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REGULARITY AND INDEX THEORY FOR ECONOMIES WITH SMOOTH PRODUCTION TECHNOLOGIES

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Using smooth profit functions to characterize production possibilities, we extend the concepts of regularity and fixed point index to economies with very general technologies, involving both constant and decreasing returns. To prove the genericity of regular economies we rely on an approach taken by Mas-Colell that utilizes the topological concept of transversality. We also generalize the index theorem given by Kehoe. Our results shed new light on the question of when an economy has a unique equilibrium.

1. INTRODUCTION

DIFFERENTIAL TOPOLOGY HAS, over the past decade, provided economists with a unified framework for studying both the local and global properties of solutions to general equilibrium models. Debreu [1] initiated this line of research with his introduction of the concept of a regular economy, a model whose equilibria are locally unique and vary continuously with the underlying economic parameters. Dierker [2] pointed out the close connection of this concept with that of the fixed point index, a concept ideally suited to the study of existence and uniqueness of equilibria. Both of these studies focused attention on pure exchange economies that allow no production. More recently a number of different researchers, among them Fuchs [**5**, **6**], Mas-Colell [**13**, **14**, **15**], Smale [**19**], and Kehoe [**9**, **10**, **11**], have extended these concepts to models with production. In all of these studies the concepts of genericity and transversality have played an important role in ruling out degenerate situations.

The approach taken in this paper is in the spirit of Mas-Colell [**15**] and Kehoe [**10**], who emphasize the development of a formula for computing the index of an equilibrium and the connection between this formula and theorems dealing with the uniqueness of equilibrium. Both of these writers model the production side of an economy as an activity analysis technology. Unfortunately, the results obtained by Mas-Colell and Kehoe are not immediately applicable to economies with more general production technologies. It is true, of course, that any constant-returns technology can be approximated in a continuous manner by an activity analysis technology. Furthermore, any decreasing-returns technology can be represented as a constant-returns technology with certain nonmarketed factors of production. As we shall see, however, the differentiable nature of our ap-

[1] Many people have influenced the ideas presented in this paper. Most of my understanding of the concept of regularity has developed as a result of conversations and correspondence that I have had with Andreu Mas-Colell. Sidney Winter taught me the importance of the concept of duality in characterizing production technologies. Franklin Fisher, David Levine, and an anonymous referee provided helpful suggestions. Above all, I am grateful to Herbert Scarf, who introduced me to index theorems and encouraged me to apply them to production economies.

proach makes an activity analysis approximation to a smooth production technology unsuitable.

Using smooth profit functions to characterize production possibilities, we are able to extend the concepts of regularity and fixed point index to economies with very general technologies, involving both constant and decreasing returns. To prove the genericity of regular economies we rely on an approach taken by Mas-Colell [**15**], which utilizes in an elegant and insightful manner the topological concept of transversality. We also generalize the index theorem given by Kehoe [**10**]. Our results shed new light on the question of when an economy has a unique equilibrium. Providing a satisfactory answer to this question is crucial to the applicability of general equilibrium models in comparative statics exercises.

2. ECONOMIES WITH SMOOTH PROFIT FUNCTIONS

We initially deal with constant-returns production technologies; we later treat decreasing returns as a special case. The model is identical to that in Kehoe [**10**] except for its description of the production technology. The consumption side of the model is completely described by an aggregate excess demand function ξ.

ASSUMPTION 1 (Differentiability): $\xi : R^n_+ \setminus \{0\} \to R^n$ is C^1.

ASSUMPTION 2 (Homogeneity): ξ is homogeneous of degree zero; $\xi(t\pi) \equiv \xi(\pi)$ for all $t > 0$.

ASSUMPTION 3 (Walras's Law): ξ obeys Walras's law; $\pi'\xi(\pi) \equiv 0$.

Kehoe [**12**] generalizes Assumption 1 to one that allows the norm of excess demand to become unbounded as some prices approach zero. For the sake of simplicity, however, we assume here that ξ is defined and continuous over all nonnegative prices except the origin.

