The Cone of Diversification of a Small, Open, Cobb-Douglas Model

$$\max \; p_1\theta_1 k_1^{\alpha_1}\ell_1^{1-\alpha_1} + p_2\theta_2 k_2^{\alpha_2}\ell_2^{1-\alpha_2}$$

$$\text{s.t. } k_1 + k_2 \le \overline{k}$$

$$\ell_1 + \ell_2 \le \overline{\ell}$$

$$k_j \ge 0\,,\; \ell_j \ge 0\,.$$

The cone of diversification — which is the set of endowments $(\overline{k}, \overline{\ell})$ for which the first-order conditions of the revenue maximization problem hold with equality — depends only on relative prices, $\kappa_1^{soe}(p_2 / p_1)$ and $\kappa_2^{soe}(p_2 / p_1)$:

$$\kappa_1^{soe}(p_2 / p_1) \ge \frac{\overline{k}}{\overline{\ell}} \ge \kappa_2^{soe}(p_2 / p_1)\,.$$

First-order conditions:

$$r = p_1\alpha_1\theta_1 k_1^{\alpha_1-1}\ell_1^{1-\alpha_1} = p_2\alpha_2\theta_2 k_2^{\alpha_2-1}\ell_2^{1-\alpha_2}$$

$$w = p_1(1-\alpha_1)\theta_1 k_1^{\alpha_1}\ell_1^{-\alpha_1} = p_2(1-\alpha_2)\theta_2 k_2^{\alpha_2}\ell_2^{-\alpha_2}\,.$$

$$\frac{w}{r} = \frac{(1-\alpha_1)k_1}{\alpha_1\ell_1} = \frac{(1-\alpha_2)k_2}{\alpha_2\ell_2}$$

$$\frac{k_2}{\ell_2} = \frac{(1-\alpha_1)\alpha_2}{\alpha_1(1-\alpha_2)}\frac{k_1}{\ell_1}$$

$$\frac{p_2}{p_1} = \frac{\alpha_1\theta_1(k_2/\ell_2)^{1-\alpha_2}}{\alpha_2\theta_2(k_1/\ell_1)^{1-\alpha_1}}$$

$$\left(\frac{k_2}{\ell_2}\right)^{1-\alpha_2} = \frac{p_2\alpha_2\theta_2}{p_1\alpha_1\theta_1}\left(\frac{k_1}{\ell_1}\right)^{1-\alpha_1}$$

$$\frac{k_2}{\ell_2} = \left(\frac{p_2\alpha_2\theta}{p_1\alpha_1\theta_1}\right)^{\frac{1}{1-\alpha_2}}\left(\frac{k_1}{\ell_1}\right)^{\frac{1-\alpha_1}{1-\alpha_2}}$$

$$\frac{(1-\alpha_1)\alpha_2}{\alpha_1(1-\alpha_2)}\frac{k_1}{\ell_1} = \left(\frac{p_2\alpha_2\theta}{p_1\alpha_1\theta_1}\right)^{\frac{1}{1-\alpha_2}}\left(\frac{k_1}{\ell_1}\right)^{\frac{1-\alpha_1}{1-\alpha_2}}$$

$$\frac{(1-\alpha_1)\alpha_2}{\alpha_1(1-\alpha_2)} = \left(\frac{p_2\alpha_2\theta}{p_1\alpha_1\theta_1}\right)^{\frac{1}{1-\alpha_2}}\left(\frac{k_1}{\ell_1}\right)^{\frac{\alpha_2-\alpha_1}{1-\alpha_2}}$$

$$\left(\frac{k_1}{\ell_1}\right)^{\alpha_1-\alpha_2}\left(\frac{(1-\alpha_1)\alpha_2}{\alpha_1(1-\alpha_2)}\right)^{1-\alpha_2}=\frac{p_2\alpha_2\theta}{p_1\alpha_1\theta_1}$$

$$\frac{k_1}{\ell_1}=\left(\frac{p_2\alpha_2\theta_2}{p_1\alpha_1\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_1(1-\alpha_2)}{(1-\alpha_1)\alpha_2}\right)^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}}$$

$$\frac{k_2}{\ell_2}=\left(\frac{p_2\alpha_2\theta}{p_1\alpha_1\theta_1}\right)^{\frac{1}{1-\alpha_2}}\left(\left(\frac{p_2\alpha_2\theta_2}{p_1\alpha_1\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_1(1-\alpha_2)}{(1-\alpha_1)\alpha_2}\right)^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}}\right)^{\frac{1-\alpha_1}{1-\alpha_2}}$$

$$\frac{k_2}{\ell_2}=\left(\frac{p_2\alpha_2\theta_2}{p_1\alpha_1\theta_1}\right)^{\frac{1}{1-\alpha_2}+\frac{1-\alpha_1}{(\alpha_1-\alpha_2)(1-\alpha_2)}}\left(\left(\frac{\alpha_1(1-\alpha_2)}{(1-\alpha_1)\alpha_2}\right)^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}}\right)^{\frac{1-\alpha_1}{1-\alpha_2}}$$

$$\frac{k_2}{\ell_2}=\left(\frac{p_2\alpha_2\theta_2}{p_1\alpha_1\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_1(1-\alpha_2)}{(1-\alpha_1)\alpha_2}\right)^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}}$$

$$\kappa_1^{soe}(p_2/p_1)=\left(\frac{\alpha_2\theta_2}{\alpha_1\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_1(1-\alpha_2)}{(1-\alpha_1)\alpha_2}\right)^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}=\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}$$

where

$$\bar{\kappa}_1^{soe}=\left(\frac{\alpha_2\theta_2}{\alpha_1\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_1(1-\alpha_2)}{(1-\alpha_1)\alpha_2}\right)^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}}=\left(\frac{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}\theta_2}{\alpha_1^{\alpha_2}(1-\alpha_1)^{1-\alpha_2}\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}},$$

and

$$\kappa_2^{soe}(p_2/p_1)=\left(\frac{\alpha_2\theta_2}{\alpha_1\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_1(1-\alpha_2)}{(1-\alpha_1)\alpha_2}\right)^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}=\bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}$$

where

$$\bar{\kappa}_2^{soe}=\left(\frac{\alpha_2^{\alpha_1}(1-\alpha_2)^{1-\alpha_1}\theta_2}{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}.$$

We have found the cone of diversification. Let us find the factor prices r and w:

$$r = p_1\alpha_1\theta_1\left(\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\right)^{\alpha_1-1} = p_1\alpha_1\theta_1(\bar{\kappa}_1^{soe})^{\alpha_1-1}\left(\frac{p_1}{p_2}\right)^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}}$$

$$w = p_1(1-\alpha_1)\theta_1\left(\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\right)^{\alpha_1} = p_1(1-\alpha_1)\theta_1(\bar{\kappa}_1^{soe})^{\alpha_1}\left(\frac{p_2}{p_1}\right)^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}}.$$

Now let us find the production allocation (k_1,ℓ_1), (k_2,ℓ_2) that solves the revenue maximization problem. Notice that

$$\frac{\bar{\kappa}_1^{soe}}{\bar{\kappa}_2^{soe}} = \frac{\left(\dfrac{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}\theta_2}{\alpha_1^{\alpha_2}(1-\alpha_1)^{1-\alpha_2}\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}}{\left(\dfrac{\alpha_2^{\alpha_1}(1-\alpha_2)^{1-\alpha_1}\theta_2}{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}}$$

$$\frac{\bar{\kappa}_1^{soe}}{\bar{\kappa}_2^{soe}} = \left(\frac{\alpha_2^{\alpha_2-\alpha_1}(1-\alpha_2)^{\alpha_1-\alpha_2}}{\alpha_1^{\alpha_2-\alpha_1}(1-\alpha_1)^{\alpha_1-\alpha_2}}\right)^{\frac{1}{\alpha_1-\alpha_2}}$$

$$\frac{\bar{\kappa}_1^{soe}}{\bar{\kappa}_2^{soe}} = \frac{\alpha_1(1-\alpha_2)}{\alpha_2(1-\alpha_1)}.$$

Feasibility conditions:

$$k_1 + k_2 = \bar{k}$$

$$\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\ell_1 + \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\ell_2 = \bar{k}$$

$$\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\ell_1 + \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}(\bar{\ell}-\ell_1) = \bar{k}$$

$$\left(\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\right)\ell_1 = \bar{k} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\bar{\ell}$$

$$\ell_1 = \frac{\bar{k} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\bar{\ell}}{\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}}$$

$$\ell_1 = \frac{\frac{\bar{k}}{\bar{\ell}} - \bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}}{\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}}\bar{\ell}$$

$$\ell_2 = \frac{\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} - \frac{\bar{k}}{\bar{\ell}}}{\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}}\bar{\ell}$$

$$k_1 = \bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\ell_1 = \bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\frac{\frac{\bar{k}}{\bar{\ell}} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}}{\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}}\bar{\ell}$$

$$k_2 = \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\ell_2 = \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\frac{\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} - \frac{\bar{k}}{\bar{\ell}}}{\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}}\bar{\ell}.$$

