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HECKSCHER-OHLIN TRADE THEORY WITH A CONTINUUM OF GOODS

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This paper studies trade theory for the case of a continuum of goods, two factors, two countries, and Cobb-Douglas demand functions. If factor endowments are similar, factor price equalization obtains and geographic patterns of production are indeterminate; nonetheless the effects of changes in factor endowments on prices and welfare in each country are well defined. Factor price equalization does not obtain if factor endowments are far apart, and the geographic pattern of specialization is then determinate. The effects of changes in endowments on the range of goods produced in each country and on prices of goods and factors are analyzed for this case, and the elasticity of substitution in production is shown to play an important role in determining comparative static outcomes.

I. INTRODUCTION

This paper studies Heckscher-Ohlin trade theory in a model with a continuum of goods. In earlier work [Dornbusch, Fischer, and Samuelson, 1977] we formulated a continuum model for the case of Ricardian technology—constant unit labor requirements and labor as the only factor—and showed that the continuum assumption offers great convenience in analyzing standard questions in trade theory. In this paper we report on some applications of the continuum model to the more complicated case of two factors of production.

Section II introduces the model and shows the derivation of the closed-economy equilibrium. Here it is shown that the equilibrium wage-rental ratio fully characterizes the equilibrium of our economy,

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even though there is an infinity of goods. In Section III we proceed to an open economy with equalized factor prices. The familiar result that the world economy under factor price equalization is like a single economy is shown, as is the fact that there is no determinate trade pattern. There are nevertheless interesting results relating to growth, world income distribution, and welfare that can be demonstrated for this case.

Section IV develops the analysis of equilibrium in the world economy when factor prices differ internationally, as they will when factor endowments are sufficiently far apart. We show first the determination of equilibrium specialization patterns, and then derive the relationship between relative factor prices and the various regional endowments of factors of production. There is now the possibility—not present in the two-commodity model—of some ambiguity about the effects of growth on relative factor and commodity prices and trade patterns. There is also the fact that specialization will always extend to a range of goods rather than to a single commodity and that accordingly, as we shall show, relative factor prices may move in opposite directions in the two countries.

To pave the way for analysis of the Heckscher-Ohlin continuum model, we recapitulate known results for the case of n goods and m factors. In particular, the case of $n > m$ is of most interest here, since the continuum model involves n as an uncountably infinite number. The exposition simplifies when we assume the following: (1) uniform (strongly quasi-concave) homothetic tastes;[1] and (2) each country has the same concave, first-degree-homogeneous production functions, of the smooth neoclassical type. In the two-factor case ($m = 2$, corresponding to capital and labor), the exposition simplifies when out of any pair of goods, one is always definitely more capital-intensive than the other at *every* factor-price ratio.

One can verify the following results:

For any pattern of factor endowments, a unique pattern of goods and factor pricing is determined. Also, there is a unique equilibrium pattern of world and regional consumptions, with consumption ratios the same in every country when transport costs of goods are literally zero.

If all countries have identical factor-endowment ratios, the pricing and consumption pattern prevailing in every country is what

1. Much of our continuum analysis will be carried out under the further simplifying assumption that uniform homothetic tastes are of the Millian or Cobb-Douglas form: regardless of prices and incomes, every region always spends a fixed function of its income on each respective good. See Samuelson [forthcoming], Wan [1977], and Wilson [forthcoming] for more general models.

would prevail in each and every country under autarky. Real factor returns are, of course, equalized (with or without goods' trade).

However, if there are more goods than factors, it need *not* be the case that each region produces its own autarky-quantum of consumption goods, although that is an admissible pattern of geographical specialization. But, with $n > m$, there will be an $(n - m)$-fold infinity of geographical patterns or production specialization.[2] The indeterminacy is inessential when transport costs are zero.[3]

With $n = m$, the geographical pattern will almost always be unique.

Now let the two countries differ slightly in factor-endowment ratios. Then factor returns will stay equalized, and uniform world pricing and consuming patterns are uniquely determined. Again, with $n > m$, the pattern of production specialization is not geographically determined in a unique way. There will still be an $(n - m)$-fold infinity of indifferent admissible patterns. However, the region with excess labor endowment on average produces goods that are of higher labor intensity than goods produced abroad.

Finally, let the two countries differ more and more in factor-endowment ratios. With n finite, eventually the country with the greater relative capital endowment will find its real rental rate (in terms of every good) falling below that of the other country; as the real rental rate r falls here and the wage-rental ratio w/r rises, the real wage also rises here; and w^*/r^*, w^*, and $1/r^*$ all move in the same direction.[4]

Here is a brief account of Heckscher-Ohlin specialization patterns in the two-factor case. Under our strong assumption, we can rank the goods in order of decreasing relative capital intensity:

$$1, 2, \ldots, j, j + 1, \ldots, n.$$

Good 1 will certainly be produced at home if ours is the capital-rich region with higher w/r. Good n will certainly be produced abroad and be imported by us. As in the analysis of Ricardian comparative advantage, from technology alone, one cannot deduce which other goods

2. See Meade [1950] for discussion of the algebra of the $n = 3 > m = 2$ case. Geometrically, when $n > m$, the production-possibility frontier of any region has straight lines or planes on it; so the tangency between two regions' frontiers, needed to define the world production-possibility frontier, can be along an infinity of points belonging to both regions' flat facets. See also Melvin [1968].

3. When transport costs are positive, the invisible hand of competitive arbitrage will solve the problem that minimizes deadweight loss from cross haulage or non-optimal specialization.

4. An increase in capital in our region, other things equal, lowers both real r and, at least initially, r^*; it raises both real w and initially real w^*; how it ultimately affects r/r^* does not seem to be uniformly predictable.

$2, \ldots, j$ the home region must also produce, and which goods $j, \ldots, n$ we definitely have to import. Where j falls between 1 and n must depend on the full general-equilibrium conditions of reciprocal demand.

In the discrete many-goods case, there are two possible types of equilibria: one in which *no* good is produced by *both* countries; and the other "limbo" case in which the borderline good *is* produced in both countries. The continuum case that we consider in this paper is special in that there is only one type of equilibrium, which we may think of as involving the joint production of the borderline commodity. We proceed now to the closed-economy continuum model.