The production technology is specified by m C^2 profit functions $a_j : R^n_+ \setminus \{0\} \to R$, which can be regarded as a mapping from $R^n_+ \setminus \{0\}$ into R^m, $a(\pi) = (a_1(\pi), \dots, a_m(\pi))$. To motivate this approach, let us consider the problem of maximizing profits when production possibilities are specified directly by a production function. Suppose that a vector of feasible net-output combinations is one that satisfies the constraints

$$f(x) = 0,$$

$$x_i \geq 0 \qquad (i = 1, \dots, h),$$

$$x_i \leq 0 \qquad (i = h + 1, \dots, n).$$

Here $f : R^n \to R$ is a constant-returns production function, homogeneous of degree one and concave, that produces the first h commodities as outputs employing the final $n - h$ commodities as inputs. Suppose that we attempt to maximize $\pi'x$ subject to the feasibility constraints where π is a fixed vector of nonnegative prices. The problem that immediately arises is that, given the assumption of constant returns, profit is unbounded if there is some feasible vector x for which $\pi' x > 0$. There are several ways to get around this difficulty. For example, if $h = 1$, that is, if f produces a single output, we can impose the additional constraint $x_1 = 1$. Another, more general, solution to this problem is to impose the constraint $\|x\| = 1$.

It is well known that $a(\pi)$ is homogeneous of degree one, convex, and continuous as long as the feasible set is nonempty, even when the optimal net-output vector is not single-valued. When a is differentiable, Hotelling's lemma says that the profit maximizing net-output vector for any vector of prices is the gradient vector Da_π (see, for example, Diewert [3]). Given the constant-returns nature of the production technology, we can consider this gradient vector as an activity analysis vector: Any nonnegative scalar multiple of it is a feasible input-output combination.

Let us now consider again the general case $a : R^n_+ \setminus \{0\} \to R^m$. The Jacobian matrix Da_π maps R^n into R^m. Define the mapping $A : R^n_+ \setminus \{0\} \to R^{n \times m}$ by the rule $A(\pi) \equiv (Da_\pi)'$. $A(\pi)$ is a generalization of the concept of an activity analysis matrix. Indeed, in the situation where each a_j is the linear function $\sum_{i=1}^n a_{ij}\pi_i$, $A(\pi)$ is a matrix of constants. The set of feasible net-output vectors corresponding to $a(\pi)$ is the production cone $Y_a = \{x \in R^n \mid x = A(\pi)y$ for some $\pi \in R^n_+ \setminus \{0\},\ y \in R^m_+\}$. Observe that Y_a contains the origin, is convex, and is closed if a is C^1. We specify the production side of our model by imposing restrictions on the mapping a.

Assumption 4 (Differentiability): $a : R^n_+ \setminus \{0\} \to R^m$ is C^2.

Assumption 5 (Homogeneity): a is homogeneous of degree one; $a(t\pi) = ta(\pi)$ for any $t > 0$.

Assumption 6 (Convexity): Each function a_j is convex; $a_j(t\pi^1 + (1-t)\pi^2) \leq ta_j(\pi^1) + (1-t)a_j(\pi^2)$ for any $0 < t < 1$.

Assumption 7 (Free Disposal): $A(\pi)$ always includes n free disposal activities, one for each commodity. Letting these activities be the first $n \leq m$, we set $a_j(\pi) = -\pi_j,\ j = 1, \ldots, n$.

Assumption 8 (Boundedness): There exists some $\pi > 0$ such that $a(\pi) \leq 0$.

The convexity and homogeneity of a imply that $\pi^{1\prime}A(\pi^1) \geq \pi^{1\prime}A(\pi^2)$ for all $\pi^1, \pi^2 \in R^n_+ \setminus \{0\}$. It is easy to use this observation to demonstrate that Assumption 8 is equivalent to the assumption that there is no output possible without any inputs in the sense that $Y_a \cap R^n_+ = \{0\}$.

Notice that the activity analysis specification used by Mas-Colell and Kehoe is a special case of this type of technology. Assumptions 5–8 are quite natural; it is the differentiability part of Assumption 4 that is restrictive. It would be possible to impose conditions on production functions that would give rise to such smoothness in the corresponding profit functions and net-output functions. These conditions would be similar to those on a consumer's utility function that imply smoothness in the corresponding indirect utility function and individual demand function. Since we have chosen to specify the production side of the economy using profit functions rather than production functions, we shall not pursue this issue. A further restriction embodied in Assumption 4 is that we require the net-output functions to be continuous even on the boundary of R^n_+. We can avoid this problem if we impose the constraint $\|x\| = 1$ rather than $x_1 = 1$. This is not an important conceptual issue, however. We shall ignore it.

An economy is specified as a pair (ξ, a) that satisfies Assumptions 1–8. Let $\mathscr{D}$ be the space of excess demand functions endowed with the uniform C^1 topology. We say that two demand functions ξ^1 and ξ^2 are close if their values and those of their partial derivatives are uniformly close on the compact set S. Similarly let $\mathscr{A}$ be the space of the profit maps endowed with the uniform C^2 topology. The space of economies $\mathscr{E} = \mathscr{D} \times \mathscr{A}$ receives the product topology induced by the topologies on $\mathscr{D}$ and $\mathscr{A}$.

DEFINITION: An *equilibrium* of an economy (ξ, a) is a price vector $\hat{\pi}$ that satisfies the following conditions: (a) $a(\hat{\pi}) \leq 0$. (b) There exists $\hat{y} \geq 0$ such that $\xi(\hat{\pi}) = A(\hat{\pi})\hat{y}$. (c) $\hat{\pi}'e = 1$ where $e = (1, \ldots, 1)$.

The condition $a(\hat{\pi}) \leq 0$ implies that at $\hat{\pi}$ no excess profits can be made. The second condition, when combined with Walras's law and the homogeneity of a,

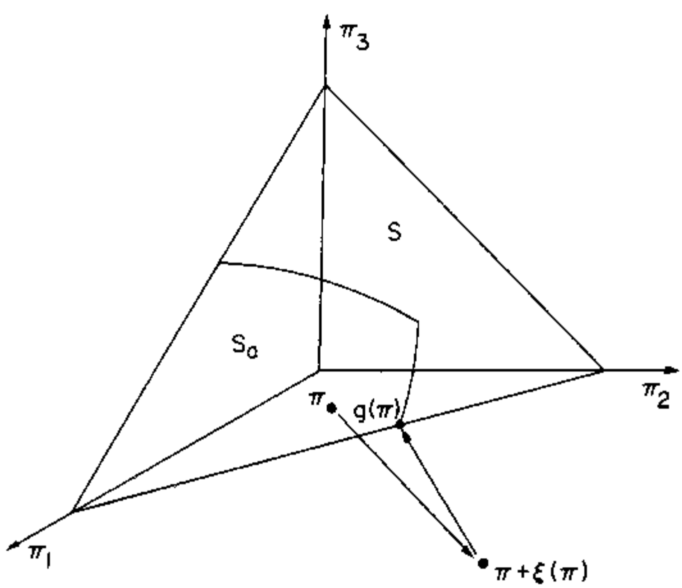

FIGURE 1.

implies that $\hat{\pi}'\xi(\hat{\pi}) = \hat{\pi}'A(\hat{\pi})\hat{y} = a(\hat{\pi})'\hat{y} = 0$. The homogeneity assumptions on ξ and a imply that, if $\hat{\pi}$ satisfies the first two equilibrium conditions, then $t\hat{\pi}$ also does for any $t > 0$. Therefore, when examining equilibria, we possess a degree of freedom that we use to impose the restriction $\hat{\pi}'e = 1$. The free disposal assumption allows us to restrict our attention even further to the unit simplex $S = \{\pi \in R^n \mid \pi \geq 0, \pi'e = 1\}$.

The proof of the existence of an equilibrium follows similar lines as those in Eaves [**4**], Todd [**20**], and Kehoe [**10**]. We define the set $S_a = \{\pi \in R^n \mid a(\pi) \leq 0, \pi'e = 1\}$. Our assumptions imply that S_a is a nonempty, closed, convex subset of S. Let p^{S_a} be the projection mapping that associates any point in R^n with the point in S_a that is closest in terms of Euclidean distance. We define the map $g: S \to S$ by the rule $g(\pi) = p^{S_a}(\pi + \xi(\pi))$.

THEOREM 1: *Fixed points $\hat{\pi} = g(\hat{\pi})$ of the map g and equilibria of (ξ, a) are equivalent.*

PROOF: At any point $\pi \in S$, $g = g(\pi)$ can be computed by solving the quadratic programming problem

$$\min 1/2(g - \pi - \xi(\pi))'(g - \pi - \xi(\pi))$$

$$\text{subject to} \quad a(g) \leq 0, \quad g'e = 1.$$

The Kuhn–Tucker theorem implies that the vector g solves this problem if and only if there exists $y \in R^m_+$ and $\lambda \in R$ such that $g - \pi - \xi(\pi) + A(g)y + \lambda e = 0$ and $a(g)'y = 0$. Consequently, $\hat{\pi} = g(\hat{\pi})$ is a fixed point if and only if $-\xi(\hat{\pi}) + A(\hat{\pi})\hat{y} + \hat{\lambda}e = 0$. Walras's law implies that this relationship holds if and only if $\hat{\lambda} = 0$. Therefore, $\xi(\hat{\pi}) = A(\hat{\pi})\hat{y}$ and $\hat{\pi} \in S_a$ is equivalent to $\hat{\pi} = g(\hat{\pi})$. *Q.E.D.*

Since S is nonempty, compact and convex and g is continuous, Brouwer's fixed point theorem implies the existence of an equilibrium of (ξ, a).

3. REGULAR PRODUCTION ECONOMIES

In the subsequent discussion we focus our attention on the partial derivatives of g at its fixed points. For proofs and more detailed discussion of many of the results presented here we refer to Kehoe [**10**]. To make matters simple, we define X as a smooth (that is, C^1) n dimensional manifold with boundary that is a compact, convex subset of R^n, chosen so that it contains S in its interior and does not contain the origin. It is easy to smoothly extend the domain of ξ to X. We are justified, therefore, in viewing X as the domain of g.

Unfortunately, g is not everywhere differentiable except in very special cases. All we need is that it is differentiable at its fixed points. To ensure this holds we need to impose two additional restrictions on (ξ, a). Consider the mapping $b: R^n_+ \setminus \{0\} \to R^k$, $0 \leq k \leq m$, made up of k of the profit functions

$(a_1(\pi), \ldots, a_m(\pi))$. Let $B : R^n_+ \setminus \{0\} \to R^{n \times k}$ be the corresponding matrix function whose columns are the gradients of the individual profit functions b_j, $j = 1, \ldots, k$.

ASSUMPTION 9: At any point $\pi \in S$ the profit functions b that satisfy $b(\pi) = 0$ are such that the columns of $B(\pi)$ are linearly independent.

ASSUMPTION 10: Suppose that b is the vector of profit functions that earn zero profit at some equilibrium $\hat{\pi}$. Then the vector $\hat{y} \in R^k_+$ is strictly positive in the equation $\xi(\hat{\pi}) = B(\hat{\pi})\hat{y}$.

We later justify these assumptions on the grounds that they hold for any open dense subset of economies in $\mathscr{E}$. Actually, Assumption 9 is stronger than needed. What we require for g to be differentiable at its fixed points is that the matrix of activities in use at every equilibrium has linearly independent columns. Assumption 9 implies that this condition holds but has the advantage of being easier to deal with in genericity arguments. Notice that this assumption does not rule out the isoquants of two production functions intersecting: At intersections both production plans are not necessarily profit maximizing. It does, however, rule out isoquants becoming tangent to each other. Assumption 10 rules out the possibility of an activity earning zero profit but not being used at equilibrium.

Suppose that $\hat{\pi}$ is an equilibrium of (ξ, a). Let C be the $n \times (k+1)$ matrix $[B(\hat{\pi}) \quad e]$ where $B(\hat{\pi})$ is the $n \times k$ matrix of activities in use at equilibrium and e is the $n \times 1$ vector whose every element is unity. Further let H be the $n \times n$ matrix formed by taking the Hessian matrices of the k profit functions b evaluated at π, multiplying them by the corresponding activity levels, then adding them together; that is, $H(\hat{\pi}) = \sum_{i=1}^{k} D^2(a_i)_{\hat{\pi}} \hat{y}_i$.

THEOREM 2: *If an economy $(\xi, a) \in \mathscr{E}$ satisfies Assumptions* 9 *and* 10, *then g is differentiable in some open neighborhood of every fixed point $\hat{\pi}$. Moreover, $Dg_{\hat{\pi}} = ((I+H)^{-1} - (I+H)^{-1}C(C'(I+H)^{-1}C)^{-1}C'(I+H)^{-1})(I + D\xi_{\hat{\pi}})$.*

PROOF: At the equilibrium $\hat{\pi}$ the conditions that determine $g(\hat{\pi})$, $\hat{y}$, and $\hat{\lambda}$ are

$$g - \pi - \xi(\pi) + B(g)y + \lambda e = 0,$$

$$b(g) = 0,$$

$$g'e = 1.$$

The continuity of g and a ensures that $a_j(g(\pi)) < 0$ for any π in some neighborhood of $\hat{\pi}$ and any a_j not included in the vector b. If we could demonstrate that y varies continuously with π in some neighborhood of $\hat{\pi}$, then Assumption 10 would imply that $b(g(\pi)) = 0$ over the neighborhood. The implicit function theorem implies that g, y, and λ all vary smoothly with π on some neighborhood of $\hat{\pi}$ if the matrix of partial derivatives obtained by differentiating the above system with respect to these variables is nonsingular. Indeed, we can find $Dg_{\hat{\pi}}$ by solving the system

$$\begin{bmatrix} I + H(\hat{\pi}) & B(\hat{\pi}) & e \\ B'(\hat{\pi}) & 0 & 0 \\ e' & 0 & 0 \end{bmatrix} \begin{bmatrix} Dg_{\hat{\pi}} \\ Dy_{\hat{\pi}} \\ D\lambda_{\hat{\pi}} \end{bmatrix} = \begin{bmatrix} I + D\xi_{\hat{\pi}} \\ 0 \\ 0 \end{bmatrix}.$$

The matrix that we must prove is invertible is

$$\begin{bmatrix} I + H & C \\ C' & 0 \end{bmatrix}.$$

Notice that $I + H$ is positive definite since I is positive definite and H is positive semi-definite. Consequently, $I + H$ is invertible. Notice too that C has full column rank since $B(\pi)$ has full column rank by Assumption 9 and $\pi' B(\pi) = 0$ while $\pi' e = 1$. Using these observations, we can compute

$$\begin{bmatrix} I + H & C \\ C' & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (I+H)^{-1} - (I+H)^{-1}C\left(C'(I+H)^{-1}C\right)^{-1}C'(I+H)^{-1} & (I+H)^{-1}C\left(C'(I+H)^{-1}C\right)^{-1} \\ \left(C'(I+H)^{-1}C\right)^{-1}(I+H)^{-1} & \left(C'(I+H)^{-1}C\right)^{-1} \end{bmatrix}.$$

Solving for $Dg_{\hat{\pi}}$, we obtain $Dg_{\hat{\pi}} = ((I + H)^{-1} - (I + H)^{-1}C(C'(I + H)^{-1} C)^{-1}C'(I + H)^{-1}(I + D\xi_{\hat{\pi}})$. *Q.E.D.*

Notice that in the activity analysis case every element of H is zero and, consequently, $Dg_{\hat{\pi}} = (I - C(C'C)^{-1}C')(I + D\xi_{\hat{\pi}})$.

Let us consider a subset of economies that satisfy Assumptions 9 and 10 and the further restriction that 0 is a regular value of $(g - I): X \to R^n$. Here, of course, I is the identity mapping.

DEFINITION: An economy $(\xi, a) \in \mathscr{E}$ that satisfies Assumptions 9 and 10 and is such that $DG_{\hat{\pi}} - I$ is nonsingular at every equilibrium is a *regular economy*. The set of regular economies is denoted $\mathscr{R}$.

Regular economies possess many desirable properties. For example, the inverse function theorem applied to $g - I$ at every equilibrium $\hat{\pi}$ implies that the equilibria of a regular economy are isolated. Since the set of equilibria lie in the compact set S and $g - I$ is continuous, this implies that a regular economy has a finite number of equilibria. Consider the equilibrium price correspondence $\Pi: \mathscr{E} \to S$ that associates any economy with the set of its equilibria. The topology on $\mathscr{E}$ is fine enough to imply that Π is an upper-semi-continuous correspondence. On $\mathscr{R}$, moreover, Π is continuous and the number of equilibria is locally constant.

4. THE INDEX THEOREM

The version of the Lefschetz fixed point theorem given by Saigal and Simon [**18**] provides us with a tool for counting the equilibria of (ξ, a). If (ξ, a) is a regular economy, then the local Lefschetz number of any fixed point of g can be calculated as $L_{\hat{\pi}}(g) = \operatorname{sgn}(\det[Dg_{\hat{\pi}} - I])$. Saigal and Simon prove that $\sum_{\pi = g(\pi)} L_{\pi}(g) = (-1)^n$. A regular economy therefore has an odd number of equilibria. Furthermore, a necessary and sufficient condition for a regular economy to have a unique equilibrium is that $L_{\hat{\pi}}(g) = (-1)^n$ at every equilibrium.

To make much economic sense of this result we need to develop alternative expressions for $\operatorname{sgn}(\det[Dg_{\hat{\pi}} - I])$. According to Theorem 2,

$$\begin{aligned} &Dg_{\hat{\pi}} - I \\ &\quad = \Big((I+H)^{-1} - (I+H)^{-1}C\big(C'(I+H)^{-1}C\big)^{-1}C'(I+H)^{-1}\Big) \\ &\quad\quad \times (I + D\xi_{\hat{\pi}}) - I. \end{aligned}$$

We can premultiply this matrix by $I + H$, which is positive definite, without changing the sign of its determinant. Since $C'(I+H)^{-1}C$ is also positive definite we can compute

$$L_{\hat{\pi}}(g) = \operatorname{sgn}\left(\det\begin{bmatrix} \big(I - C\big(C'(I+H)^{-1}C\big)^{-1}(I+H)^{-1}\big)(I + D\xi_{\hat{\pi}}) - I - H & 0 \\ C'(I+H)^{-1}(I + D\xi_{\hat{\pi}}) & C'(I+H)^{-1}C \end{bmatrix}\right).$$

Performing elementary row operations, adding the second row of this matrix premultiplied by $C(C'(I+H)^{-1}C)^{-1}$ to the first row, then subtracting the first row of the resulting matrix premultiplied by $C'(I+H)^{-1}$ from the second row, we obtain the expression

$$L_{\hat{\pi}}(g) = \operatorname{sgn}\left(\det\begin{bmatrix} D\xi_{\hat{\pi}} - H & C \\ C' & 0 \end{bmatrix}\right).$$

DEFINITION: If $\hat{\pi}$ is an equilibrium of a regular economy (ξ, a), then *index*$(\hat{\pi})$ is defined as

$$(-1)^n \operatorname{sgn}(\det[Dg_{\hat{\pi}} - I]) = (-1)^n \operatorname{sgn}\left(\det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} - H(\hat{\pi}) & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix}\right).$$

The following theorem is an immediate consequence of the Lefschetz fixed point theorem and the definition of index($\hat{\pi}$).

THEOREM 3: *Suppose that* (ξ, a) *is a regular economy. Then* $\sum_{\pi \in \Pi(\xi, a)} \operatorname{index}(\pi) = +1$.

Notice that the existence of an equilibrium for (ξ, a) follows directly from this theorem.

In applications of this theorem other expressions for index($\hat{\pi}$) are useful. They can be derived using simple matrix manipulations. Let the matrix $\bar{J}$ be formed by deleting from $D\xi_{\hat{\pi}}$ all rows and columns corresponding to commodities with zero prices at equilibrium $\hat{\pi}$ and then deleting one more row and corresponding column. Let $\bar{H}$ be formed similarly by deleting the same rows and columns from $H(\hat{\pi})$. Let $\bar{B}$ be formed by deleting the same rows from $B(\hat{\pi})$ as well as all columns corresponding to disposal activities. It is easy to verify that

$$\operatorname{index}(\hat{\pi}) = \operatorname{sgn}\left(\det\begin{bmatrix} -\bar{J} + \bar{H} & \bar{B} \\ -\bar{B}' & 0 \end{bmatrix}\right).$$

If only one commodity has positive price at $\hat{\pi}$, then index($\hat{\pi}$) = +1.

