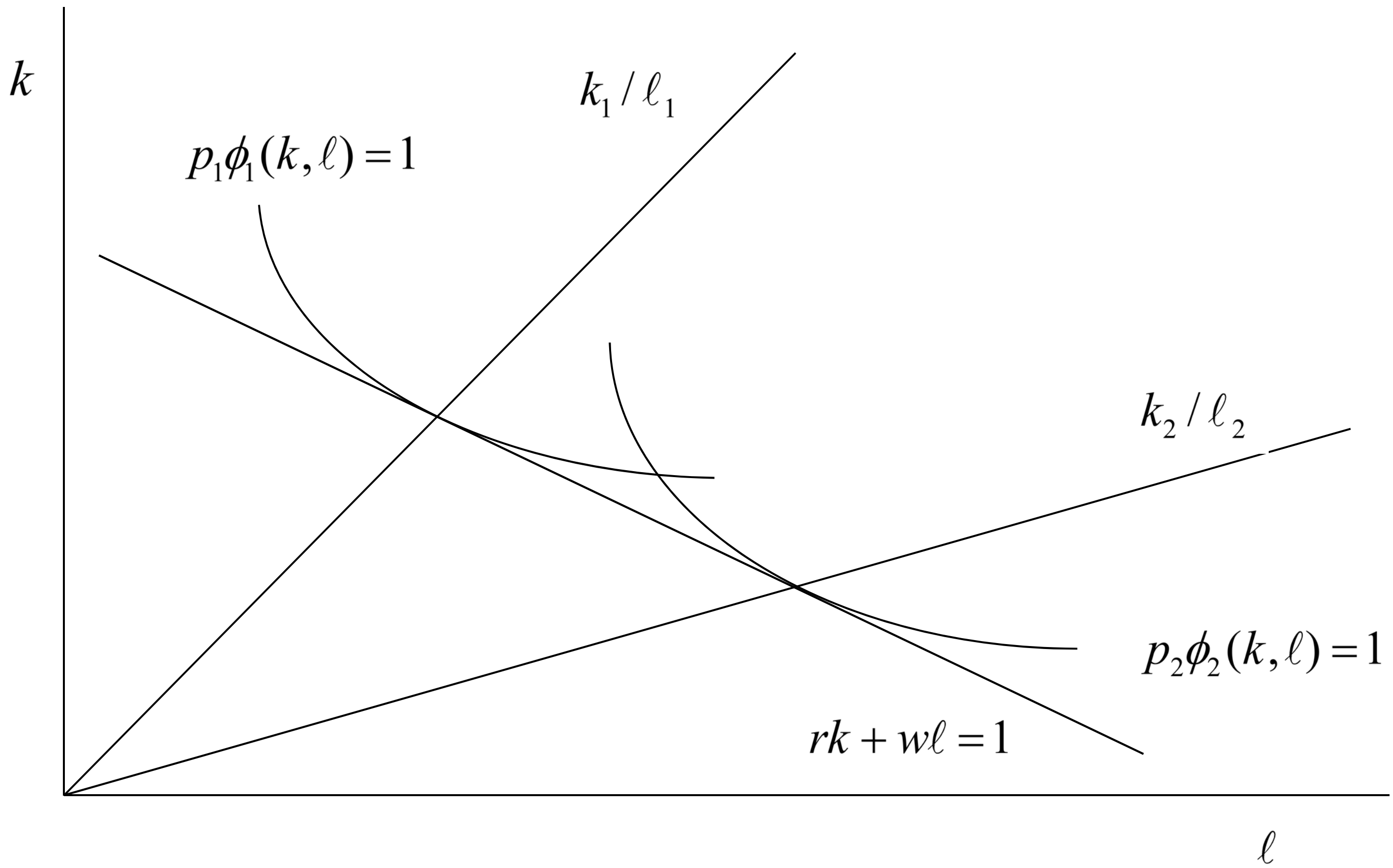
Patterns of Specialization in Heckscher-Ohlin Models

To sort out the different cases for production and specialization, we use the Lerner diagram.