II. The Closed Economy

There is a continuum of goods, indexed by z (z runs from zero to one), and two factors of production, capital and labor. Supplies of the two factors are fixed exogenously at levels K and N, respectively. Production functions for each of the goods are regularly neoclassical, satisfying the Inada conditions. The goods are indexed in order of decreasing capital intensity of production; there are no factor intensity reversals so that, for instance, good zero is the most capital-intensive in production at all wage-rental ratios.

We assume *uniform* homothetic demand. Our earlier paper's assumption of Millian or Cobb-Douglas demand is again postulated: C-D demand means that the function of income spent on good z, $b(z)$, is the same at all prices and in both countries. The integral over $b(z)$ on the interval $0 \leq z \leq 1$ is thus unity, as all income is spent.

1. The Equilibrium

We now examine the existence and properties of the equilibrium of this economy. For every good, the condition of equality of supply and demand is

$$P(z)Q(z) = b(z)(wN + rK), \tag{1}$$

where $Q(z)$ is the output of good z, $P(z)$ is the price, w is the wage rate, r is the rental rate on capital.

Full employment is assumed, so that total income is equal to the bracketed expression on the right-hand side of (1). By virtue of the assumed constant returns to scale, the value of output of commodity z is equal to $[wN(z) + rK(z)]$, where $N(z)$ and $K(z)$ are the quantities of labor and capital employed in the production of z. Simple manipulation results in (2) as an equivalent statement of the equilibrium condition in market z:

$$x(z) \equiv \frac{N(z)}{N} = \frac{b(z)(\sigma + k)}{\sigma + k(\sigma;z)}, \tag{2}$$

where $\sigma \equiv w/r$ is the wage-rental ratio, k is the aggregate capital-labor ratio, and $k(\sigma;z)$ is the capital-labor ratio in industry z, a function of σ, the wage-rental ratio.[5]

Labor and capital must be fully employed in equilibrium. This fact implies, for labor, that

$$\int_0^1 N(z)dz = N \tag{3}$$

or

$$\int_0^1 x(z)dz = 1 = \int_0^1 \frac{b(z)\,(\sigma + k)}{\sigma + k(z)}dz. \tag{4}$$

The full-employment condition for capital is, analogously,

$$\int_0^1 x(z)k(z)dz = k = \int_0^1 \frac{b(z)k(z)(\sigma + k)}{\sigma + k(z)}dz. \tag{5}$$

Combining (4) and (5), we obtain the fundamental equilibrium condition for the closed economy:

$$\phi(\sigma;k) \equiv \int_0^1 \frac{b(z)(\sigma + k)}{\sigma + k(z)}[k(z) - k]dz = 0. \tag{6}$$

To demonstrate the existence of a unique equilibrium, we consider first the case of Cobb-Douglas production functions. With such production functions distributive shares are constant, and we can substitute in (4) the constant ratios of income shares $\Psi(z) \equiv k(z)/\sigma$ to obtain

$$1 = \int_0^1 b(z)\frac{1 + k/\sigma}{1 + \Psi(z)}dz \equiv \xi\left(\frac{k}{\sigma}\right). \tag{4}$$

It is immediately apparent that the right-hand side is a uniformly decreasing function of the wage-rental ratio that runs the gamut from $+\infty$ to a number less than one, and that accordingly there exists a unique $\bar{\sigma}$ that solves (4)′ as shown in Figure I.

For the more general case of production functions not restricted to unit elasticities of substitution, it is convenient to look at (6). That at least one equilibrium for this economy exists is established by ob-

5. In what follows, we shall simplify notation and write $k(z)$ instead of $k(\sigma,z)$.

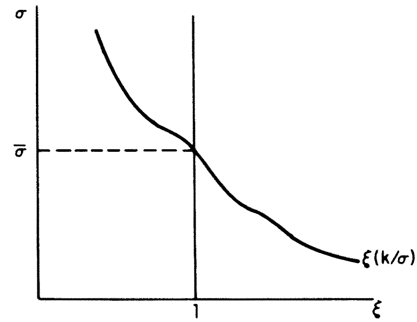

FIGURE I

serving first that, by virtue of the assumption that the production functions are neoclassical, $\phi(\ )$ is continuous in σ; and second, that because each production function satisfies the Inada conditions, there are values of σ such that $k(z)$ is greater than k for all z, and hence that $\phi(\sigma,k)$ is positive at those values of σ, and similarly, that there are values of σ at which $k(z)$ is less than k for all z, so that $\phi(\ )$ is negative at those values of σ. Given that $\phi(\sigma,k)$ is continuous in σ, there is some $\bar{\sigma}$ such that $\phi(\ )$ is equal to zero; $\bar{\sigma}$ is an equilibrium wage-rental ratio.

At such a wage-rental ratio there is full employment of both factors, and the production of each good can be determined from (2) and (3). The equilibrium is in fact unique. To show this, we establish that the derivative of $\phi(\ )$ with respect to σ is positive. Differentiating (6) yields

$$(7)\quad \frac{\partial \phi}{\partial \sigma} = \int_0^1 \frac{b(z)}{[\sigma + k(z)]^2}\left[(k(z) - k)^2 + (\sigma + k)^2 \frac{\partial k(z)}{\partial \sigma}\right] dz > 0.$$

The existence of a unique equilibrium $\bar{\sigma}$ under our strong assumptions is thus established. Given the wage-rental ratio, the rental rate in terms of any good can be determined from the production functions. The relative price structure and the level of income are accordingly determined, as is the demand for each good. We next examine relative price behavior in more detail.

Defining the relative price of commodity z in terms of commodity z', we have

$$\frac{P(z)}{P(z')} = \frac{a(z)w + c(z)r}{a(z')w + c(z')r} = \frac{a(z)[\sigma + k(z)]}{a(z')[\sigma + k(z')]},$$

where $a(z)$ is the unit labor requirement and $c(z)$ the unit capital requirement for commodity z. Taking logs and differentiating, we obtain the familiar result,

$$\frac{d \ln[P(z)/P(z')]}{d\sigma} = \frac{[k(z') - k(z)]}{[\sigma + k(z)][\sigma + k(z')]},$$

so that an increase in the wage-rental ratio lowers the relative price of the capital-intensive good.

