Roll-Over Crises in Sovereign Debt Markets*

Juan Carlos Conesa

Stony Brook University

Timothy J. Kehoe

University of Minnesota, Federal Reserve Bank of Minneapolis,
and National Bureau of Economic Research

June 2022

Abstract

We present a simple dynamic model in which the government of a small, open economy can experience a financial crisis during which it defaults on its sovereign debt. Whether or not a crisis occurs can depend on the expectations of international investors as well as on fundamental factors like changes in national income and changes in the international interest rate. In our model, if investors expect a crisis to occur, they do not purchase new bonds issued by the government and the government is unable to refinance its existing debt. We refer to the crises that depend on fundamentals as solvency crises and the crises that depend on self-fulfilling expectations of investors as liquidity crises. As in other models of debt crises, the government can eliminate the possibility of a crisis by lowering its total stock of debt. Unlike in models in which crises depend only on changes in fundamentals, the government can eliminate the possibility of a liquidity crisis occurring by lengthening the maturity of its debt and thereby reducing its refinancing needs.

Key words: Sovereign debt; Debt maturity; Liquidity crisis; Financial panic

JEL codes: E44, F34, G15, H63

*The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1. Introduction

In this article we develop a simple version of Cole and Kehoe's (1996, 2000) model for the analysis of liquidity crises in debt markets. This type of debt crisis is common and yet has received much less attention from researchers than debt crises that depend solely on fundamental factors. Our version of the model is based on the work of Conesa and Kehoe (2017). The model is simple in that it does not have private investment and capital as is the case in Cole and Kehoe (1996, 2000) and it does not have stochastic income as in Conesa and Kehoe (2017).

In our model, the government of a small open economy may experience a financial crisis that results in a sovereign debt default. Whether or not a crisis occurs depends on the expectations of international investors, unlike other models in the tradition of Eaton and Gersovitz (1981), such as Hamann (2002), Aguiar and Gopinath (2006), Arellano (2009), and many others. others, in which fundamental factors such as changes in national income or interest rates are the only determinants of debt crises. García Rodríguez (2022) provides an excellent explanation of how these models work. In our model if investors expect a crisis, they do not buy new debt issues and the government can't refinance maturing debt. Our model also allows for crises based on fundamental factors. We refer to crises that depend on fundamentals as solvency crises, and crises that depend on investors' self-fulfilling expectations as liquidity crises. As in other models, the government can reduce the probability of a debt crisis by reducing the total amount of debt. Unlike other models, the government can reduce the probability of a crisis by lengthening the maturity of the debt, thus reducing refinancing needs.

A debt crisis occurs when a sovereign government does not meet the established payments. In anticipation of this possibility, it is natural for debtors to demand a risk premium, a higher return on that debt that compensates for the possibility of default. Therefore, the risk premium reflects the probability that international investors assign to the possibility of a default. From this derive the differences over time and between countries in the risk premium. Spain pays a higher risk premium than Germany because international investors estimate that the probability of default is higher in Spain than in Germany. In the summer of 2012, the Spanish risk premium soared because investors perceived a radical increase in the probability of a debt crisis in Spain. Therefore, an increase in the risk premium does not constitute a debt crisis, but rather it is an indicator that investors have increased their expectations that a debt crisis will occur.

The elements that make a debt crisis possible are:

- The government does not have the capacity to commit ex-ante to honor scheduled payments. If this commitment capacity existed, or what is the same, if the penalty for non-payment were infinite, then we would never observe non-payments and consequently there would be no risk premium in such contracts. For this point, the inexistence of a legal power that has jurisprudence over all sovereign governments and can impose punishments and conditions in case of payment problems is crucial. This aspect of sovereign debt marks a great difference with respect to debt contracts between individuals of the same legal jurisdiction, where the debtor can legally claim payment and the courts of justice have jurisprudence on cases of non-payment.

- The government does not have the capacity to write debt contracts contingent on all potential eventualities. It would be materially impossible to write debt contracts that set out what the payment structure would be in the event of a recession of a certain degree of severity, a war in Ukraine for so many months, that there is a drought in California and a government of a certain type comes to power in anywhere in the world, and millions of other circumstances. It is not that all these events are unpredictable, which they are, but rather that a contract could never be formulated that would include all these potential contingencies.

Debt contracts are (with few exceptions) non-contingent nominal contracts. They stipulate a specific payment in, for example, euros at a stipulated future date, regardless of the circumstances that may occur on that date. This opens the possibility that in certain circumstances the government decides that it cannot make the payments, or what is the same as even being able to make them, it is better for the government itself and/or for the citizens not to make them.

An additional peculiarity of debt contracts is that they are made up of contracts at different time horizons. Governments auction weekly treasury bills for 3 months, one year, ten or thirty years. In some historical periods some countries have even issued debt in perpetuity. The structure of the pre-existing debt (that which is still pending "maturity") determines the periodicity and magnitude of the payments, what we call the debt service, which is the key element in determining whether a country or an administration able or willing to meet such payments. Thus, debt service depends on interest rates, but it also crucially depends on the structure of the debt.

In practice, governments must continuously refinance a non-trivial fraction of their total debt, and this depends on their structure. The shorter the maturity of the debt, the higher the debt service and the more frequently such debts must be refinanced. Furthermore, the maturity of the debt is not the same as its duration. A bond that pays 1 euro in one year, 1 euro in two years, and 5 euros in three years, generates a different payment structure than a bond that pays an amount of 2 euros per year for three years. The maturity of both bonds is the same (three years), but the second bond represents a greater concentration of payments in the recent future compared to the first bond, that is, the second bond has a lower duration, understood as the weighted average number of years of expected payments.

There are two different mechanisms by which a debt crisis can be triggered:

- Solvency crises: International investors are willing to refinance and even increase debt, at a price that compensates them for the probability of future defaults. However, normally due to a persistent drop in the government's fiscal resources, servicing current and/or future debt becomes excessively costly for the government and it prefers to suspend payments.

- Liquidity crises: These occur when fiscal resources are sufficient to guarantee payments as long as the debt markets are operational. For some reason, however, the distrust of international investors limits the usual access to credit by the government, and this ends up generating defaults given the impossibility of refinancing the pre-existing debt.

Observe how solvency crises occur due to changes in the fundamentals of the economy: a severe and persistent economic crisis, a fall in tax revenues, or an unexpected increase in government spending needs. In contrast, liquidity crises can occur in the absence of any change in the fundamentals of the economy. By their very nature, they are self-fulfilling crises: investors do not want to invest in this country's debt for fear of a default, with which the country loses the ability to refinance the fraction of debt that must be serviced at a given time, and that can end up resulting in a default. That is why this type of crisis is also called a rollover crisis.