There are four possible patterns of specialization given that we have called the capital intensive good good 1 and the capital intensive country country 1:

1. $y_1^1 > 0$, $y_2^1 > 0$, $y_1^2 > 0$, $y_2^2 > 0$.

2. $y_1^1 > 0$, $y_2^1 = 0$, $y_1^2 = 0$, $y_2^2 > 0$.

3. $y_1^1 > 0$, $y_2^1 > 0$, $y_1^2 = 0$, $y_2^2 > 0$.

4. $y_1^1 > 0$, $y_2^1 = 0$, $y_1^2 > 0$, $y_2^2 > 0$.

We refer to each of these patterns as a case.

To determine the region of the space of endowments $(\bar{k}^1, \bar{\ell}^1, \bar{k}^2, \bar{\ell}^2)$ in which each case occurs in equilibrium, we follow the procedure:

1. Guess the case.
2. Determine the equilibrium prices (p_1, p_2).
3. Use the prices (p_1, p_2) to determine the optimal capital-labor ratios k_1 / ℓ_1 and k_2 / ℓ_2 in the Lerner diagram.
4. Use the requirements imposed by the case that we have assumed in step 1 to obtain inequality restrictions on $\bar{k}^1 / \bar{\ell}^1$ and $\bar{k}^2 / \bar{\ell}^2$. If, for example, we have guessed that we are in case 3, we require that $\bar{k}^1 / \bar{\ell}^1 \geq k_1 / \ell_1$ and $k_1 / \ell_1 \geq \bar{k}^2 / \bar{\ell}^2 \geq k_2 / \ell_2$.

It would be useful to prove that the Lerner diagram does indeed characterize the different cases. That is, for example, if the prices (p_1, p_2) are such that $\bar{k}^1 / \bar{\ell}^1 \geq k_1 / \ell_1$, then it cannot be profit maximizing to set $y_2^1 > 0$.

Case 1

We use the conditions

$$\frac{p_2}{p_1} = \frac{a_2 \theta_1 k_1^{\alpha_1} \ell_1^{1-\alpha_1}}{a_1 \theta_2 k_2^{\alpha_2} \ell_2^{1-\alpha_2}}$$
$$p_1 \alpha_1 \theta_1 \left(\frac{k_1}{\ell_1} \right)^{\alpha_1 - 1} = p_2 \alpha_2 \theta_2 \left(\frac{k_2}{\ell_2} \right)^{\alpha_2 - 1}$$
$$p_1 (1-\alpha_1) \theta_1 \left(\frac{k_1}{\ell_1} \right)^{\alpha_1} = p_2 (1-\alpha_2) \theta_2 \left(\frac{k_2}{\ell_2} \right)^{\alpha_2}$$
$$k_1 + k_2 = \bar{k}^1 + \bar{k}^2$$
$$\ell_1 + \ell_2 = \bar{\ell}^1 + \bar{\ell}^2$$

to obtain expressions for p_2 / p_1, k_1, ℓ_1, k_2, ℓ_2:

$$k_1 = \frac{a_1 \alpha_1}{a_1 \alpha_1 + a_2 \alpha_2} (\bar{k}^1 + \bar{k}^2)$$
$$k_2 = \frac{a_2 \alpha_2}{a_1 \alpha_1 + a_2 \alpha_2} (\bar{k}^1 + \bar{k}^2)$$

$$\ell_1 = \frac{a_1(1-\alpha_1)}{a_1(1-\alpha_1)+a_2(1-\alpha_2)}(\bar{\ell}^1+\bar{\ell}^2)$$
$$\ell_2 = \frac{a_2(1-\alpha_2)}{a_1(1-\alpha_1)+a_2(1-\alpha_2)}(\bar{\ell}^1+\bar{\ell}^2)$$
$$\frac{p_2}{p_1} = \frac{\theta_1\left(a_1(1-\alpha_1)+a_2(1-\alpha_2)\right)\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}}{\theta_2\left(a_1\alpha_1+a_2\alpha_2\right)\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}}(\bar{k}^1+\bar{k}^2)^{\alpha_1-\alpha_2}(\bar{\ell}^1+\bar{\ell}^2)^{\alpha_2-\alpha_1}.$$