Check:

$$k_1 + k_2 = \frac{\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\frac{\bar{k}}{\bar{\ell}} - \bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} + \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\bar{\kappa}_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\frac{\bar{k}}{\bar{\ell}}}{\bar{\kappa}_1^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}} - \bar{\kappa}_2^{soe}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}}\bar{\ell}$$

$$k_1 + k_2 = \frac{\overline{\kappa}_1^{soe} \left(\frac{p_2}{p_1} \right)^{\frac{1}{\alpha_1 - \alpha_2}} \frac{\overline{k}}{\overline{\ell}} - \overline{\kappa}_2^{soe} \left(\frac{p_2}{p_1} \right)^{\frac{1}{\alpha_1 - \alpha_2}} \frac{\overline{k}}{\overline{\ell}}}{\overline{\kappa}_1^{soe} \left(\frac{p_2}{p_1} \right)^{\frac{1}{\alpha_1 - \alpha_2}} - \overline{\kappa}_2^{soe} \left(\frac{p_2}{p_1} \right)^{\frac{1}{\alpha_1 - \alpha_2}}} \overline{\ell}$$

$$k_1 + k_2 = \overline{k} \ .$$

To find consumption c_1 and c_2, we solve the representative consumer's problem and obtain

$$c_1 = a_1 \frac{w\overline{\ell} + r\overline{k}}{p_1}$$

$$c_2 = a_2 \frac{w\overline{\ell} + r\overline{k}}{p_2} .$$

Notice that the determination of the production plan has nothing to do with the solution to the consumer's problem. In particular, a_1 and a_2 show up only in the determination of c_1 and c_2, not in r, w, (k_1, ℓ_1), and (k_2, ℓ_2).

The Cone of Diversification of an Integrated World Economy, Cobb-Douglas Model

$$\max \; a_1 \log \theta_1 k_1^{\alpha_1} \ell_1^{1-\alpha_1} + a_2 \log \theta_2 k_2^{\alpha_2} \ell_2^{1-\alpha_2}$$

$$\text{s.t. } k_1 + k_2 \leq \bar{k}$$

$$\ell_1 + \ell_2 \leq \bar{\ell}$$

$$k_j \geq 0 \,,\; \ell_j \geq 0 \,.$$

The cone of diversification is determined by the capital-labor ratios of the solution to the autarky equilibrium of the world economy where $\bar{k} = \bar{k}^1 + \bar{k}^2$ and $\bar{\ell} = \bar{\ell}^1 + \bar{\ell}^2$.

The solution to this problem is always interior, and the first-order conditions are

$$\frac{\alpha_1 a_1 \theta_1 k_1^{\alpha_1 - 1} \ell_1^{1-\alpha_1}}{\theta_1 k_1^{\alpha_1} \ell_1^{1-\alpha_1}} = \frac{a_1 \alpha_1}{k_1} = \frac{a_2 \alpha_2}{k_2} = r$$

$$\frac{(1-\alpha_1) a_1 \theta_1 k_1^{\alpha_1} \ell_1^{-\alpha_1}}{\theta_1 k_1^{\alpha_1} \ell_1^{1-\alpha_1}} = \frac{a_1(1-\alpha_1)}{\ell_1} = \frac{a_2(1-\alpha_2)}{\ell_2} = w$$

$$a_2 \alpha_2 k_1 = a_1 \alpha_1 k_2 = a_1 \alpha_1 (\bar{k} - k_1)$$

$$k_1 = \frac{a_1 \alpha_1}{a_1 \alpha_1 + a_2 \alpha_2} \bar{k}$$

$$k_2 = \frac{a_2 \alpha_2}{a_1 \alpha_1 + a_2 \alpha_2} \bar{k}$$

$$\ell_1 = \frac{a_1(1-\alpha_1)}{a_1(1-\alpha_1) + a_2(1-\alpha_2)} \bar{\ell}$$

$$\ell_2 = \frac{a_2(1-\alpha_2)}{a_1(1-\alpha_1) + a_2(1-\alpha_2)} \bar{\ell}$$

$$\kappa_1^{ge}(\bar{k} / \bar{\ell}) = \frac{k_1}{\ell_1} = \frac{\alpha_1 (a_1(1-\alpha_1) + a_2(1-\alpha_2))}{(1-\alpha_1)(a_1 \alpha_1 + a_2 \alpha_2)} \frac{\bar{k}}{\bar{\ell}}$$

$$\kappa_2^{ge}(\bar{k} / \bar{\ell}) = \frac{k_2}{\ell_2} = \frac{\alpha_2 (a_1(1-\alpha_1) + a_2(1-\alpha_2))}{(1-\alpha_2)(a_1 \alpha_1 + a_2 \alpha_2)} \frac{\bar{k}}{\bar{\ell}}$$

$$\kappa_1^{ge}(\bar{k}/\bar{\ell}) = \bar{\kappa}_1^{ge}\frac{\bar{k}}{\bar{\ell}} = \frac{\alpha_1(a_1(1-\alpha_1)+a_2(1-\alpha_2))}{(1-\alpha_1)(a_1\alpha_1+a_2\alpha_2)}\frac{\bar{k}}{\bar{\ell}}$$

where

$$\bar{\kappa}_1^{ge} = \frac{\alpha_1(a_1(1-\alpha_1)+a_2(1-\alpha_2))}{(1-\alpha_1)(a_1\alpha_1+a_2\alpha_2)},$$

and

$$\kappa_2^{ge}(\bar{k}/\bar{\ell}) = \bar{\kappa}_2^{ge}\frac{\bar{k}}{\bar{\ell}} = \frac{\alpha_2(a_1(1-\alpha_1)+a_2(1-\alpha_2))}{(1-\alpha_2)(a_1\alpha_1+a_2\alpha_2)}\frac{\bar{k}}{\bar{\ell}}$$

where

$$\bar{\kappa}_2^{ge} = \frac{\alpha_2(a_1(1-\alpha_1)+a_2(1-\alpha_2))}{(1-\alpha_2)(a_1\alpha_1+a_2\alpha_2)}.$$

We have found the cone of diversification and the production plans

$$k_1 = \frac{a_1\alpha_1}{a_1\alpha_1+a_2\alpha_2}\bar{k}$$

$$k_2 = \frac{a_2\alpha_2}{a_1\alpha_1+a_2\alpha_2}\bar{k}$$

$$\ell_1 = \frac{a_1(1-\alpha_1)}{a_1(1-\alpha_1)+a_2(1-\alpha_2)}\bar{\ell}$$

$$\ell_2 = \frac{a_2(1-\alpha_2)}{a_1(1-\alpha_1)+a_2(1-\alpha_2)}\bar{\ell}.$$

Let us find the prices p_2, w, and r, where we normalize $p_1 = 1$;

$$w = (1-\alpha_1)\theta_1\left(\frac{k_1}{\ell_1}\right)^{\alpha_1} = (1-\alpha_1)\theta_1\left(\bar{\kappa}_1^{ge}\frac{\bar{k}}{\bar{\ell}}\right)^{\alpha_1}$$

$$r = \alpha_1\theta_1\left(\frac{k_1}{\ell_1}\right)^{\alpha_1-1} = \alpha_1\theta_1\left(\bar{\kappa}_1^{ge}\frac{\bar{k}}{\bar{\ell}}\right)^{\alpha_1-1}$$

$$w = (1-\alpha_1)\theta_1\left(\bar{\kappa}_1^{ge}\frac{\bar{k}}{\bar{\ell}}\right)^{\alpha_1} = p_2(1-\alpha_2)\theta_2\left(\bar{\kappa}_2^{ge}\frac{\bar{k}}{\bar{\ell}}\right)^{\alpha_1}$$

$$p_2 = \frac{(1-\alpha_1)\theta_1\left(\bar{\kappa}_1^{ge}\dfrac{\bar{k}}{\bar{\ell}}\right)^{\alpha_1}}{(1-\alpha_2)\theta_2\left(\bar{\kappa}_2^{ge}\dfrac{\bar{k}}{\bar{\ell}}\right)^{\alpha_1}} = \frac{(1-\alpha_1)\theta_1(\bar{\kappa}_1^{ge})^{\alpha_1}}{(1-\alpha_2)\theta_2(\bar{\kappa}_2^{ge})^{\alpha_1}}$$

Once again, the solution to the consumer's problem is

$$c_1 = a_1\frac{w\bar{\ell}+r\bar{k}}{p_1}$$
$$c_2 = a_2\frac{w\bar{\ell}+r\bar{k}}{p_2}.$$

Unlike the small open economy model, this is a general equilibrium model, and the parameters of the consumer's utility function a_1 and a_2 show up in the determination of all of the variables.