In practice, it is difficult to discern one type of crisis from another. The only thing we observe is that a certain country has suspended payments, but we have very little information about the availability or not of international investors to refinance that debt. One aspect that is often used is the existence of failed debt auctions. In most countries, an important part of the new debt issuances are carried out in public auctions. A failed auction is often an indicator of panic among investors. In a recent work, Bocola and Dovis (2019) propose to identify the two types of crisis based on the response of governments in relation to decisions on the maturity of the new debt issued. Liquidity crises can be triggered unexpectedly at any time, without the fundamentals having materially worsened. At the same time, something that radically changes the expectations of international investors can suddenly eradicate the possibility of a liquidity crisis. This is our interpretation of the impact of Draghi's famous declaration in the summer of 2012: "whatever it takes". If international investors perceive that an institution, in this case the European Central Bank, is absolutely committed to providing the necessary liquidity to refinance Spain's debt, then these investors change their expectations regarding the possibility of a debt crisis and the premium risk should drop immediately.

2. Model

In this model, we study the optimal strategic behavior of a government that does not have the capacity to commit to meeting payments, and that sells debt in markets where investors can panic for exogenous reasons at any time.

The three types of agents that exist in this model are:

- The government (or rather successive governments).
- International investors; there is a continuum of them in the interval [0,1].
- Consumers; there is also a continuum of them in the interval [0,1].

The government acts strategically and chooses how much to spend on the provision of public goods, how much to borrow and whether to declare a default. We assume that tax revenue is a constant fraction of output. Much our analysis focuses on the government decision problem. International investors buy government bonds. These investors are risk neutral and have no liquidity restrictions. The assumptions of risk neutrality and the absence of liquidity restrictions are compatible with the existence of risk-averse investors but with well-diversified portfolios. Since the country is small, the debt purchases of this country constitute a minimal part of their portfolios. International investors coordinate their expectations with a stochastic variable of the "sunspot" type. Consumers value the consumption of private and public goods, and have a utility function $u(c,g)$ and an intertemporal utility of $E\sum_{t=0}^{\infty}\beta^t u(c_t,g_t)$, where $\beta \in (0,1)$ is the discount factor.

Assuming a continuum of agents is the formal procedure that ensures any individual agent (individual 0.4317…, for example) has zero mass with respect to the total number of individuals. This guarantees that their individual decisions do not affect the aggregate, and therefore there is no possibility of strategic behavior on the part of these individual agents. In this model, only the government will have a strategic behavior, in the sense of internalizing that its actions have an impact on the actions of the other agents. In fact, the lack of government commitment implies that the current government does not have the capacity to commit future government to a certain plan of action, so it is a strategic dynamic game between successive governments. The state of the economy at a given moment in time is given by the vector: $s=(B,z_{-1},\zeta)$, where.

- B is the existing outstanding debt.
- z_{-1} is an indicator of the existence of a debt crisis in the past or not: $z_{-1}=1$ no, $z_{-1}=0$ yes.
- ζ is the realization of a stochastic variable not related to the fundamentals called a sunspot.

Total available output in this economy is $y(z)=Z^{1-z}\overline{y}$ where $1>Z>0$ is a parameter and $(1-Z)$ is interpreted as the output loss from past debt default.

3. Sunspots

The sunspot variable serves to coordinate the expectations of international investors. If the level of debt is low enough, the value of the sunspot does not matter since the government will service its debt regardless of what international investors do. Similarly, if the level of debt is high enough the value of the sunspot does not matter because the government will default regardless of what international investors do. Otherwise, for intermediate values of debt, sunspots have real effects. Think, for example, of the ratings that agencies like Moody's, Fitch or S&P assign to the debt of different countries (although we do not want to imply in any way that these ratings are random). A change in such ratings can coordinate the expectations of international investors, beyond the impact that changes in fundamental variables may have.

Suppose that the sunspot variable, $\zeta \in U[0,1]$, comes from of a uniform distribution in the interval $[0,1]$. If the value of this variable exceeds a certain threshold, $\zeta \geq 1-\pi$, that is, if the news is bad enough, then investors panic and expect that a crisis will be triggered. Here π, $0<\pi<1$, is an exogenously given arbitrary number, and is the probability of triggering a panic among investors. Whether this panic is self-fulfilling and ends up generating a crisis depends on the fundamental variables. If an individual is certain that a country is not going to default even if all investors together panic and do not buy the country's new debt issues, then this individual has no reason not to buy that country's debt. If all investors think alike (hence the coordination of expectations), then the panic is not self-fulfilling. On the other hand, if an individual suspects that, if the government fails to refinance its debt, it will declare a default, then the individual will not go to the debt auction. If all investors act in the same way, then a default will occur, validating investors' expectations.

Cass and Shell (1983) were the first to model this type of random variables in a dynamic economic model with the possibility of multiple equilibria. They called it a sunspot in reference to the study by Jevons (1878) which, given the scant historical evidence available, had established a relationship between business cycles in the United Kingdom and sunspot activity. The reference to such sunspots is somewhat unfortunate: Substantial increases in sunspot activity, which occur every 11 to 14 years, significantly affect agriculture. Harrison (1976), for example, presents evidence of a relationship between sunspot activity and agricultural crop yields. Furthermore, geophysicists such as Xu et al. (2021) model sunspots as the result of complex and nonlinear dynamics, but deterministic, and not as stochastic variables.

4. Strategic problem of the government

We assume that the government is benevolent in the sense that it maximizes consumers' utility subject to its budget constraint and the reaction of international investors. We also assume that the government, the consumers, and the investors discount the future by the same factor. In this case, consumers play a passive role, but later we discuss extents to which consumers make labor supply decisions in the face of changes in tax rates. We also discuss a generalization of the model in which agents have different discount factors.

Consider the optimization problem of a benevolent government, which acts in the best interests of its citizens. We express this by a utility function, $u(c,g)$, which depends on private consumption and public consumption. We make the assumptions that the function $u(c,g)$ is separable, concave and continuously differentiable, strictly increasing in both consumption and public spending, and strictly concave in public spending. These assumptions are useful to characterize the optimal behavior of the government.

Given tax revenues of $\theta y(z)$, where θ is the tax rate that we take as given and $y(z)$ is the level of output, the government chooses c, g, B', z to solve the problem written recursively:

$$V(s) = \max \; u(c,g) + \beta E V(s')$$

$$\text{s.t. } c = (1-\theta) y(z)$$

$$g + zB = \theta y(z) + q(B', s) B'$$

$$z \in \{0,1\}$$

$$z = 0 \text{ if } z_{-1} = 0 .$$

Here, $V(s)$ represents the value function, as a function of the vector state variables s, which is the utility function in this period plus the discounted expected value of the value function in the next period (as a function of the state of the economy in the next period, s'), with β, $1 > \beta > 0$, as discount factor. To start with the simplest problem we have assumed that the debt has a maturity of one period.