Notice that, in this case, and only this case, we do not need to know p_2/p_1 to calculate the optimal capital-labor ratios k_1/ℓ_1 and k_2/ℓ_2.

$$\frac{k_1}{\ell_1} = \frac{\alpha_1\left(a_1(1-\alpha_1)+a_2(1-\alpha_2)\right)}{(1-\alpha_1)\left(a_1\alpha_1+a_2\alpha_2\right)}\frac{(\bar{k}^1+\bar{k}^2)}{(\bar{\ell}^1+\bar{\ell}^2)}$$
$$\frac{k_2}{\ell_2} = \frac{\alpha_2\left(a_1(1-\alpha_1)+a_2(1-\alpha_2)\right)}{(1-\alpha_2)\left(a_1\alpha_1+a_2\alpha_2\right)}\frac{(\bar{k}^1+\bar{k}^2)}{(\bar{\ell}^1+\bar{\ell}^2)}.$$

The restrictions that characterize the set of endowments that are in case 1 are

$$\frac{\alpha_1\left(a_1(1-\alpha_1)+a_2(1-\alpha_2)\right)}{(1-\alpha_1)\left(a_1\alpha_1+a_2\alpha_2\right)}\frac{(\bar{k}^1+\bar{k}^2)}{(\bar{\ell}^1+\bar{\ell}^2)} > \frac{\bar{k}^i}{\bar{\ell}^i} > \frac{\alpha_2\left(a_1(1-\alpha_1)+a_2(1-\alpha_2)\right)}{(1-\alpha_2)\left(a_1\alpha_1+a_2\alpha_2\right)}\frac{(\bar{k}^1+\bar{k}^2)}{(\bar{\ell}^1+\bar{\ell}^2)},\ i=1,2.$$

Case 2

We can easily find p_2/p_1:

$$\frac{p_2}{p_1} = \frac{a_2\theta_1(\bar{k}^1)^{\alpha_1}(\bar{\ell}^1)^{1-\alpha_1}}{a_1\theta_2(\bar{k}^2)^{\alpha_2}(\bar{\ell}^2)^{1-\alpha_2}}.$$

We now use

$$p_1\alpha_1\theta_1\left(\frac{k_1}{\ell_1}\right)^{\alpha_1-1} = p_2\alpha_2\theta_2\left(\frac{k_2}{\ell_2}\right)^{\alpha_2-1}$$
$$p_1(1-\alpha_1)\theta_1\left(\frac{k_1}{\ell_1}\right)^{\alpha_1} = p_2(1-\alpha_2)\theta_2\left(\frac{k_2}{\ell_2}\right)^{\alpha_2}$$

to calculate k_1/ℓ_1 and k_2/ℓ_2:

$$\frac{k_1}{\ell_1}=\left(\frac{p_2\theta_2}{p_1\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{a_2}{\alpha_1-\alpha_2}}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1-a_2}{\alpha_1-\alpha_2}}$$
$$\frac{k_2}{\ell_2}=\left(\frac{p_2\theta_2}{p_1\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{a_1}{\alpha_1-\alpha_2}}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1-a_1}{\alpha_1-\alpha_2}}.$$

Plugging in p_2 / p_1, we obtain

$$\frac{k_1}{\ell_1}=\left(\frac{a_2(\overline{k}^1)^{\alpha_1}(\overline{\ell}^1)^{1-\alpha_1}}{a_1(\overline{k}^2)^{\alpha_2}(\overline{\ell}^2)^{1-\alpha_2}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{a_2}{\alpha_1-\alpha_2}}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1-a_2}{\alpha_1-\alpha_2}}$$
$$\frac{k_2}{\ell_2}=\left(\frac{a_2(\overline{k}^1)^{\alpha_1}(\overline{\ell}^1)^{1-\alpha_1}}{a_1(\overline{k}^2)^{\alpha_2}(\overline{\ell}^2)^{1-\alpha_2}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{a_1}{\alpha_1-\alpha_2}}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1-a_1}{\alpha_1-\alpha_2}}.$$

