^{10}, \hat{c}_1^{20}, (\hat{c}_1^{11}, \hat{c}_2^{11}), (\hat{c}_1^{21}, \hat{c}_2^{21}), (\hat{c}_2^{12}, \hat{c}_3^{12}), (\hat{c}_2^{22}, \hat{c}_3^{22}) \ \ldots$, such that

- Consumer $i0$ chooses $\hat{c}_1^{i0}$, $i = 1, 2$, to solve

$$\max \ u_{i0}(c_1^{i0})$$
$$\text{s.t} \quad \hat{p}_1 c_1^{i0} \leq \hat{p}_1 w_2^i + m^i$$
$$c_1^{i0} \geq 0 .$$

- Given $\hat{r}_t$, consumer it, $i = 1, 2$, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^{it}, \hat{c}_{t+1}^{it})$ and $\hat{b}_t^{it}$ to solve

$$\max \ u_i(c_t^{it}, c_{t+1}^{it})$$
$$\text{s.t.} \quad \hat{p}_t c_t^{it} + \hat{p}_{t+1} c_{t+1}^{it} \leq \hat{p}_t w_1^i + \hat{p}_{t+1} w_2^i$$
$$c_t^{it}, c_{t+1}^{it} \geq 0 .$$

- $\hat{c}_t^{1t-1} + \hat{c}_t^{2t-1} + \hat{c}_t^{1t} + \hat{c}_t^{2t} = w_2^1 + w_2^2 + w_1^1 + w_1^2$, $t = 1, 2, \ldots$.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_1, \hat{r}_2, \ldots$, an allocation $\hat{c}_1^{10}, \hat{c}_1^{20}, (\hat{c}_1^{11}, \hat{c}_2^{11}), (\hat{c}_1^{21}, \hat{c}_2^{21}), (\hat{c}_2^{12}, \hat{c}_3^{12}), (\hat{c}_2^{22}, \hat{c}_3^{22})$ …, and asset holdings $\hat{b}_1^{11}, \hat{b}_1^{21}, \hat{b}_2^{12}, \hat{b}_2^{22} \ldots$ such

- Consumer $i0$ chooses $\hat{c}_1^{i0}$, $i = 1, 2$, to solve

$$\begin{gathered}\max \; u_{i0}(c_1^{i0}) \\ \text{s.t} \;\; c_1^{i0} \le w_2^i + m^i \\ c_1^{i0} \ge 0 .\end{gathered}$$

- Given $\hat{r}_t$, consumer it, $i = 1, 2$, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^{it}, \hat{c}_{t+1}^{it})$ and $\hat{b}_t^{it}$ to solve

$$\begin{gathered}\max \; u_i(c_t^{it}, c_{t+1}^{it}) \\ \text{s.t.} \quad c_t^{it} + \hat{b}_t^{it} \le w_1^i \\ c_{t+1}^{it} \le w_2^i + (1 + \hat{r}_t)\hat{b}_t^{it} \\ c_t^{it}, c_{t+1}^{it} \ge 0 .\end{gathered}$$

- $\hat{c}_t^{1t-1} + \hat{c}_t^{2t-1} + \hat{c}_t^{1t} + \hat{c}_t^{2t} = w_2^1 + w_2^2 + w_1^1 + w_1^2$, $t = 1, 2, \ldots$.

- $\hat{b}_1^{11} + \hat{b}_1^{21} = m^1 + m^2$
  $\hat{b}_t^{1t} + \hat{b}_t^{2t} = \left[\prod_{\tau=1}^{t-1} (1 + \hat{r}_\tau)\right](m^1 + m^2)$, $t = 2, 3, \ldots$.

3. (a) The Euler conditions are

$$\beta^t \frac{1}{c_t} - p_t = 0$$

$$-p_{t-1} + p_t \alpha\theta k_t^{\alpha-1} = 0 ,$$

which can be combined with the feasibility constraint into

$$-\frac{1}{\theta k_{t-1}^{\alpha} - k_t} + \frac{\beta\alpha\theta k_t^{\alpha-1}}{\theta k_t^{\alpha} - k_{t+1}} = 0 .$$

The transversality condition is

$$\lim_{t\to\infty} p_t k_{t+1} = \lim_{t\to\infty} \frac{\beta^t k_{t+1}}{\theta k_t^\alpha - k_{t+1}} = 0 .$$