The first constraint simply states that individuals consume their disposable income. The second constraint is the government budget constraint, where the expenditures on the left (government spending and debt service) are financed by the resources on the right (tax revenues and proceeds from the sale of debt). The debt that the government sells in this period is denoted B', and $q(B', s)$ is the equilibrium price of the debt, which depends on how much debt is sold today and the state of the economy today. The government can decide to default, $z = 0$, which causes the current debt B to disappear from the budget constraint, allowing a higher level of public spending today. If this happens, output falls immediately and forever from $\overline{y}$ to $Z\overline{y}$, and debt markets are closed forever (or put another way, the price at which the government can place debt becomes zero forever).

These simplifications make the problem much easier to solve. In reality, defaults rarely imply the total disappearance of the debt and the permanent exclusion of the financial markets. In most cases of default, a negotiation process opens with creditors or with international institutions that ends up resulting in a restructuring of the debt, which usually consists of a partial reduction in the amount of the debt and the extension of the payments (maturity lengthens). In turn, the countries regain access to credit, sometimes immediately and sometimes after some time. There are many studies on the topic of debt renegotiation in case of default, see Yue (2010), Cruces and Trebesch (2013), Arellano and Bai (2014), Almeida et al. (2018), among others.

5. International investors

We assume that there is a continuum (of measure 1) of international investors. These investors have linear preferences and discount the future by the same factor β. Consequently, they arel be willing to agree to buy any amount of debt that has an expected return equal to the return they can obtain on a safe asset with a fixed return.

Therefore, the valuation of the bonds is.

$$q(B',s) = \beta \times Ez(B',s',q(B'(s'),s')) .$$

$$\text{price of the bond} = \text{price of a risk free bond} \times \text{probability de default}$$

Here the function $z(B,s,q)$ determines the variable that takes the value 1 in the case of repayment. Therefore, the expected value of this function is the probability of repayment in the following period, which is a function of the amount of debt in the following period and the price at which the government in the following period can borrow, $q' = q(B'(s'),s')$

6. Timing of decisiones

In these types of strategic maximization problems, the timing of decision making plays a fundamental role. Schematically, this is the timing that we assume within a period:

1. Once ζ_t is realized, the state of the economy at the beginning of period t is $s_t = (B_t, z_{t-1}, \zeta_t)$

$$\downarrow$$

2. The government decides on the amount of debt to offer in the public auction, B_{t+1}

$$\downarrow$$

3. The investors decide whether or not to buy B_{t+1}, and the price q_t is determined

↓

4. The government decides whether or not to repay its debt B_t, z_t, and this determines (y_t, c_t, g_t)

When the government decides in phase 2 how much debt to offer for sale, it knows the price structure of the debt, since it is a function of the debt to be placed on the market and the state of the economy. The government decides whether it prefers to place one million euros at 2% or whether to place five million euros at 3%. Here the interest rate is $1/q-1$, and it is increasing with the level of debt. Finally, the government does not have the capacity to commit not to declare a default in phase 4. This implies that the government, once it has obtained the resources derived from the sale of the debt in phase 3, could declare the suspension of payments in phase 4. Precisely for this reason, in phase 3, investors have to anticipate whether the government is going to decide to default in phase 4, only in the event of a negative answer will investors be willing to buy the debt offered by the government. And investors, given their information, perfectly anticipate whether the government is going to declare a default in phase 4 or not. This specification of timing, first proposed by Cole and Kehoe (1996), is convenient for examining liquidity crises. Investors do or do not lend to the government based on their expectations of a default, and in turn whether or not the government can sell debt determines its incentives to declare a default. This yields the possibility of two equilibria (under certain fundamental values) in which expectations are self-fulfilling and which occur within the same period under the same information. There is no new information that is necessary for the existence of these two equilibriam (investors and governments have all the necessary information).

In models where the emphasis is on solvency crises, the time sequence is usually different. This occurs in the analyzes of Eaton and Gersovitz (1981), Hamman (2002), Aguiar and Gopinath (2006), or Arellano (2008), and many other later studies. In these cases, the government observes the state of the economy, decides whether or not to declare a default, and only then decides how much debt to put up for sale for the following period. This does not mean that liquidity crises cannot be introduced with this type of temporal sequence, but then they represent an intertemporal problem where investors must form expectations in this period about events that will determine the default decision in the following period.

7. Definition of a recursive equilrium

Given our timing, the decisions constitute a dynamic game, where the government in phase 2 takes into account how investors are going to behave in phase 3 and how the government is going to behave in phase 4. Likewise, investors in phase 3 also take into account how the government is going to behave in phase 4. The concept of equilibrium in dynamic games is subgame perfect equilibrium, and there are usually many of these (or a continuum, since we are dealing with a dynamic game with infinite horizon). As usual in this literature, we are going to restrict our analysis to Markov equilibria, which constitute a special case of perfect equilibrium in subgames where we restrict the strategies of the individuals to be functions exclusively of the value of the state variables in each period of the game. This eliminates the possibility, undoubtedly interesting, of more complex solutions where the actions at a given moment are a function of the history of actions of individuals, and where reputational aspects play a crucial role. Restricting our analysis to Markov equilibria is attractive because it allows the equilibrium to be defined and computed recursively.

In equilibrium, the value of π is arbitrary, which implies that there is a continuum of equilibria, one for each value of π. If we let the vector of state variables include the name of the year, there are even more possibilities: We can imagine that a panic can only occur in odd-numbered years or prime-numbered years.

Definition: An equilibrium consists of a value function $V(s)$, associated policy functions, $B'(s)$, $z(B',s,q)$, and $g(B',s,q)$, and a debt price function $q(B',s)$ such that:

1. At the beginning of the period, given the debt price function $q(B',s)$ and the policy functions of the government at the end of the period, $z(B',s,q)$ and $g(B',s,q)$, the value function $B'(s)$ is the solution to the government's problem of choosing B' to solve

$$V(s) = \max\ u(c,g) + \beta EV(s')$$
$$\text{s.t.}\quad c = (1-\theta)y(z(B',s,q(B',s))$$
$$g(B',s,q(B',s)) + z(B',s,q(B',s))B = \theta y(z) + q(B',s)B'.$$

2. The bond price function clears the market for bonds:

$$q(B',s) = \beta E z(B'(s), s', q(B'(s'), s')) .$$

3. At the end of the period, given the value function $V(s') = V(B', z, \zeta')$, the bond price $q = q(B'(s), s)$ the new debt issued $B' = B'(s)$, the policy functions $z(B', s, q)$ and $g(B', s, q)$ are the solution to the government's problem of choosing z and g to solve

$$\max \ u(c, g) + \beta E V(B', a', z, \zeta')$$
$$\text{s.t. } c = (1-\theta) y(a, z)$$
$$g + zB = \theta y(a, z) + qB'$$
$$z \in \{0,1\}, \ z = 0 \text{ if } z_{-1} = 0 .$$

Some clarifications are worthwhile. First of all, notice that, in part 2, the price of the debt that is sold this period, B', is a function of the price of the debt that is going to be sold the following period, $q(B'(s'), s')$. Second, in part 3, q and B' have already been determined, and therefore are numbers and not functions.