The restrictions that characterize the set of endowments that are in case 2 are

$$\frac{\overline{k}^1}{\overline{\ell}^1}\geq\left(\frac{a_2(\overline{k}^1)^{\alpha_1}(\overline{\ell}^1)^{1-\alpha_1}}{a_1(\overline{k}^2)^{\alpha_2}(\overline{\ell}^2)^{1-\alpha_2}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{a_2}{\alpha_1-\alpha_2}}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1-a_2}{\alpha_1-\alpha_2}}$$
$$\frac{\overline{k}^2}{\overline{\ell}^2}\leq\left(\frac{a_2(\overline{k}^1)^{\alpha_1}(\overline{\ell}^1)^{1-\alpha_1}}{a_1(\overline{k}^2)^{\alpha_2}(\overline{\ell}^2)^{1-\alpha_2}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{a_1}{\alpha_1-\alpha_2}}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1-a_1}{\alpha_1-\alpha_2}}.$$

Case 3

We use the conditions

$$\frac{p_2}{p_1}=\frac{a_2\theta_1(k_1^1)^{\alpha_1}(\ell_1^1)^{1-\alpha}}{a_1\left(\theta_2(k_2^1)^{\alpha_2}(\ell_2^1)^{1-\alpha_2}+\theta_2(\overline{k}^2)^{\alpha_2}(\overline{\ell}^2)^{1-\alpha_2}\right)}$$
$$p_1\alpha_1\theta_1\left(\frac{k_1^1}{\ell_1^1}\right)^{\alpha_1-1}=p_2\alpha_2\theta_2\left(\frac{k_2^1}{\ell_2^1}\right)^{\alpha_2-1}$$
$$p_1(1-\alpha_1)\theta_1\left(\frac{k_1^1}{\ell_1^1}\right)^{\alpha_1}=p_2(1-\alpha_2)\theta_2\left(\frac{k_2^1}{\ell_2^1}\right)^{\alpha_2}$$
$$k_1^1+k_2^1=\overline{k}^1$$
$$\ell_1^1+\ell_2^1=\overline{\ell}^1.$$

The calculation of k_1^1, ℓ_1^1, k_2^1, ℓ_2^1 become

$$\kappa_1(p_2/p_2)=\frac{k_1^1}{\ell_1^1}=\left(\frac{p_2\theta_2}{p_1\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{a_2}{\alpha_1-\alpha_2}}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1-a_2}{\alpha_1-\alpha_2}}$$

$$\kappa_2(p_2/p_2)=\frac{k_2^1}{\ell_2^1}=\left(\frac{p_2\theta_2}{p_1\theta_1}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{a_1}{\alpha_1-\alpha_2}}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1-a_1}{\alpha_1-\alpha_2}}.$$

$$\ell_1^1=\frac{\overline{k}^1-\kappa_2(p_2/p_2)\overline{\ell}^1}{\left(\kappa_2(p_2/p_2)-\kappa_2(p_2/p_2)\right)}$$

$$\ell_2^1=\overline{\ell}^1-\ell_1^1=\frac{\kappa_1(p_2/p_2)\overline{\ell}^1-\overline{k}^1}{\left(\kappa_2(p_2/p_2)-\kappa_2(p_2/p_2)\right)}$$

$$k_1^1=\frac{\overline{k}^1-\kappa_2(p_2/p_2)\overline{\ell}^1}{\left(\kappa_2(p_2/p_2)-\kappa_2(p_2/p_2)\right)}\kappa_1(p_2/p_2)$$

$$k_2^1=\frac{\kappa_1(p_2/p_2)\overline{\ell}^1-\overline{k}^1}{\left(\kappa_2(p_2/p_2)-\kappa_2(p_2/p_2)\right)}\kappa_2(p_2/p_2),$$

which provides us with a nonlinear equation for determining p_2/p_1:

$$\frac{p_2}{p_1}=\frac{a_2\theta_1\kappa_1(p_2/p_1)^{\alpha_1}\dfrac{\overline{k}^1-\kappa_2(p_2/p_2)\overline{\ell}^1}{\left(\kappa_2(p_2/p_2)-\kappa_2(p_2/p_2)\right)}}{a_1\left(\theta_2\kappa_2(p_2/p_1)^{\alpha_2}\dfrac{\kappa_1(p_2/p_2)\overline{\ell}^1-\overline{k}^1}{\left(\kappa_2(p_2/p_2)-\kappa_2(p_2/p_2)\right)}+\theta_2(\overline{k}^2)^{\alpha_2}(\overline{\ell}^2)^{1-\alpha_2}\right)}.$$