(b) The Bellman equation is

$$\begin{aligned} V(k) &= \max \log c + \beta V(k') \\ &\text{s.t. } c + k' \le \theta k^\alpha \\ &c, k' \ge 0 \\ &k \text{ fixed.} \end{aligned}$$

Guessing $V(k) = a_0 + a_1 \log k$. This becomes

$$\begin{aligned} a_0 + a_1 \log k &= \max \log c + \beta(a_0 + a_1 \log k') \\ &\text{s.t. } c + k' \le \theta k^\alpha \\ &c, k' \ge 0 \\ &k \text{ fixed.} \end{aligned}$$

The first order conditions for this problem are

$$\frac{1}{c} - p = 0$$

$$\frac{\beta a_1}{k'} - p = 0 ,$$

which imply that

$$k' = g(k) = \frac{\beta a_1}{1+\beta a_1} \theta k^\alpha .$$

We can plug these solutions back into the Bellman equation to obtain

$$a_0 + a_1 \log k = \log\left(\frac{1}{1+\beta a_1} \theta k^\alpha \right) + \beta \left[a_0 + a_1 \log\left(\frac{\beta a_1}{1+\beta a_1} \theta k^\alpha \right) \right].$$

Collecting all the terms on the right-hand side that involve $\log k$, we can solve for a_1 :

$$a_1 = \alpha + \alpha\beta a_1$$

$$a_1 = \frac{\alpha}{1-\alpha\beta} ,$$

which implies that

$$k' = g(k) = \alpha\beta\theta k^{\alpha}$$

$$c = (1-\alpha\beta)\theta k^{\alpha} .$$

[We could also solve for a_0:

$$a_0 = \frac{1}{1-\beta}\left[\log\left(\frac{\theta}{1+\beta a_1}\right) + \beta a_1 \log\left(\frac{\beta a_1 \theta}{1+\beta a_1}\right)\right]$$

$$a_0 = \frac{1}{1-\beta}\left[\log\left((1-\alpha\beta)\theta\right) + \frac{\alpha\beta}{1-\alpha\beta}\log\left(\alpha\beta\theta\right)\right].$$

but this is tedious and it is not called for.]

(c) Euler conditions: The condition for consumption is

$$\beta^t \frac{1}{c_t} - p_t = \frac{\beta^t}{(1-\alpha\beta)\theta k_t^{\alpha}} - p_t = 0$$

substituting this into the condition for

$$-\frac{\beta^{t-1}}{(1-\alpha\beta)\theta k_{t-1}^{\alpha}} + \frac{\beta^t \alpha\theta k_t^{\alpha-1}}{(1-\alpha\beta)\theta k_t^{\alpha}} = 0$$

$$-\frac{\beta^{t-1}}{(1-\alpha\beta)\theta k_{t-1}^{\alpha}} + \frac{\beta^t \alpha\theta k_t^{\alpha-1}}{(1-\alpha\beta)\theta k_t^{\alpha}} = 0$$

$$-k_t + \alpha\beta\theta k_{t-1}^{\alpha} = 0,$$

which is satisfied because it is just the policy function.

Transversality condition:

$$\lim_{t\to\infty} p_t k_{t+1} = \lim_{t\to\infty} \frac{\beta^t \alpha\beta\theta k_t^{\alpha}}{\theta k_t^{\alpha} - \alpha\beta\theta k_t^{\alpha}} = \frac{\alpha\beta}{1-\alpha\beta}\lim_{t\to\infty} \beta^t = 0$$

(d) Suppose that there is a representative consumer with the utility function and the endowment $\bar{k}_0$ of capital in period 0 and the endowment 1 of labor in every period. The production function is

$$y_t = \theta k_t^{\alpha} \ell_t^{1-\alpha} .$$

Capital depreciates completely every period.

With sequential market markets structure, there are markets for goods, capital services, labor services, and bonds open every period. Consumers sell labor services and rent capital to the firm. They buy goods from the firm, some of which they consume and some of which they save as capital. They trade bonds among themselves.

A **sequential markets equilibrium** is sequences of wages $\hat{w}_0, \hat{w}_1, \ldots$, rental rates $\hat{r}_0^k, \hat{r}_1^k, \ldots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, labor levels $\hat{\ell}_0, \hat{\ell}_1, \ldots$, capital stocks $\hat{k}_0, \hat{k}_1, \ldots$, and bond holdings $\hat{b}_0, \hat{b}_1, \ldots$, such that

- Given $\hat{w}_0, \hat{w}_1, \ldots$, $\hat{r}_0^k, \hat{r}_1^k, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \ldots$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$, $\hat{k}_0, \hat{k}_1, \ldots$, and $\hat{b}_0, \hat{b}_1, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$
$$\text{s.t. } c_t + k_{t+1} + b_{t+1} \le \hat{w}_t + (1 + \hat{r}_t^b) b_t,\ t = 0,1,\ldots$$
$$k_0 = \bar{k}_0,\ b_0 = 0$$
$$c_t, k_t \ge 0,\ b_t \ge -B.$$

- $\hat{w}_t = (1-\alpha)\theta \hat{k}_t^{\alpha} \hat{\ell}_t^{-\alpha}$, $t = 0,1,\ldots$
- $\hat{r}_t^k = \alpha\theta \hat{k}_t^{\alpha-1} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$
- $\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}$, $t = 0,1,\ldots$
- $\hat{\ell}_t = 1$, $t = 0,1,\ldots$
- $\hat{b}_t = 0$, $t = 0,1,\ldots$

[An alternative would be to specify an Arrow-Debreu equilibrium:

With Arrow-Debreu markets, there are futures markets of goods, capital, labor services, and capital services, open in period 0. Consumers buy goods from firms. Who makes the capital accumulation decision can be modeled different ways. We could have consumers buy and sell future claims to capital and sell claims to capital services to firms, or we could have consumers sell their initial capital to firms and have firms buy and sell future claims to capital and sell claims to capital services to other firms.

An **Arrow-Debreu equilibrium** is sequences of prices of goods $\hat{p}_0, \hat{p}_1, \ldots$, wage rates $\hat{w}_0, \hat{w}_1, \ldots$, rental rates $\hat{r}_0, \hat{r}_1, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, and capital stocks $\hat{k}_0, \hat{k}_1, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \ldots$, $\hat{w}_0, \hat{w}_1, \ldots$, and $\hat{r}_0$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$
$$\text{s.t. } \sum_{t=0}^{\infty} (\hat{p}_t c_t + \hat{p}_t k_{t+1}) \le \sum_{t=0}^{\infty} (\hat{w}_t + \hat{r}_t k_t)$$
$$c_t, k_t \ge 0 \,,\; k_0 = \bar{k}_0 .$$

- $\hat{w}_t = \hat{p}_t (1-\alpha)\theta \hat{k}_t^{\alpha} \hat{\ell}_t^{-\alpha}$, $t = 0,1,\ldots$
- $\hat{r}_t = \hat{p}_t \alpha\theta \hat{k}_t^{\alpha-1} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$
- $\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}$, $t = 0,1,\ldots$]

To calculate the sequential markets equilibrium, we just run the first order difference equation

$$k_{t+1} = g(k_t) = \alpha\beta\theta k_t^{\alpha}$$

forward, starting at $k_0 = \bar{k}_0$. [Notice that this problem actually has an analytical solution:

$$k_t = \alpha\beta\theta k_{t-1}^{\alpha} = \alpha\beta\theta\left(\alpha\beta\theta k_{t-2}^{\alpha}\right)^{\alpha} = \left(\alpha\beta\theta\right)^{\sum_{\tau=0}^{t-1}\alpha^{\tau}} \bar{k}_0^{\alpha^t} = \left(\alpha\beta\theta\right)^{\frac{1-\alpha^t}{1-\alpha}} \bar{k}_0^{\alpha^t} ,$$

but you do not need to work with this solution.]

We set

$$c_t = (1-\alpha\beta)\theta k_t^{\alpha}$$
$$b_t = 0$$
$$r_t^k = \alpha\theta k_t^{\alpha-1}$$
$$r_t^b = \alpha\theta k_t^{\alpha-1} - 1$$
$$w_t = (1-\alpha)\theta k_t^{\alpha} .$$

[In an Arrow-Debreu equilibrium, we set

$$p_t = \beta^t \left(\frac{c_0}{c_t} \right)$$
$$w_t = p_t (1-\alpha)\theta k_t^{\alpha}$$
$$r_t = p_t \alpha\theta k_t^{\alpha-1} .]$$