8. Characterization of equilibrium

Notice that the equilibrium condition in part 2 of the definition of an equilibrium is potentially a difficult problem of finding a fixed point in the space of the bond price functions $q(B', s)$: The same function shows up on both sides of the equation, on the left-hand side for the price this period and on the right-hand side for the expected price in the next period. We now show that our assumption that international investors are risk-neutral allows us to simplify this problem to finding two thresholds for debt, the lower threshold $\bar{b}$ and the upper threshold $\bar{B}(\pi)$. The investors expect that, if $B > \bar{b}$, government will default if the value of the sunspot satisfies $\zeta > 1-\pi$, and, if so, the government defaults. Similarly, the government defaults if $B > \bar{B}(\pi)$, and if so, the investors expect it.

The characterization of the equilibrium in terms of $\bar{b}$ and $\bar{B}(\pi)$ is the principal theoretical contribution of this article, so it is worth repeating in different words:

- If debt is less than the lower threshold, $B \leq \bar{b}$, then the government does not declare a default in any case, even if investors panic and the government cannot refinance its debt. Since international investors understand that the government always rolls over debt if $B \leq \bar{b}$ in equilibrium they ignore sunspots, even when $\zeta > 1-\pi$. Similarly, if there is no crisis in the current period, and if $B' \leq \bar{b}$, then investors are willing to pay the risk-free price $q(B',s)=\beta$ at the debt auction. Cole and Kehoe call the zone of debt where $B \leq \bar{b}$ the safe zone.

- At the other extreme, if the debt is greater than the upper threshold, $B > \bar{B}(\pi)$, then to maximize the government's objective function it is better to declare default even if there are investors willing to continue lending to this government. The debt service is too great and the government prefers to suffer the cost of a default to eliminate its debts.

- For intermediate levels of debt, $\bar{b} < B \leq \bar{B}(\pi)$, however, the government does not default as long as it continues to have access to credit, but does default if there is a panic among investors, which occurs when the sunspot variable satisfies $\zeta > 1-\pi$. Cole and Kehoe called this area the crisis zone, since it is the zone where investors are likely to panic and, given the level of debt, the panic is self-fulfilling.

Now we characterize the price function $q(B',s)$. Once a government has declared a default there is no incentive to repay debt, which is an immediate consequence of the fact that the costs of a default are permanent. Recall that the only reason why a government without commitment capacity repays the debt is to avoid the present and future costs of a default. Once the government has already defaulted and already paid the costs, nothing provides the government with any incentive to repay, and as a consequence the price of the debt is zero: $q(B',(B,0,\zeta))=0$. Put more simply, no investor is willing to lend to the government.

If the country enters a debt crisis, that is, if $B > \bar{b}$ and $\zeta \geq 1-\pi$, then also $q(B',(B,1,\zeta))=0$. When the outstanding debt is in the crisis zone, if there is a panic the government will default, so investors are not willing to buy any new debt.

In any other case, the price of the debt q depends exclusively on the amount of debt that the government wants to place on the market B'.

$$q(B',(B,1,\zeta)) = \begin{cases} \beta & \text{if } B' \le \overline{b} \\ \beta(1-\pi) & \text{if } \overline{b} < B' \le \overline{B}(\pi) \\ 0 & \text{if } \overline{B}(\pi) < B' \end{cases}.$$

The intuition is simple. If, in the next period, the debt is going to be below the lower threshold, then the government is going to repay with probability one. That debt is perfectly safe, and therefore its price is the discount factor, or in other words, the interest rate is the risk-free interest rate: $1/\beta - 1$ (the risk premium is zero).

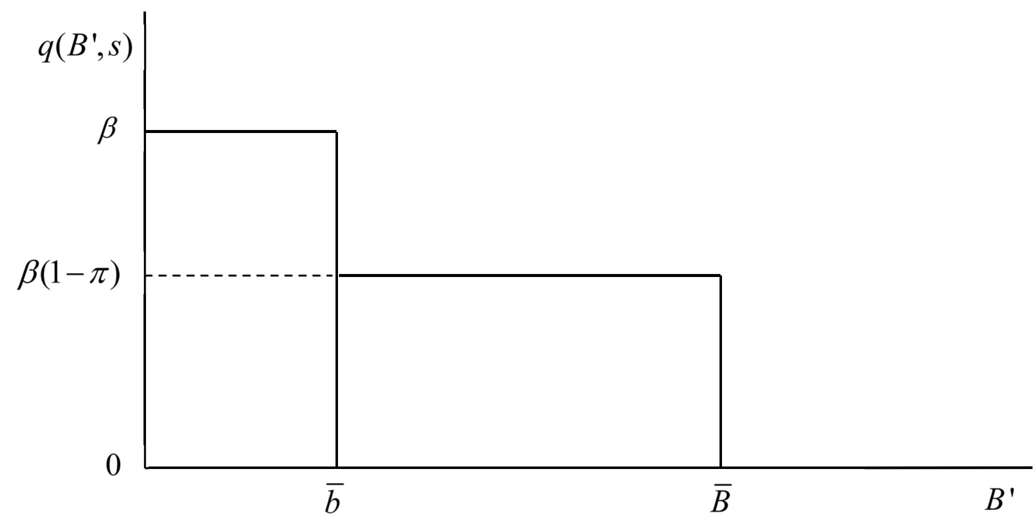

Figure 1. The price of new debt issued $q(B',s)$ as a function of the quantity offered for sale B'.

If the amount of debt offered for sale exceeds the upper threshold $\overline{B}(\pi)$, the government declares a default even in the absence of panics. Since default is a sure case, the price of the debt is $q = 0$. Put another way, investors are not willing to buy such a large level of debt.

Finally, if the government sells a level of debt within the crisis zone, in the following period the government will pay only in the absence of a panic, which is occurs with probability $1-\pi$.

9. Optimal debt policy

Given the assumptions we have made, we can compute the lower threshold explicitly. First, we calculate the objective function in case of default, if $s=(B,z_{-1},\zeta)=(B,0,\zeta)$, then the value function is

$$V(B,0,\zeta)=\frac{u((1-\theta)Z\overline{y},\theta Z\overline{y})}{1-\beta}.$$

This is a steady state with utility is that of constant private consumption equal to income net of taxes and public consumption equal to tax collection. The output is $Z\overline{y}<\overline{y}$. This situation is permanent, and hence the denominator of $1-\beta=1/\sum_{t=0}^{\infty}\beta^{t}$.

Consequently, the lower threshold $\overline{b}$ can be calculated using the condition

$$u((1-\theta)\overline{y},\theta\overline{y}-\overline{b})+\frac{\beta u((1-\theta)\overline{y},\theta\overline{y})}{1-\beta}=\frac{u((1-\theta)Z\overline{y},\theta Z\overline{y})}{1-\beta}$$

That is, $\overline{b}$ is the maximum level of debt that the government is willing to repay 100% to avoid a debt crisis if there is a panic among investors. In the event that investors do not buy the new debt B', if the government repays 100% of its debt $B=\overline{b}$, public spending is $g=\theta\overline{y}-\overline{b}$ and the following period the debt is zero. In that case, as of the following period, the debt would always be zero, since the government has no incentive to issue debt to transfer future consumption to the present (or vice versa) when $q=\beta$. Note that this cannot happen in equilibrium: Investors know that the government is going to repay and do not panic if $B\leq\overline{b}$ even if $\zeta\geq 1-\pi$.

If the debt were a little higher, $B=\overline{b}+\varepsilon$, the right side would be higher than the left, indicating that in the event of an investor panic, the government would prefer a default to repaying 100% of the debt. Given this lower threshold, the optimal policy for lower debt levels $B\leq\overline{b}$ is to refinance this debt in perpetuity. When $q=\beta$, the government wants to keep private and public consumption constant. There is no incentive to increase the debt (decreasing consumption) or decrease it (increasing consumption). Therefore, if $s=(B,z_{-1},\zeta)=(B,1,\zeta)$, the optimal policy is $B'(B,1,\zeta)=B$ and the value function is

$$V(B,1,\zeta)=\frac{u((1-\theta)\overline{y},\theta\overline{y}-(1-\beta)B)}{1-\beta}.$$

Along this path, sunspots are irrelevant and have no real consequences.

We can characterize the optimal policy of the government in the crisis zone using the government's first-order condition for B_{t+1}:

$$q_t u_g(g_t) = \beta(1-\pi)u_g(g_{t+1})$$

Our assumptions on $u(c,g)$ imply that, while B_{t+1} in the crisis zone, it is optimal to hold g_t constant. If the government increases its debt holding g_t constant, then implies that the debt eventually passes reaches the upper threshold, where the government declares a default, and which cannot be optimal. Consequently, the government reduces its debt until it reaches the safe zone or keeps it constant. It is easy to show that if the government reduces debt, it reduce the debt to $\bar{b}$, not to a strictly lower level in the safe zone. Suppose the government reduces the debt to $\bar{b}$ in T periods, $T = 1, 2, ..., \infty$. The possibility that $T = \infty$ allows the government to keep the debt constant. The level of constant expenditure g_t throughout the time sequence is

$$g_t = g^T(B_0).$$

Using the sequence of government budget constraints, we can calculate

$$g^T(B_0) = \theta\bar{y} - \frac{1-\beta(1-\pi)}{1-(\beta(1-\pi))^T}\left(B_0 - (\beta(1-\pi))^{T-1}\beta\bar{b}\right).$$

Notice that

$$g^\infty(B_0) = \lim_{T\to\infty} g^T(B_0) = \theta\bar{y} - (1-\beta(1-\pi))B_0 .$$

So, we compute $V^T(B_0)$ as

$$\begin{aligned} V^T(B_0) = {} & \frac{1-(\beta(1-\pi))^T}{1+\beta(1-\pi)} u((1-\theta)\bar{y}, g^T(B_0)) \\ & + \frac{1-(\beta(1-\pi))^{T-1}}{1+\beta(1-\pi)} \frac{\beta\pi u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} \\ & + (\beta(1-\pi))^{T-2} \frac{\beta u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{b})}{1-\beta} \end{aligned},$$

Cole and Kehoe (1996) and Conesa and Kehoe (2017) do the algebra to calculate this expression. To calculate the upper threshold $\bar{B}(\pi)$, we calculate the expression

$$V(\overline{B}(\pi)) = \max\left[V^1(\overline{B}(\pi)), V^2(\overline{B}(\pi)), ..., V^\infty(\overline{B}(\pi))\right]$$
$$= u((1-\theta)Z\overline{y}, \theta Z\overline{y} + \beta(1-\pi)\overline{B}(\pi)) + \beta\frac{u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{1-\beta}\cdot$$

That is, the upper threshold is the level of debt such that the government is indifferent between repaying or collecting the money from the debt issue (to the maximum possible) and then declaring a default.

Summarizing, we can write the value function as

$$V(B,1,\zeta) = \begin{cases} \dfrac{u((1-\theta)\overline{y}, \theta\overline{y} - (1-\beta)B)}{1-\beta} & \text{if } B \le \overline{b} \\ \max\left[V^1(B), V^2(B), ..., V^\infty(B)\right] & \text{if } \overline{b} < B \le \overline{B}(\pi),\ \zeta \le 1-\pi \\ \dfrac{u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{1-\beta} & \text{if } \overline{b} < B \le \overline{B}(\pi),\ \zeta > 1-\pi \\ \dfrac{u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{1-\beta} & \text{if } \overline{B}(\pi) < B \end{cases}$$

In Figure 2, we show possibilities for the optimal debt policies depending on the initial debt.

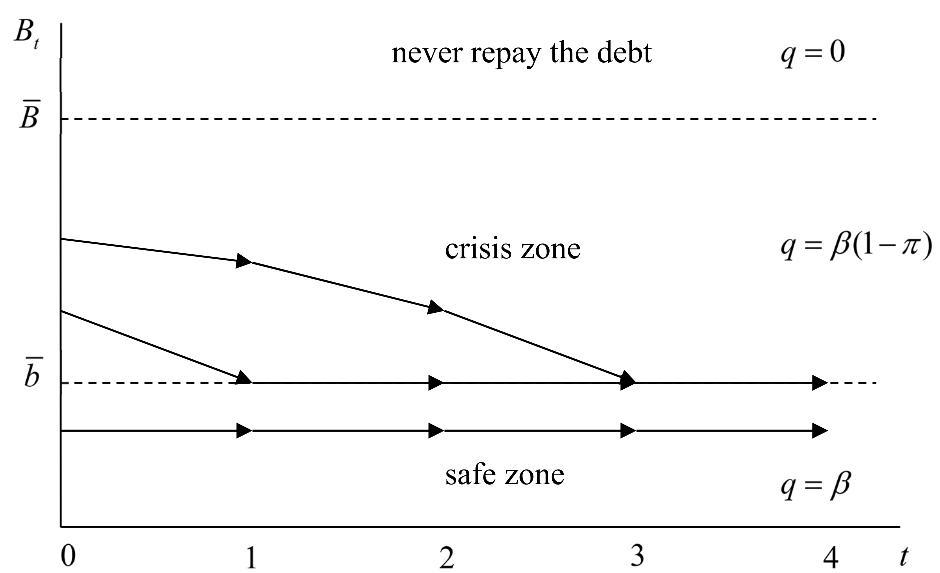

Figure 2. Optimal behavior of debt depending on the initial level of debt B_0.