Real factor returns are also determined by the wage-rental ratio. An increase in the wage-rental ratio raises the real wage in terms of all commodities and lowers the real rental on capital in terms of all goods.[6]

2. *Comparative Statics*

The equilibrium condition and the relative price structure can now be used to discuss questions of comparative statics. We consider here two applications, changes in the factor endowments, and changes in tastes.

Endowment Changes. We would expect an increase in the capital-labor ratio to reduce the relative return to capital. That result can be demonstrated by differentiating the equilibrium condition in (6), noting that

$$\left.\frac{\partial \phi}{\partial k}\right|_{\phi=0} = -(\sigma + k) \int_0^1 \frac{b(z)}{\sigma + k(z)}\, dz = -1$$

and hence

$$\frac{d\bar{\sigma}}{dk} = 1 \Big/ \frac{\partial \phi}{\partial \sigma} > 0.$$

The increase in the equilibrium wage-rental ratio induced by a higher capital-labor ratio implies changes in relative prices, real factor returns, and resource allocation.[7] The relative price of labor-intensive goods will rise, and the real wage will increase in terms of all com-

6. Defining the real wage and the real rental in terms of commodity z yields

$$\frac{w}{P(z)} = \frac{w}{wa(z) + rc(z)} = \frac{\sigma/a(z)}{\sigma + k(z)}; \quad \frac{r}{P(z)} = \frac{1/a(z)}{\sigma + k(z)}.$$

The elasticities of the real factor prices with respect to the wage-rental ratio are

$$\frac{d \ln(w/P(z))}{d \ln \sigma} = \frac{k(z)}{\sigma + k(z)} \quad \text{and} \quad \frac{d \ln[r/P(z)]}{d \ln \sigma} = -\frac{\sigma}{\sigma + k(z)}.$$

7. For the special case of Cobb-Douglas demand *and* production functions, it can be shown that the elasticity of σ with respect to k is unity. This is directly apparent from (4)′.

modities, while the real return to the more plentiful capital declines in terms of all commodities.

Next we ask how an increase in the capital stock affects the output of each good. The output of good z (per capita) is given by

$$q(z) = \frac{b(z)(w + rk)}{P(z)} = \frac{b(z)(\sigma + k)f(k(z))}{\sigma + k(z)}. \tag{8}$$

The effect of an increased k therefore is

$$\frac{dq(z)}{dk} = \frac{b(z)f(k(z))}{\sigma + k(z)}\left[1 + \frac{d\sigma}{dk}\frac{k(z) - k}{\sigma + k(z)}\right]. \tag{9}$$

For all low z industries, for which $k(z) > k$, an increase in the aggregate capital stock unambiguously increases physical production. However, it is possible that output in high z industries falls.[8] Note from (9) that if output for any industry falls, it also falls for all those with lower capital intensity. These results simply reflect the income and price effects induced by an expansion in the capital-labor ratio. The income effect raises demand for all goods. The induced increase in the wage-rental ratio lowers the relative price of capital-intensive goods—goods where $k(z) > k$—and thus adds to the demand for those commodities, while reducing demand for relatively labor-intensive goods.

Given that $d\sigma/dk > 0$, capital intensity of production rises in every industry when the overall capital-labor ratio rises. Accordingly, an increase in k must mean that labor is withdrawn in absolute quantity from those high z industries (if any) whose output falls. But it is not necessarily true that labor input is increased for *all* low z industries; that depends on the derivative $\partial k(z)/\partial\sigma$. The more the industry's capital-output ratio responds to a change in the wage-rental ratio (i.e., the higher the elasticity of substitution at that point), the less likely is it that labor is shifted into a particular low z industry. But, on average, labor is shifted into low z industries.

Shifts in Tastes. We ask next about a shift in tastes toward the low z industries. Since we are increasing the demand for capital-intensive goods, we should expect the wage-rental ratio to fall. We conveniently parameterize the change in tastes by adding to each $b(z)$ an amount $\lambda\Delta b(z)$, where λ is a positive scalar, $\int \Delta b(z)dz = 0$, and $\Delta b(z) > 0$ for $0 \leq z \leq \hat{z}$; and $\Delta b(z) < 0$ for $\hat{z} \leq z \leq 1$, where $k(\hat{z}) = k$.

8. Of course, it is possible that the output of *all* goods rises. That *must* happen with Cobb-Douglas production functions for every good: in that case a rise in K can be shown to leave real income measured in wage units unchanged. Hence costs, measured in wage units, fall for every good and output increases.

Then

$$\frac{\partial \phi}{\partial \lambda} = \int_0^1 \frac{\Delta b(z)(\sigma + k)(k(z) - k)}{\sigma + k(z)} dz > 0,$$

and accordingly

$$\frac{d\sigma}{d\lambda} < 0.$$

An interesting implication of the shift in demand, and a perhaps surprising one, is that the net effect is not necessarily to raise equilibrium output of all the goods toward which demand shifts. The reason again is the relative price effect, which in this instance implies an increase in the relative price of capital-intensive goods. Since the relative price increases with capital intensity, the substitution effect works against increases in the output of highly capital-intensive commodities and thus for the most capital-intensive goods potentially offsets the shift in demand.

We shall not pursue the properties of the closed-economy equilibrium further because our main interest in it is as a prelude to the open-economy model, to which we now turn.

III. Trading Equilibrium with Equalized Factor Prices

We now put the closed economy of Section II together with another such economy, which differs only in its factor supplies: the foreign capital-labor ratio k^* is smaller than ours. Their labor force is of size N^*.

Initially assume that after the opening of the economies to trade, factor price equalization obtains. The costs of production of all goods will accordingly be identical in both countries after trade opens. Any particular good z can be produced as well (cheaply) in one country as in the other. Given zero transport costs, prices of goods are the same in both countries. Patterns of production are not, however, totally indeterminate, since full employment of both factors requires us, on balance, to produce more capital-intensive goods.