Once we solve for p_2/p_1, we check that

$$\kappa_1(p_2/p_2)>\frac{\overline{k}^1}{\overline{\ell}^1}>\kappa_2(p_2/p_2)$$

$$\kappa_2(p_2/p_2)>\frac{\overline{k}^1}{\overline{\ell}^1}.$$

Case 4

We use the conditions

$$\frac{p_2}{p_1}=\frac{a_2\left(\theta_1(\overline{k}^1)^{\alpha_1}(\overline{\ell}^1)^{1-\alpha}+\theta_1(k_1^2)^{\alpha_1}(\ell_1^2)^{1-\alpha}\right)}{a_1\theta_2(k_2^2)^{\alpha_2}(\ell_2^2)^{1-\alpha_2}}$$

$$p_1\alpha_1\theta_1\left(\frac{k_1^2}{\ell_1^2}\right)^{\alpha_1-1}=p_2\alpha_2\theta_2\left(\frac{k_2^2}{\ell_2^2}\right)^{\alpha_2-1}$$

$$p_1(1-\alpha_1)\theta_1\left(\frac{k_1^2}{\ell_1^2}\right)^{\alpha_1} = p_2(1-\alpha_2)\theta_2\left(\frac{k_2^2}{\ell_2^2}\right)^{\alpha_2}$$
$$k_1^2 + k_2^2 = \bar{k}^2$$
$$\ell_1^2 + \ell_2^2 = \bar{\ell}^2.$$

Everything is symmetric with case 3. We just reverse the roles of countries 1 and 2 and of goods 1 and 2.

Characterizing endowments

Let us fix $\bar{k} = \bar{k}^1 + \bar{k}^2$ and $\bar{\ell} = \bar{\ell}^1 + \bar{\ell}^2$. The endowment $(\bar{k}^1, \bar{\ell}^1, \bar{k}^2, \bar{\ell}^2)$ can be represented as a point $(\bar{k}^1, \bar{\ell}^1)$ in a box where $(\bar{k}^2, \bar{\ell}^2) = (\bar{k} - \bar{k}^1, \bar{\ell} - \bar{\ell}^1)$. We want to divide this box up into regions for which each of the cases applies. Notice that we have only considered the upper triangle of this box where $\bar{k}^1 / \bar{\ell}^1 \geq (\bar{k} - \bar{k}^1)/(\bar{\ell} - \bar{\ell}^1)$. To characterize the endowments in the entire box, we have to allow 3 additional cases, 2', 3', and 4', which are just cases 2, 3, and 4 with the roles of countries 1 and 2 reversed.

Notice that the region in which case 1 applies is easily defined:

$$\frac{\alpha_1\left(a_1(1-\alpha_1)+a_2(1-\alpha_2)\right)}{(1-\alpha_1)\left(a_1\alpha_1+a_2\alpha_2\right)}\frac{(\bar{k}^1+\bar{k}^2)}{(\bar{\ell}^1+\bar{\ell}^2)} > \frac{\bar{k}^i}{\bar{\ell}^i} > \frac{\alpha_2\left(a_1(1-\alpha_1)+a_2(1-\alpha_2)\right)}{(1-\alpha_2)\left(a_1\alpha_1+a_2\alpha_2\right)}\frac{(\bar{k}^1+\bar{k}^2)}{(\bar{\ell}^1+\bar{\ell}^2)},\ i=1,2.$$

This is a system of 4 linear inequalities. Letting each inequality bind one by one, we can determine the boundary of region endowments that satisfy all four.