We can motivate this formula for the index by considering the equations that locally determine an equilibrium:

$$\xi(\pi) - B(\pi)y = 0,$$

$$b(\pi) = 0.$$

Suppose $\hat{\pi}$ is strictly positive. We can set $\pi_1 = \hat{\pi}_1$ by homogeneity and use Walras's law to ignore the first equation $\xi_1(\pi) - \sum(\partial b_j/\partial \pi_1)(\pi)y_j = 0$. Differentiating this system with respect to π and y then yields

$$\begin{bmatrix} \bar{J} - \bar{H} & -\bar{B} \\ \bar{B} & 0 \end{bmatrix}.$$

If this matrix is nonsingular, then the inverse function theorem implies that $\hat{\pi}$ is an isolated equilibrium, and the implicit function theorem implies that this equilibrium varies continuously with the parameters of (ξ, a). In other words, (ξ, a) is a regular economy. The index theorem implies that the sign of the determinant of this matrix is crucial for counting the number of equilibria.

Another formula for index($\hat{\pi}$) can be computed as follows: Choose an $n \times (n-k)$ matrix V whose columns span the null space of the columns of $B(\hat{\pi})$. Let E be the $n \times n$ matrix whose every element is unity. Then it is possible to demonstrate that

$$\operatorname{index}(\hat{\pi}) = \operatorname{sgn}\left(\det\left[V'(E + H(\hat{\pi}) - D\xi_{\hat{\pi}})V\right]\right).$$

For the derivations of these, and other, formulas for index($\hat{\pi}$) see Kehoe [**9**].

5. A TRANSVERSALITY APPROACH TO REGULARITY

Mas-Colell [15] has developed an interpretation of the definition of regular economy based on the concept of transversality. Using his interpretation, we can reduce the demonstration that regularity is a generic property of the space of economies to an argument dealing with transversal intersections of manifolds. Actually, Mas-Colell [13] proves the genericity of regular economies for economies with smooth production technologies. The approach used there, however, is more complicated than that of Mas-Colell [15] and cannot be used to develop an index theorem. References for the technical concepts employed here are the books on differential topology by Guillemin and Pollack [7] and Hirsch [8].

In the subsequent discussion we find it convenient to change our normalization rule for prices. By Assumptions 2 and 5 the vector $\hat{p} = (1/\|\hat{\pi}\|)\hat{\pi}$ satisfies the conditions $a(\hat{p}) \leq 0$ and $\xi(\hat{p}) = A(\hat{p})\hat{y}$ for some $\hat{y} \in R^m_+$ if and only if the vector $\hat{\pi} = (1/e'\hat{p})\hat{p}$ is an equilibrium of (ξ, a). Furthermore, $D\xi_{\hat{p}} = \|\hat{\pi}\| D\xi_{\hat{\pi}}$, $A(\hat{p}) = A(\hat{\pi})$, and $H(\hat{p}) = \|\hat{\pi}\| H(\hat{\pi})$. Let Σ be the unit sphere $\{p \in R^n \mid \|p\| = 1\}$. We redefine X so that it now contains $\Sigma \cap R^n_+$ in its interior but otherwise has the same properties as before. Let P be the intersection of Σ and the interior of X. P is an $n-1$ dimensional smooth manifold with boundary. To avoid problems with the boundary of R^n, we extend both ξ and A to C^1 maps on X. Define $a(p) = A'(p)p$ at points not in R^n_+.

The requirement that (ξ, a) is a regular economy involves three conditions. Let us first turn our attention to Assumption 9. Let M be the power set of the integers $\{1, \ldots, m\}$. For every $M_j \in M$, $j = 1, \ldots, 2^m$, define the set $Q(M_j) = \{x \in R^m \mid x_i = 0 \text{ if } i \in M_j\}$. In other words, $Q(M_j)$ is the coordinate subspace of R^m on which the coordinates whose indices are elements of M_j are zero. The following result is an immediate application of the definition of transversality.

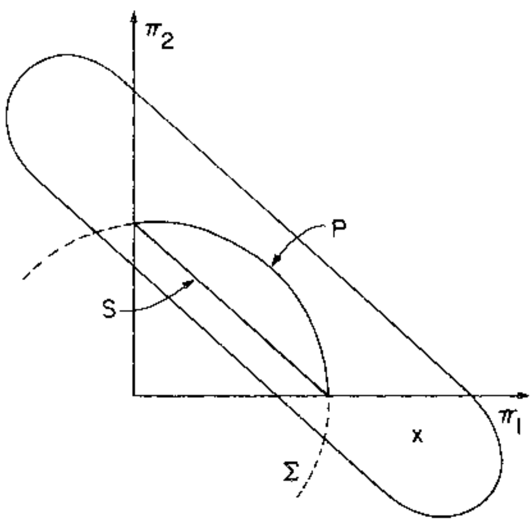

FIGURE 2.