We leave to the Appendix the demonstration that the equilibrium conditions for the world economy are precisely those of the closed economy except that now the capital-labor ratio is $\hat{k} \equiv \pi k + (1 - \pi)k^*$, that is, the respective country's capital-labor ratios weighted by the shares in the world labor force, $\pi \equiv N/(N + N^*)$. The opening of trade with the resulting establishment of a common wage-rental ratio implies, using the result $d\sigma/dk > 0$ of the previous part, that in the capital poor country the wage-rental ratio will rise, while at home it will decline. Prices will be equalized and in the home country the relative price of labor-intensive goods will fall by comparison with autarky, while abroad it will rise.

Production patterns will be indeterminate except in one important respect. Only in a probabilistic sense are we, the capital-rich country, exporters of a *particular* capital-intensive commodity. On *average,* however, we do export capital-intensive commodities, while foreigners export labor-intensive commodities. Equivalently, we are *net* exporters of capital services and *net* importers of labor services. It is this fact that is used now to discuss the effects of endowment changes on income distribution and welfare.

Consider first the benefits from the opening of trade. We noted above that the opening of trade will equalize wage-rental ratios, lowering the relative price of labor in the capital-rich country and increasing the relative price of labor in the capital-poor country. Now we want to show that the opening of trade raises "welfare" for both countries.

We define the welfare function corresponding to our Cobb-Douglas tastes:

$$U = \int_0^1 b(z)[\ln D(z)]dz, \tag{10}$$

where $D(z)$ denotes per capita consumption of commodity z. Substituting for $D(z)$, we have the indirect utility function:

$$U = \int_0^1 b(z)\left[\ln b(z)\frac{\sigma + k}{\sigma + k(z)}f(k(z))\right]dz = U(\sigma,k). \tag{10$'$}$$

The opening of trade will reduce our wage-rental ratio, since by assumption, we are the capital-rich country. To assess the effect on welfare, we take the derivative of (10′) with respect to σ to obtain

$$\frac{dU}{d\sigma} = \int_0^1 b(z)\left[\frac{1}{\sigma + k} - \frac{1}{\sigma + k(z)}\right]dz. \tag{11}$$

Using (6), with k replaced by $\hat{k}$, and noting that $\int_0^1 b(z)\,dz = 1$, we have

$$\frac{dU}{d\sigma} = \frac{\hat{k} - k}{(\sigma + k)(\sigma + \hat{k})}. \tag{11$'$}$$

For the home country, $\hat{k} < k$, and σ falls on the opening of trade; thus (11′) shows that the opening of trade makes us better off. For the foreigners, an expression similar to (11′), but with k replaced by k^*, applies. For them, $\hat{k} > k^*$, and σ rises on the opening of trade, which also makes them better off. The magnitude of the gain will depend on the extent of the differences in factor endowments.

We turn next to the income distribution and welfare effects of changes in factor endowments. Consider first the distribution of income. Our share in world income is equal to

$$\frac{wN + rK}{wN + rK + wN^* + rK^*} = \pi \frac{\sigma + k}{\sigma + \hat{k}}. \tag{12}$$

A change in factor endowments would therefore change the international distribution of income. First, a larger labor force would raise the capital-rich country's share in world income. Second, an increase in the foreign capital-labor ratio raises the world capital-labor ratio and thus the wage-rental ratio. Income, therefore, is redistributed away from capital and toward labor. That redistribution must benefit the capital-poor country.

What are the welfare effects of a change in our own capital-labor ratio? Again using the indirect utility function and taking the derivative with respect to k, noting that $\sigma = \sigma(\hat{k})$, we have

$$\frac{\partial U}{\partial k} = \frac{1}{\sigma + k} + \pi \frac{\hat{k} - k}{(\sigma + k)(\sigma + \hat{k})} \frac{d\sigma}{d\hat{k}}. \tag{13}$$

The first term measures the welfare gain of an increase in capital in the closed economy and is of course positive. The second term is negative for the capital-rich country and arises from the international redistribution of income toward the capital-poor country. An increase in the world capital-labor ratio worsens the terms of trade of capital and may do so sufficiently to more than offset the gains from increased productive resources. It can be shown, though, that this possibility cannot arise with elasticities of substitution in production of unity or more.[9]

In summary, the home country, being capital rich and therefore being a net exporter of capital services and importer of labor services, will benefit from a rise in the world wage-rental ratio due to demand shifts or foreign endowment changes. The foreign country, being labor rich, will benefit if the home country accumulates capital or experiences balanced growth or if demand shifts raise the equilibrium wage-rental ratio.

The interesting conclusion of the model with factor price equalization is that, the absence of a determinate production pattern notwithstanding, we can make judgments about welfare effects of

9. Proof of this proposition is omitted for brevity. However, examination of (13) suggests that if $d\sigma/dk$ is low, the second term is less likely to dominate. And $d\sigma/dk$ is inversely related to individual elasticities of substitution in production.

growth or demand changes. The welfare effects are determinate because there is no ambiguity about the fact that the home country is a *net* exporter of capital services and importer of labor services and there is a unique relation between endowments and relative factor and commodity prices.

IV. Trade with Complete Specialization

Now we consider the case where endowments differ sufficiently to give rise to complete specialization. Assuming, as before, that the foreign capital-labor ratio falls short of ours ($k^* < k$), the equilibrium is one where we produce in the capital-intensive range of goods, while foreigners specialize, without overlap, in the range of labor-intensive commodities. The borderline $\bar{z}$ is one of the variables determined as part of the equilibrium along with the two wage-rental ratios. In the equilibrium, we shall produce goods z, $0 \leq z \leq \bar{z}$, and foreigners produce those z in the interval $\bar{z} \leq z \leq 1$.

1. The Equilibrium

The equilibrium is determined by the conditions of trade balance equilibrium and factor market equilibrium in each country. In addition, the cost of production of the borderline commodity $\bar{z}$ is equalized between the two countries. Cost and price equalization at the margin requires that

$$P(\bar{z}) = wa(\bar{z}) + rc(\bar{z}) = P^*(\bar{z}) = w^*a^*(\bar{z}) + r^*c^*(\bar{z}) \tag{14}$$

or

$$\frac{r}{r^*} = \frac{a^*(\bar{z})}{a(\bar{z})} \frac{\sigma^* + k^*(\bar{z})}{\sigma + k(\bar{z})}. \tag{14'}$$

Since the wage-rental ratios differ, with $\sigma > \sigma^*$, the borderline commodity will be produced with a more capital-intensive technique at home compared to that used abroad.

Next we examine the condition of trade balance: the value of our imports is equal to the value of our exports. Our total income is $wN + rK$, and we import all those goods we do not produce, i.e., goods lying in the interval $\bar{z} \leq z \leq 1$. Similarly, foreigners import goods in the range $0 \leq z \leq \bar{z}$. The condition of trade balance is therefore[10]

10. After a little manipulation, (15) can be written in the form

$$\frac{w}{w^*} = \frac{N^*}{N} \frac{(1 + k^*/\sigma^*)}{(1 + k/\sigma)} B(\bar{z}),$$

which is similar to (10′) in our 1977 Ricardian paper.

$$(15) \quad (wN + rK) \int_{\bar{z}}^{1} b(z)dz = (w^*N^* + r^*K^*) \int_{0}^{\bar{z}} b(z)dz.$$

Using (14)′, we can rewrite the trade balance condition as

$$(16) \quad \mathrm{B}(\bar{z}) \equiv \frac{\int_0^{\bar{z}} b(z)dz}{\int_{\bar{z}}^{1} b(z)dz} = \frac{\pi}{1-\pi}\left(\frac{\sigma + k}{\sigma^* + k^*}\right)\left(\frac{a^*(\bar{z})}{a(\bar{z})}\right)\left(\frac{\sigma^* + k^*(\bar{z})}{\sigma + k(\bar{z})}\right).$$

The equilibrium conditions are completed by the requirement of factor market equilibrium in each country:[11]

$$(17) \qquad 0 = \int_0^{\bar{z}} \frac{b(z)}{\sigma + k(z)}\,[k(z) - k]dz,$$

and

$$(18) \qquad 0 = \int_{\bar{z}}^{1} \frac{b(z)}{\sigma^* + k^*(z)}\,[k^*(z) - k^*]\,dz.$$

It goes almost without saying that prices of goods are equal in the two countries in the assumed absence of transport costs, and that the costs of production of goods that we do not produce would be higher here than they are abroad.

The equilibrium conditions in (16) to (18) determine the equilibrium wage-rental ratios in each country, $\bar{\sigma}$ and $\bar{\sigma}^*$, and the equilibrium competitive margin $\bar{z}$. The relative price structure, incomes, commodity demands, and factor allocation are associated with the equilibrium relative factor prices. The relative prices and the competitive margin determine trade patterns.

That there is a unique equilibrium of equations (16), (17), and (18), with its implied pricing relationships that we have not written out explicitly, can be established by considering the normative problem that is satisfied by, and only by, the competitive solution. This is shown in Appendix 1.

11. To obtain (17), start from the labor market equilibrium condition,

$$N = \int_0^{\bar{z}} [a(z)b(z)(wN + rK) + (w^*N^* + r^*K^*)]/P(z)\,dz,$$

use (15), and obtain

$$\theta = \int_0^{\bar{z}} \frac{b(z)(\sigma + k)}{\sigma + k(z)}\,dz,$$

where $\theta \equiv \int_0^{\bar{z}} b(z)dz$. Similar manipulation, starting from the full employment condition for capital shows that

$$\theta k = \int_0^{\bar{z}} \frac{b(z)(\sigma + k)}{\sigma + k(z)}\,k(z)dz.$$

Equation (17) follows.

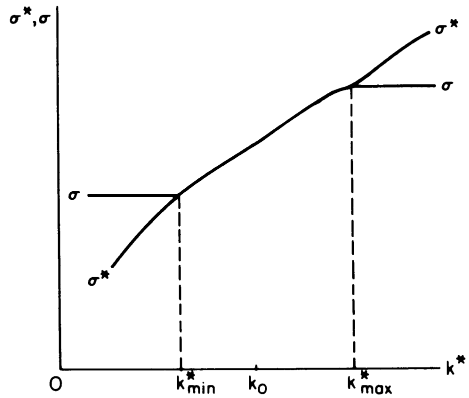

FIGURE II

2. *Comparative Statics*

We turn now to questions of comparative statics. Specifically we want to establish the effect of endowment changes on relative factor and commodity prices and on trade patterns. In our 1977 paper we showed how growth in one country changes the *equilibrium* dividing line $\bar{z}$ with the growing country increasing the range of goods produced but experiencing a decline in the equilibrium relative wage and terms of trade. In the present model there is a broader range of questions, since in addition to labor growth we can consider balanced growth or growth in the capital-labor ratio. As we shall see presently, there is ambiguity in the effect of growth on factor and commodity prices and in the effect of growth on trade patterns.

Before establishing these results, it is worth asking what the effects of growth are in the standard two-commodity Heckscher-Ohlin model. Figure II summarizes these results in plotting the equilibrium wage-rental ratios as a function of the foreign capital-labor ratio, given our capital stock k_0. The range of factor price equalization is $k^*_{\min}$ to $k^*_{\max}$. For that range of their capital-labor ratio, relative commodity and hence factor prices are such that we are nonspecialized.

A rise in their capital-labor ratio at constant relative prices of goods would, via the Rybczynski effect, create a world excess supply of the capital-intensive good and hence lower its equilibrium relative price and therefore raise the common equilibrium wage-rental ratio. As foreign k^* goes outside the range ($k^*_{\min} \leq k^* \leq k^*_{\max}$), we become completely specialized in one sector, and consequently our wage-rental ratio is determined by the relative marginal productivities in the producing sector. For the foreign country, though, the wage-rental

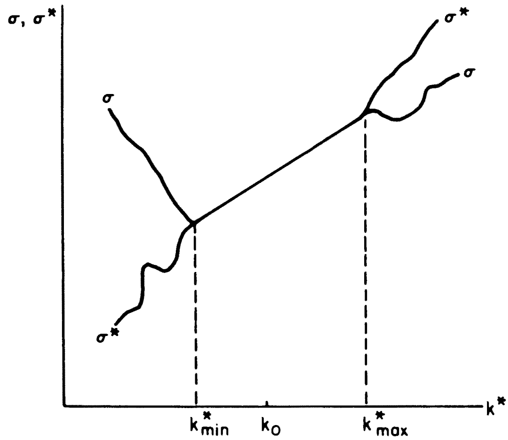

FIGURE III

ratio would continue to be an increasing function of the capital-labor ratio.[12]

One important difference in a multi-commodity model, and in particular in our continuum model, is the fact that we never achieve complete specialization in one single good and that accordingly the relation between relative factor prices and endowments will differ from that in Figure II.[13] The comparative static results of the continuum case are laid out in Table I and are illustrated in Figure III, which is to be compared with Figure II.

Consider first the case where our capital-labor ratio is larger than that abroad, $k^* < k_{\min}$, and where we have complete specialization and unequalized factor returns. A rise in the foreign capital-labor ratio, as Figure III shows, will reduce our wage-rental ratio but will have an ambiguous effect abroad. The interpretation of this result is that at constant relative factor and commodity prices, and hence a given $\bar{z}$, the increased foreign income leads to increased spending on all goods with production concentrated on only the range $\bar{z} \leq z \leq 1$. There will accordingly be excess demand for our goods and an excess supply of foreign goods. In the home country the adjustment takes the form of a decline in the wage-rental ratio and therefore an increase in the relative price of our goods that are capital-intensive. The decline in the wage-rental ratio also leads to the use of more labor-intensive techniques. This adjustment is appropriate since, as Table I shows,

12. For an exposition of these relations see Johnson [1957].
13. On the multi-commodity model see Jones [1974].

TABLE I
COMPARATIVE STATIC RESULTS ($k > k^*$)

		Change in k	k^*	π
Effect on	σ	+	−	+
	σ^*	?	?	+
	$\bar{z}$	?	−	+

we shall concentrate on a narrower range of goods and therefore raise the average capital intensity in the economy. The decline in the wage-rental ratio serves to offset the resulting excess supply of labor and excess demand for capital. Finally, the narrowing of the range of goods produced at home implies that there will be increased output levels so that, in conjunction with the increased relative price, there will be an adjustment to higher foreign real income and spending on our goods.

Consider next the foreign country. Real income has increased, and therefore demand has risen for foreign-produced goods. The increased supply of capital raises potential output, but there is now an excess demand for labor. Suppose for a moment that the elasticity of substitution in production were zero. Then the foreign adjustment could take the form of a shift in the competitive margin, thus increasing capital use, and of a rise in the wage-rental ratio so as to raise the relative price of labor-intensive goods and thus reduce labor demand. It can indeed be shown that for low elasticities of substitution these conclusions hold. If elasticities of substitution are larger than one, though, it becomes possible that an increase in the wage-rental ratio creates an excess demand for capital, as substitution of capital for labor becomes important. This is the source of the ambiguity in the effect of foreign capital growth on their equilibrium wage-rental ratio.

We look now at the case where foreigners are the capital-rich country and experience further growth in their capital-labor ratio. Note that now they produce in the range $0 < z < \bar{z}$, i.e., they produce the relatively capital-intensive goods. Now the foreign wage-rental ratio will rise, but there is uncertainty about our wage-rental ratio and about the change in trade patterns. Abroad, the increase in the wage-rental ratio induces more capital-using techniques, and it lowers the relative price of relatively capital-intensive goods. Both effects operate to absorb the increased capital, but they leave us uncertain about the demand for labor. With no substitutability labor demand would decline, and we would find the foreign country expanding its range of production to encompass more labor-intensive goods. With high substitutability, by contrast, there may be an excess demand for labor requiring a narrowing of the production range. The uncertainty about the effects on trade patterns also leaves us uncertain about the change in the equilibrium wage-rental ratio at home. Again, if substitution elasticities are small, abroad we expect an expansion in the foreign production range, and therefore at home an increase in the wage-rental ratio in order to accommodate ourselves to a more labor-intensive range of products.

Consider finally a balanced increase in our own endowment. This leads to a rise in both countries' wage-rental ratios and to an expansion in the range of goods we produce. The increased range of goods production leaves us with more labor-intensive production that we offset through a rise in the wage-rental ratio. Abroad where the production range shrinks to devote resources more intensively to meet the increased world demand, the more labor-intensive output mix requires a rise in the wage-rental ratio to retain factor market equilibrium.

In the case of balanced growth, analysis of terms-of-trade effects is straight-forward. Here the goods we produce fall in price relative to the borderline commodity, which is the most labor intensive. Abroad all commodities rise in price relative to the borderline commodity, which is the least labor intensive abroad. Accordingly, our terms of trade unambiguously deteriorate.

3. The Borderline Case: Complete Specialization with Factor Price Equalization

We now briefly investigate the borderline case where both countries are completely specialized but factor prices remain equalized. This is the dividing line between the case of overlapping production ranges when endowments are close together and the case of the previous section. The relevant equations are (16) simplified to take account of the common wage-rental ratio and thus the common choice of technique:

$$B(\bar{z}) = \frac{\pi}{1 - \pi} \frac{\sigma + k}{\sigma^* + k^*}, \tag{16$'$}$$

as well as the conditions of factor market equilibrium in (17) and (18) with $\sigma^* = \sigma$.

What can be said of the relationship between k and k^* such that we remain on the borderline? The natural conjecture is that an increase in our capital-labor ratio requires an accompanying increase in the foreign relative supply of capital so as to preserve factor price equalization. Perhaps surprisingly that result cannot be demonstrated in general.

It is true that if all production functions are Cobb-Douglas, the borderline has a positive slope. However, an ambiguity arises when elasticities of substitution at home are small and those abroad are large. In such a case an increase in the domestic capital stock may tend to result in a larger increase in σ^* than in σ, requiring a *fall* in k^* to maintain factor price equilibrium (see the discussion of Section IV above).

V. Concluding Remarks

This paper has extended our Ricardian continuum model to the case of two-factor Heckscher-Ohlin trade theory. The extension is rewarding because it introduces, in a still manageable fashion, the scope for relative price changes within a country rather than only changes in relative wages and the terms of trade of the Ricardian composite commodities that emerged in our earlier analysis.

The changes in relative factor and commodity prices induced by such disturbances as demand shifts or endowment changes cannot be determined without ambiguity outside the range of factor price equalization. This ambiguity arises because changes in relative factor supplies can be accommodated either by a shifting of the competitive margin toward more or less capital-intensive goods or by a change in relative factor prices and hence commodity prices. Even under our strong demand assumptions we cannot be certain about the distribution—or even direction—of adjustment between relative factor price changes and changes in trade patterns. Hence the elasticity of substitution in production plays an important role. With elasticities of substitution that are low (equal to or less than unity) the increase in a country's capital-labor ratio will raise the wage-rental ratio, as would be true in the two-commodity model, a closed economy, or under factor price equalization. With high elasticities of substitution, that regularity is no longer assured.

Under factor price equalization the model shows the well-known indeterminacy of geographic specialization. But we can make use of the fact that the capital-rich country is a net exporter of capital services to derive results concerning income distribution and welfare as they are affected by the opening of trade or changes in factor endowments.

The Heckscher-Ohlin variant of our continuum model remains manageable under our demand assumptions because, the multiplicity of goods notwithstanding, the equilibrium can be reduced to three variables—wage-rental ratios in each country and the equilibrium competitive margin. The Stolper-Samuelson theorem ensures that the single wage-rental ratio in each country determines the entire price structure for goods produced in that country, a result only slightly more complex than the Ricardian model.

APPENDIX 1: THE EXISTENCE OF EQUILIBRIUM

The normative problem that is solved by the competitive equilibrium is

$$(A.1)\quad \Phi(K,N,K^*,N^*) = \max_{\{K(z),N(z),K^*(z),N^*(z)\}} \int_0^1 b(z)\log\{F[K(z),N(z)] + F[K^*(z),N^*(z)]\},$$

subject to

$$\left[\int_0^1 K(z)dz,\ \int_0^1 N(z)dz,\ \int_0^1 K^*(z)dz,\ \int_0^1 N^*(z)dz\right] = [K,N,K^*,N^*] = [y_1,\ldots,y_4] = y.$$

Denoting partial derivatives by numerical subscripts,

$$\frac{\partial\Phi(y_1,y_2,y_3,y_4)}{\partial y_i} = \Phi_i(y), \quad (i = 1,\ldots,4).$$

we can express our equilibrium unknowns by

$$(A.2)\quad \begin{aligned} \omega &= w/w^* = \Phi_2(y)/\Phi_4(y) \\ \sigma &= w/r = \Phi_2(y)/\Phi_1(y) \\ \sigma^* &= w^*/r^* = \Phi_4(y)/\Phi_3(y). \end{aligned}$$

$\bar{z}$ is the root for z in

$$\omega = \left(a^*(z) + \frac{c^*(z)}{\sigma^*}\right)\Big/\left(a(z) + \frac{c(z)}{\sigma}\right).$$

The Kuhn-Tucker conditions, necessary and sufficient for the maximum and the equilibrium, can be expressed compactly with the help of the following notations:

$$[K(z),N(z),K^*(z),N^*(z)] \equiv [y_1(z),\ldots,y_4(z)] = y(z)$$

$$F(y_1(z),y_2(z)) + F(y_3(z),y_4(z)) \equiv F(y(z))$$

$$\frac{\partial F[y_1(z),\ldots,y_4(z)]}{\partial y_i(z)} \equiv F_i(y(z)), \qquad i = 1,\ldots,4.$$

Then (A.1) implies that

$$(A.3)\quad \overline{L}[y(z);\mu] = \frac{b(z)F_i[y(z)]}{F[y(z)]} - \mu_i \geq 0 \qquad (i = 1,\ldots,4)$$

$$L_{4+i}[y(z);\mu] = y_i - \int_0^1 y_i(z)dz \geq 0; \quad \mu \geq 0, \quad y(z) \geq 0$$

$$\sum_1^4 L_i[y(z);\mu]y_i - \sum_1^4 L_{4+j}\,[y(z);\mu]\mu_j = 0.$$

The μ's are dual variables, proportional to real factor prices of (K,N,K^*,N^*), respectively.

Appendix 2: The Equilibrium with Factor Price Equalization

Let $\alpha(z) \geq 0$ denote the fraction of the world output of good z that we produce; $(1 - \alpha(z))$ is produced by foreigners. Denote our proportion of the world labor force, $N/(N + N^*)$, by π. Equilibrium requires that trade balances. The condition for trade balance is that our income be equal to the value of our production:

$$\text{(A.4)} \quad wN + rK = \int_0^1 \alpha(z)P(z)[Q(z) + Q^*(z)]dz = \int_0^1 \alpha(z)b(z)[(wN + rK) + (wN^* + rK^*)]dz.$$

Rearranging terms, and recalling that $\int_0^1 b(z)dz = 1$, we may write the trade balance as

$$\text{(A.5)} \quad \int_0^1 b(z)(1 - \alpha(z))(\sigma + k)dz = \frac{N^*}{N}\int_0^1 \alpha(z)b(z)(\sigma + k^*)dz = \frac{1-\pi}{\pi}\int_0^1 \alpha(z)b(z)(\sigma + k^*)dz.$$

The remaining equilibrium conditions are those for the factor markets in the two countries:

$$\text{(A.6)} \quad \int_0^1 \frac{k(z)b(z)\alpha(z)[\sigma + \pi k + (1 - \pi)k^*]}{\sigma + k(z)}dz = \pi k$$

and

$$\text{(A.7)} \quad \int_0^1 \frac{k(z)b(z)(1 - \alpha(z))[\sigma + \pi k + (1 - \pi)k^*]}{\sigma + k(z)}dz = (1 - \pi)k^*.$$

The world capital stock per capita is

$$\text{(A.8)} \quad \hat{k} \equiv \frac{K + K^*}{N + N^*} = \pi k + (1 - \pi)k^*.$$

Adding (A.6) and (A.7), we can summarize the two capital market equilibrium conditions by

$$\text{(A.9)} \quad \int_0^1 \frac{k(z)b(z)[\sigma + \hat{k}]}{\sigma + k(z)}dz = \hat{k}.$$

Similarly, the two labor market equilibrium conditions reduce to

$$\text{(A.10)} \quad \int_0^1 \frac{b(z)[\sigma + \hat{k}]}{\sigma + k(z)}dz = 1.$$

Now these are precisely the same equilibrium conditions as equations (5) and (6). A unique equilibrium σ therefore exists. Because

$k > k > k^*$ and we showed earlier that $d\sigma/dk > 0$, we conclude that the wage-rental ratio after the opening of trade is between the two closed economy wage-rental ratios. Furthermore, an increase in the capital stock in either country will increase the equilibrium σ, as long as factor-price equalization continues to obtain.

APPENDIX 3: COMPARATIVE STATICS UNDER COMPLETE SPECIALIZATION

This appendix sketches the derivation of comparative static results for the case of complete specialization without factor price equalization. The equilibrium conditions, repeated here for convenience, are those for balanced trade and equilibrium in factor markets:

$$B(\bar{z}) = \frac{\pi}{1-\pi}\left(\frac{\sigma + k}{\sigma^* + k^*}\right)\left(\frac{a^*(\bar{z})}{a(\bar{z})}\right)\left(\frac{\sigma^* + k^*(\bar{z})}{\sigma + k(\bar{z})}\right) \tag{A.11}$$

$$0 = \int_0^{\bar{z}} \frac{b(z)}{\sigma + k(z)}\,[k(z) - k]dz \tag{A.12}$$

$$0 = \int_{\bar{z}}^1 \frac{b(z)}{\sigma^* + k^*(z)}\,[k^*(z) - k^*]\,dz. \tag{A.13}$$

Differentiating (A.11)–(A.13) yields the following system:

$$[A]\begin{bmatrix} d\bar{z} \\ d\sigma \\ d\sigma^* \end{bmatrix} = [B]\begin{bmatrix} dk \\ dk^* \\ d\pi \end{bmatrix}, \tag{A.14}$$

where $[A]$ and $[B]$ are each 3×3 matrices, with elements a_{ij}, b_{ij}, respectively.

The elements are given by

$$a_{11} = \frac{B'(\bar{z})}{B(\bar{z})} = \frac{b(\bar{z})}{[\int_{\bar{z}}^1 b(z)dz]^2} > 0; \quad a_{12} = \frac{k - k(\bar{z})}{(\sigma + k)(\sigma + k(\bar{z}))} > 0;$$

$$a_{13} = \frac{k^*(\bar{z}) - k^*}{(\sigma^* + k^*)(\sigma^* + k^*(\bar{z}))} > 0;$$

$$a_{21} = \frac{b(\bar{z})}{\sigma + k(\bar{z})}\,[k(\bar{z}) - k] < 0;$$

$$a_{22} = \int_0^{\bar{z}} \frac{b(z)}{[\sigma + k(z)]^2}\left[\frac{\partial k(z)}{\partial \sigma}(\sigma + k) - (k(z) - k)\right]dz > 0;$$

$$a_{23} = 0;$$

$$a_{31} = \frac{-b(\bar{z})}{\sigma^* + k^*(\bar{z})}\,[k^*(\bar{z}) - k^*] < 0; \quad a_{32} = 0;$$

$$a_{33} = \int_{\bar{z}}^{1} \frac{b(z)}{[\sigma^* + k^*(z)]^2}$$

$$= \left[\frac{\partial k^*(z)}{\partial \sigma^*}(\sigma^* + k^*) - (k^*(z) - k^*)\right] dz > 0;$$

$$b_{11} = \frac{1}{\sigma + k} > 0; \quad b_{12} = -\frac{1}{\sigma^* + k^*} < 0; \quad b_{13} = \frac{1}{\pi(1 - \pi)} > 0;$$

$$b_{21} = \int_0^{\bar{z}} \frac{b(z)}{\sigma + k(z)} dz > 0; \quad b_{22} = 0 = b_{23};$$

$$b_{31} = 0; \quad b_{32} = \int_{\bar{z}}^{1} \frac{b(z)}{\sigma^* + k^*(z)} dz > 0; \quad b_{33} = 0.$$

Now, substituting the elements a_{ij} and b_{ij} into (A.4) makes it possible to derive the results shown in Table I of the text.

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REFERENCES

Dornbusch, R., S. Fischer, and P. A. Samuelson, "Comparative Advantage, Trade and Payments in a Ricardian Model with a Continuum of Goods," *American Economic Review,* LXVII (Dec. 1977), 823–29.

Johnson, H. G., "Factor Endowments, International Trade and Factor Prices," *The Manchester School,* XXV, No. 3 (Sept. 1957), 270–83.

Jones, R. W., "The Small Country in a Many-Commodity World," *Australian Economic Papers,* XXXVII (Dec. 1974), 225–36.

Meade, J., "The Equalization of Factor Prices: The Two-Country, Two-Factor, Three-Product Case," *Metroeconomica,* II (Dec. 1950), 129–33.

Melvin, J. R., "Production and Trade with Two Factors and Three Goods," *American Economic Review,* LVIII (Dec. 1968), 1249–68.

Samuelson, P. A., "Two Country Ricardian Analysis for Discrete Goods and for a Continuum of Goods," *Journal of International Economics,* forthcoming.

Wan, H. Y., Jr., "Three Variations on a Theme of Samuelson," mimeo, Cornell University, 1977.

Wilson, C., "On the General Structure of Ricardian Models with a Continuum of Goods: Applications to Growth, Tariff Theory and Technical Change," *Econometrica,* forthcoming.