10. A version of the model with bonds of long maturity

During the EMU crisis in 2010–13, countries like Spain had debt with an average maturity of six years or more. In contrast, during the Mexican debt crisis in December 1994 and January 1995, the average maturity of the debt was only nine months. Conesa and Kehoe (2017) present a version of the model calibrated to the Spanish economy in which one period is one year. To model maturity longer than one period, they generalize the model that we have presented so far with the analytical technique introduced by Hatchondo and Martinez (2009) and Chaterjee and Eyigungor (2012). Cole and Kehoe (1996) build a model of the Mexican economy in which a period is nine months. They argue that the Mexican government could have prevented the 1994–95 crisis by buying short-term debt and issuing long-term debt, but their argument is based on complicated modeling of long-term debt.

In the long maturity bond model, following the approach of Hatchondo and Martinez (2009) and Chaterjee and Eyigungor (2012), we write the government's problem as is to choose c, g, B', z to solve

$$V(s) = \max \; u(c,g) + \beta EV(s')$$

$$\text{s.a.} \quad c = (1-\theta)y(z)$$

$$g + z\delta B = \theta y(z) + q(B',s)\big(B' - (1-\delta)B\big)$$

$$z \in \{0,1\}$$

$$z = 0 \text{ if } z_{-1} = 0 .$$

Here δ, $0 \le \delta \le 1$, is the fraction of total debt due each period. This formulation of bonds of longer maturity allows the debt service to be a fraction of the debt, and, at the same time, it is not necessary to take into account the structure of the public debt as a whole, which would make the problem computationally intractable. The higher maturity debt implies a lower value of δ: In fact, the average maturity of the debt is $1/\delta$.

This formulation is particularly attractive because it provides a tractable recursive formula for calculating the price of bonds:

$$q(B',s) = \begin{cases} \beta\left[\delta + (1-\delta)Eq(B'(B',s),s')\right] & \text{if } B' \le \bar{b} \\ \beta(1-\pi)\left[\delta + (1-\delta)Eq(B'(B',s),s')\right] & \text{if } \bar{b} < B' \le \bar{B}(\pi) \\ 0 & \text{if } \bar{B}(\pi) < B' \end{cases} .$$

Notice that the price of bonds in the current period depends on the expected price of bonds in the subsequent period.

In this case, the model does not have an analytical solution, so we turn to a numerical solution. To do this, we calibrate the model following the same strategy as in Conesa and Kehoe (2017). To solve the model, we choose the following utility function:

$$u(c,g) = \log(c) + \gamma \log(g - \overline{g}).$$

A period corresponds to a year, and the parameters we use are those in Table 1.

parameter	value	target
$\overline{y}$	100.0	normalization
Z	0.95	default cost of 5% of output (1996)
β	0.98	risk-free interest rate of 2%
π	0.04	interest rate in the crisis zone of 6%
γ	0.20	arbitrary value
θ	0.36	fiscal revenue as fraction of output
$\overline{g}$	25.0	mínimum public spending

Table 1: Parameters in the quantitative model

See Conesa and Kehoe (2017) for a detailed discussion of the chosen values.

We can vary δ to understand the impact of increasing debt maturity. For this, we evaluate the numerical results for the values of $\delta \in \{1, 1/6, 1/30\}$, corresponding to a debt of one, six or thirty years of maturity, respectively. We choose six years since it is the average maturity of the debt of countries such as Spain or Italy.

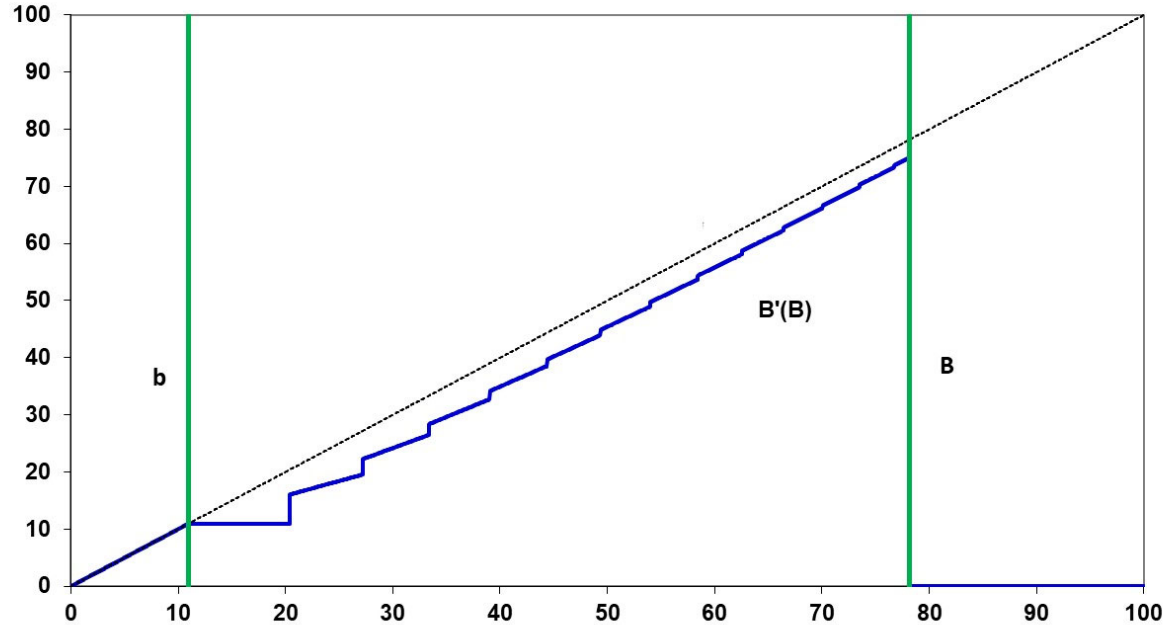

Figure 3: Optimal debt policy and equilibrium thresholds when $\delta = 1$.