We now need to divide up the region for which case 1 does not apply into cases 2, 3, and 4. To do this, we use the inequalities that define case 2,

$$\frac{\bar{k}^1}{\bar{\ell}^1} \geq \left(\frac{a_2(\bar{k}^1)^{\alpha_1}(\bar{\ell}^1)^{1-\alpha_1}}{a_1(\bar{k}^2)^{\alpha_2}(\bar{\ell}^2)^{1-\alpha_2}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{a_2}{\alpha_1-\alpha_2}}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1-a_2}{\alpha_1-\alpha_2}}$$
$$\frac{\bar{k}^2}{\bar{\ell}^2} \leq \left(\frac{a_2(\bar{k}^1)^{\alpha_1}(\bar{\ell}^1)^{1-\alpha_1}}{a_1(\bar{k}^2)^{\alpha_2}(\bar{\ell}^2)^{1-\alpha_2}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{a_1}{\alpha_1-\alpha_2}}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1-a_1}{\alpha_1-\alpha_2}}.$$

Let us let the first inequality bind:

$$(\bar{k}^1)^{\alpha_2}(\bar{\ell}^1)^{1-\alpha_2} = \left(\frac{a_1\alpha_1^{a_2}(1-\alpha_1)^{1-a_2}}{a_2\alpha_2^{a_2}(1-\alpha_2)^{1-a_2}}\right)(\bar{k}-\bar{k}^1)^{a_2}(\bar{\ell}-\bar{\ell}^1)^{1-a_2}$$

$$\frac{\overline{k}^1}{\overline{k}-\overline{k}^1} = \left(\frac{a_1\alpha_1^{a_2}(1-\alpha_1)^{1-a_2}}{a_2\alpha_2^{a_2}(1-\alpha_2)^{1-a_2}}\right)^{\frac{1}{a_2}} \left(\frac{\overline{\ell}-\overline{\ell}^1}{\overline{\ell}^1}\right)^{\frac{1-a_2}{a_2}}$$

$$\overline{k}^1 = \frac{\overline{k}}{\left(\frac{a_2\alpha_2^{a_2}(1-\alpha_2)^{1-a_2}}{a_1\alpha_1^{a_2}(1-\alpha_1)^{1-a_2}}\right)^{\frac{1}{a_2}} \left(\frac{\overline{\ell}^1}{\overline{\ell}-\overline{\ell}^1}\right)^{\frac{1-a_2}{a_2}} + 1},$$

which defines a curve in the box. Similarly, letting the second inequality bind, we obtain another curve:

$$\overline{k}^1 = \frac{\overline{k}}{\left(\frac{a_1\alpha_1^{a_1}(1-\alpha_1)^{1-a_1}}{a_2\alpha_2^{a_1}(1-\alpha_2)^{1-a_1}}\right)^{\frac{1}{a_1}} \left(\frac{\overline{\ell}^1}{\overline{\ell}-\overline{\ell}^1}\right)^{\frac{1-a_1}{a_1}} + 1}.$$

Reversing the roles of countries 1 and 2 give us

$$\overline{k}-\overline{k}^1 = \frac{\overline{k}}{\left(\frac{a_2\alpha_2^{a_2}(1-\alpha_2)^{1-a_2}}{a_1\alpha_1^{a_2}(1-\alpha_1)^{1-a_2}}\right)^{\frac{1}{a_2}} \left(\frac{\overline{\ell}-\overline{\ell}^1}{\overline{\ell}^1}\right)^{\frac{1-a_2}{a_2}} + 1}$$

$$\overline{k}-\overline{k}^1 = \frac{\overline{k}}{\left(\frac{a_1\alpha_1^{a_1}(1-\alpha_1)^{1-a_1}}{a_2\alpha_2^{a_1}(1-\alpha_2)^{1-a_1}}\right)^{\frac{1}{a_1}} \left(\frac{\overline{\ell}-\overline{\ell}^1}{\overline{\ell}^1}\right)^{\frac{1-a_1}{a_1}} + 1}.$$

Consider the example in which $a_1 = a_2 = 1/2$, $\theta_1 = \theta_2 = 1$, $\alpha_1 = 2/3$, $\alpha_2 = 1/3$: