MACROECONOMIC THEORY | **T. J. KEHOE**
ECON 8105 | **FALL 2021**

PROBLEM SET #4

1. Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital as well as labor. The consumer's utility function is

$$(1/\rho)\sum_{t=0}^{\infty}\beta^t(c_t^{\rho}-1)$$

Here $0<\beta<1$ and $-\infty<\rho<1$, $\rho\neq 0$. The consumer is endowed with 1 unit of labor in every period and $\bar{k}_0$ units of capital in period 0. Feasible allocations satisfy

$$c_t + k_{t+1} - (1-\delta)k_t \leq \theta k_t^{\alpha}\ell_t^{1-\alpha}$$

$$c_t, k_t \geq 0 \ .$$

Here $\theta>0$, $0<\alpha<1$, and $0\leq\delta\leq 1$.

a) Use l'Hôpital's Rule to calculate the utility function in the limit where $\rho\to 0$.

b) Formulate the problem of maximizing the representative consumer's utility subject to feasibility conditions as a dynamic programming problem. Write down the appropriate Bellman's equation.

c) Let $K=[0,\tilde{k}]$. Explain how you can use the feasibility conditions to choose $\tilde{k}$ to be the maximum sustainable capital stock. Let $C(K)$ be the space of continuous bounded functions on K. Endow $C(K)$ with the topology induced by the sup norm

$$d(V,W)=\sup_{k\in K}\left|V(k)-W(k)\right| \text{ for all } V,W\in C(K) \ .$$

Define a contraction mapping $T:C(K)\to C(K)$.

d) State Blackwell's sufficient conditions for T to be a contraction. (You do not need to prove that these conditions are sufficient for T to be a contraction.)

e) Using Bellman's equation from part a, set up the mapping that defines the value function iteration algorithm:

$$V_{n+1}=T(V_n)\ ,$$

where $T:C(K)\to C(K)$. (You do not need to prove that $T(V)\in C(K)$ if $V\in C(K)$.) Using Blackwell's sufficient conditions, prove that T is a contraction.

f) Specify an economic environment for which the solution to the social planner's problem in part a is a Pareto efficient allocation/production plan. Define a sequential markets equilibrium for this environment. Explain how you could use the value function iteration algorithm $V_{n+1} = T(V_n)$ to calculate the unique sequential markets equilibrium. (You do not have to prove that this equilibrium is unique.)

2. Consider the optimal growth problem

$$\max \sum_{t=0}^{\infty} (0.6)^t \log c_t$$
$$\text{s.t. } c_t + k_{t+1} \leq 10 k_t^{0.4}$$
$$c_t, k_t \geq 0$$
$$k_0 = \bar{k}_0 .$$

a) Write down the Euler conditions and the transversality condition for this problem. Calculate the steady state values of *c* and *k*.

b) Write down the functional equation that defines the value function for this problem. Guess that the value function has the form $a_0 + a_1 \log k$. Calculate the value function and the policy function. Verify that the policy function generates a path for capital that satisfies the Euler conditions and transversality condition in part a.

3. Let capital take values for the discrete grid (2, 4, 6, 8, 10). Make the original guess $V_0(k) = 0$ for all *k*, and perform the first three steps of the value function iteration

$$V_{i+1}(k) = \max \log(10k^{0.4} - k') + 0.6 V_i(k') .$$

a) Perform the value function iterations until

$$\max_k \left| V_{i+1}(k) - V_i(k) \right| < 10^{-5}.$$

Report the value function and the policy function that you obtain. Compare these results with what you obtained in question 4. (Hint: you probably want to use a computer.)

b) Repeat part a for the grid of capital stocks $(0.05, 0.10, \ldots, 9.95, 10)$. Compare your answer with those of question 4 and of part a. (Hint: you need to use a computer).

c) Repeat part b for the problem

$$\max \sum_{t=0}^{\infty} (0.6)^t \log c_t$$
$$\text{s.t. } c_t + k_{t+1} - 0.5 k_t \leq 10 k_t^{0.4}$$
$$c_t, k_t \geq 0$$
$$k_0 = \bar{k}_0 .$$

(There is now no comparison with an analytical answer to be made, however.)