Figure 3 shows that with one-year debt, the lower threshold is just over 10% of product, and that the upper threshold is almost 80% of product. With relatively low levels of debt, this economy would be exposed to liquidity crises. This is normal, since in each period the government must refinance the total of its public debt, so even low levels of this are impossible to face without access to credit. In the crisis zone, we clearly see how the optimal policy involves reducing the debt in a finite number of periods to the lower threshold.

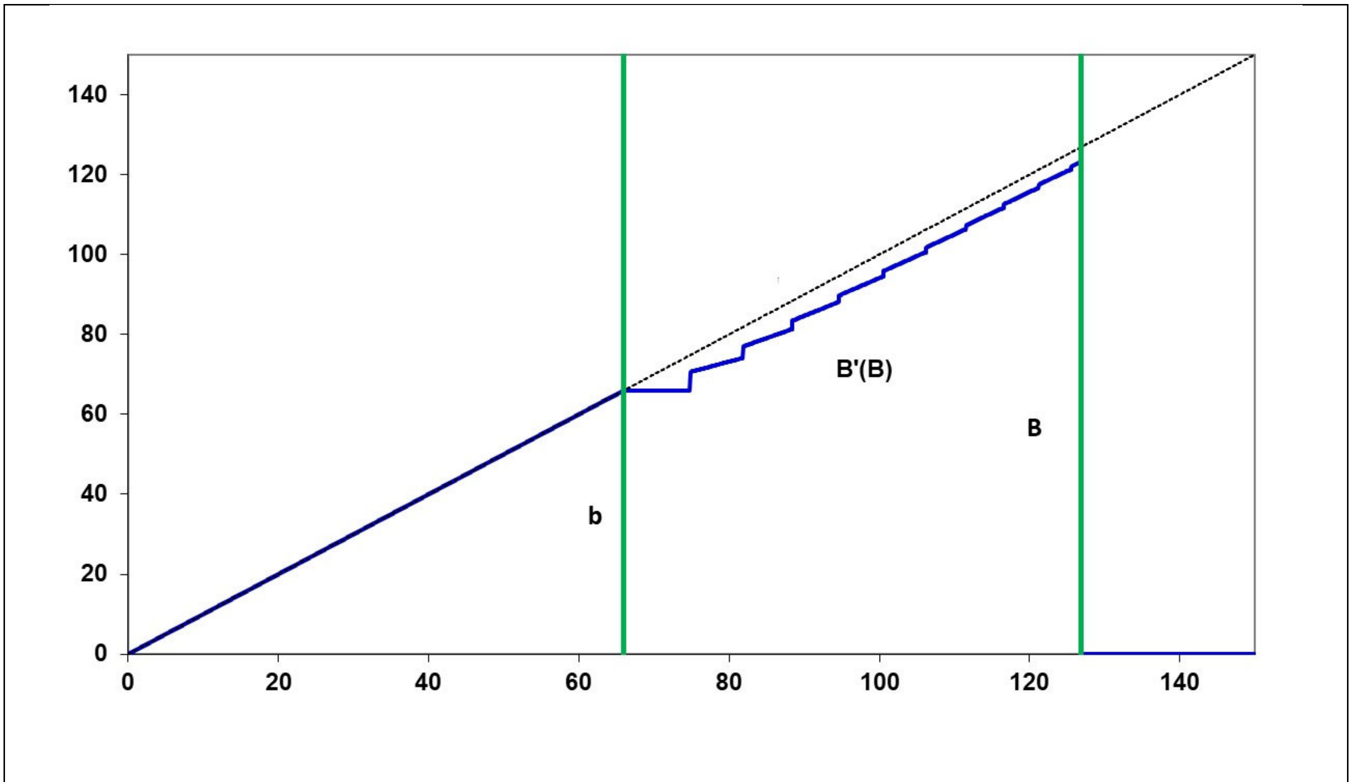

Figure 4: Optimal debt policy and equilibrium thresholds when $\delta = 1/6$.

When the debt has a maturity of six years, the thresholds change considerably. We observe in Figure 4 that the lower threshold is around 65% of the product, and the upper threshold reaches almost 130% of the product. In addition, the crisis area is proportionally smaller than in the case of $\delta = 1$.

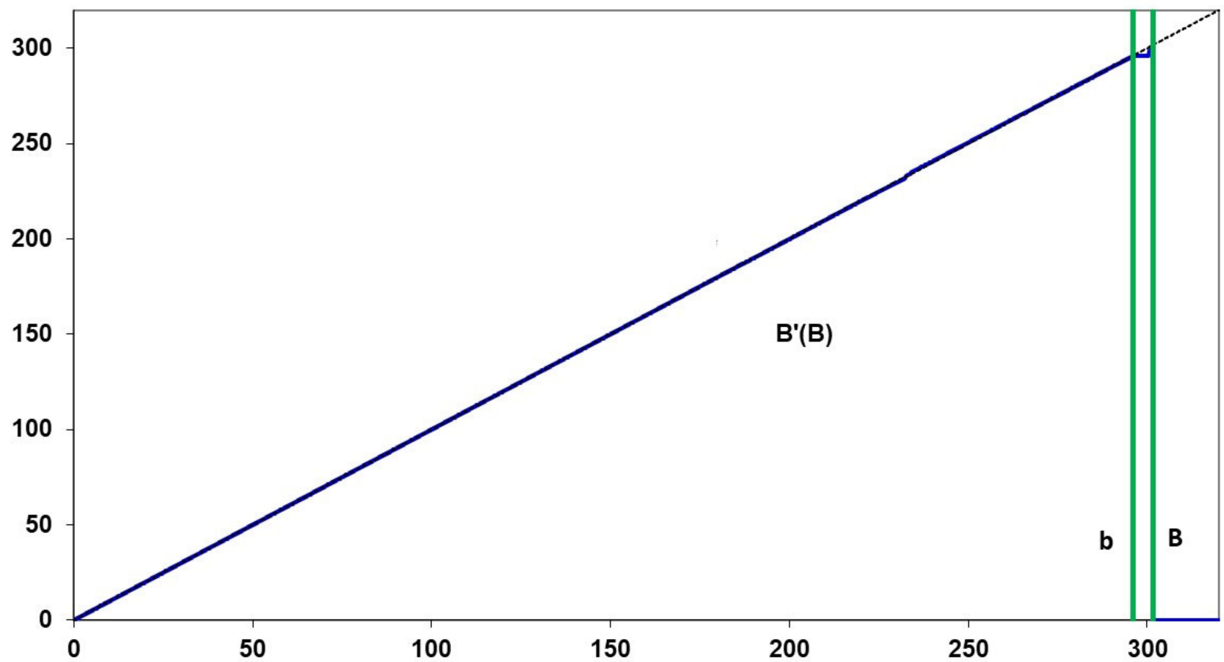

Figure 5: Optimal debt policy and equilibrium thresholds when $\delta = 1$.

Finally, with a debt maturity of thirty years, we observe in Figure 5 how the lower threshold converges to the upper threshold towards 300% of the product. That is, with debt of thirty or more years, liquidity crises practically disappear.

11. A government more impatient than the investors

Suppose now that the government is substantially more impatient than international investors. This can happen, for example, because a certain government discounts the future also by the probability of winning the next elections. Thus, the lower the probability of winning the next elections, the more impatient this government will be. We assume that the government has a discount factor of 0.90, much lower than that of international investors of 0.98. In this numerical experiment, we maintain the maturity of the debt at one period.

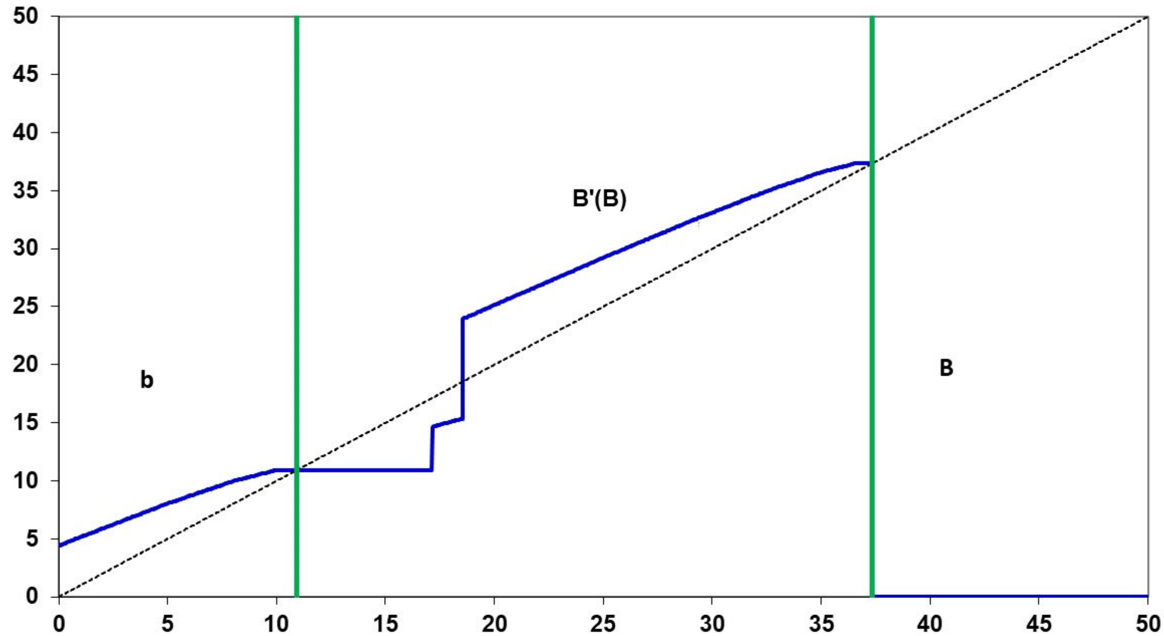

Figure 6: Optimal debt policy and equilibrium thresholds with an impatient government.

In this case, we see how the government decides to increase its debt levels both within the safe zone and in the upper part of the crisis zone. Only for debt levels close to the lower threshold does the government find it optimal to reduce its debt level. We also see how the upper threshold drops substantially, from almost 80% of output with a government as patient as investors to 37% with an impatient government..

Referencias

Aguiar, Mark and Gita Gopinath (2007), "Emerging market business cycles: The cycle is the trend," *Journal of Political Economy*, 115(1), 69–102.

Almeida, Victor, Carlos Esquivel, Timothy J. Kehoe, and Juan Pablo Nicolini (2018), "Default and Interest Rate Shocks: Renegotiation Matters," Federal Reserve Bank of Minneapolis, unpublished manuscript.

Arellano, Cristina (2008), "Default risk and income fluctuations in emerging economies," *American Economic Review*, 98(3), 690–712.

Arellano, Cristina and Yan Bai (2014), "Renegotiation policies in sovereign defaults," *American Economic Review*, 104(5), 94–100.

Bocola, Luigi and Alessandro Dovis (2019), "Self-fulfilling debt crises: A quantitative analysis," *American Economic Review*, 109(12), 4343–4377.

Cass, David and Karl Shell (1983), "Do sunspots matter?" *Journal of Political Economy*, 91(2), 193–227.

Chatterjee, Satyajit and Burcu Eyigungor (2012), "Maturity, indebtedness, and default risk," *American Economic Review*, 102(6), 2674–2699.

Cole, Harold L. and Timothy J. Kehoe (1996), "A self-fulfilling model of Mexico's 1994–1995 debt crisis," *Journal of International Economics*, 41(3–4), 309–330.

Cole, Harold L. and Timothy J. Kehoe (2000), "Self-fulfilling debt crises," *Review of Economic Studies*, 67(1), 91–116.

Conesa, Juan Carlos and Timothy J. Kehoe (2017), "Gambling for redemption and self-fulfilling debt crises," *Economic Theory*, 64(4), 707–740.

Cruces, Juan J. and Christoph Trebesch (2013), "Sovereign defaults: The price of haircuts," *American economic Journal: Macroeconomics*, 5(3), 85–117.

Eaton, Jonathan and Mark Gersovitz (1981), "Debt with potential repudiation: Theoretical and empirical analysis," *Review of Economic Studies*, 48(2), 289–309.

García Rodríguez, Marta (2022), "Riesgo de impago y su interacción con los fundamentales: El caso de Grecia," *Cuadernos Económicos de I.C.E.*, this number.

Hamann, Franz (2002), "Sovereign risk and macroeconomic fluctuations," Banco de la Republica de Colombia, Borradores de Economía, 225.

Harrison, Virden L. (1976), "Do sunspot cycles affect crop yields?" Economic Research Service, U.S. Department of Agriculture, Agricultural Economic Report No. 327.

Hatchondo, Juan Carlos, and Leonardo Martinez (2009), "Long-duration bonds and sovereign defaults," *Journal of International Economics*, 79(1), 117–125.

Jevons, W. Stanley (1878), "Commercial crises and sun-spots," *Nature*, 19(472), 33–37.

Xu, Hengyu, Yu Fei, Chun Li, Jiajuan Liang, Xinan Tian, and Zhongjie Wan (2021), "The North–South asymmetry of sunspot relative numbers based on complex network technique," *Symmetry*, 13(11): article 2228.

Yue, Vivian Z (2010), “Sovereign default and debt renegotiation,” *Journal of International Economics*, 80(2): 176–187.