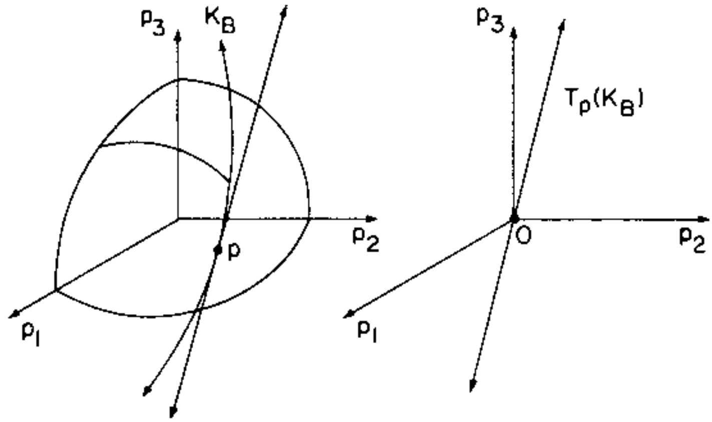

FIGURE 3.

LEMMA 1: *Consider the vector of profit functions $a \in \mathscr{A}$ extended to a C^1 map from* $\operatorname{int} X$ *into R^m. Suppose that, for every $M_j \in M$, a is transversal to $Q(M_j)$. Then a satisfies Assumption* 9.

Observe that, since Db_π is $k \times n$ and $Db_\pi \pi = 0$ if $b(\pi) = 0$, the rank of Db_π can be no greater than $n - 1$. If the number of elements in M_j is greater than $n - 1$, then the only way that it is possible for $a \pitchfork Q(M_j)$ is for $a^{-1}(Q(M_j))$ to be empty. In other words, there can never be more than $n - 1$ profit functions that earn zero profit at some price vector if the transversality condition is met. We are justified, therefore, in considering only cases where $k \leq n - 1$. Assumption 9 is simply the requirement that 0 is a regular value of $b : \operatorname{int} X \to R^k$, which implies that $b^{-1}(0)$ is a smooth manifold of dimension $n - k$. Let us define

$$K_B = \{ p \in \operatorname{int} X \mid p'B(p) = 0, p'p = 1 \}.$$

By Assumptions 5 and 9 K_B is a smooth $n - k - 1$ dimensional manifold. At any point $p \in K_B$,

$$T_p(K_B) = \{ v \in R^n \mid v'B(p) = 0, p'v = 0 \}.$$

Let B be an $n \times k$ matrix of net-output functions and let B^* be an $n \times k^*$, $k \leq k^* \leq n - 1$, matrix that includes B as a submatrix. Define the sets

$$T(P) = \{ (p,x) \in P \times R^n \mid p'x = 0 \},$$

$$\operatorname{graph}(\xi) = \{ (p,x) \in P \times R^n \mid x = \xi(p) \},$$

$$L(B^*, B)$$

$$= \{ (p,x) \in P \times R^n \mid p'B^*(p) = 0,\ x = B(p)y \text{ for some } y \in R^k \}.$$

$T(P)$ is a smooth $2(n - 1)$ dimensional manifold called the tangent bundle of P.

Walras's law implies that graph(ξ) is a smooth $n-1$ dimensional submanifold of $T(P)$. At any $(p,x) \in$ graph(ξ)

$$T_{(p,x)}(\text{graph}(\xi)) = \{(v,u) \in R^n \times R^n \mid p'v = 0, u = D\xi_p v\}.$$

If $a \in A$ satisfies Assumption 9, then $L(B^*, B)$ is a smooth $(n-1)-(k^*-k)$ dimensional manifold for any submatrices B and B^* of the matrix of net-output functions A. It too is a submanifold of $T(P)$. Notice that, if $\hat{p}$ is an equilibrium of (ξ, a), then $(\hat{p}, \xi(\hat{p})) \in \text{graph}(\xi) \cap L(B^*, B)$ where B^* is the matrix of activities that earn zero profit at $\hat{p}$ and B is the matrix of activities that are actually in use.

Any vector $(v,u) \in T_{(p,x)}(L(B^*,B))$ must clearly satisfy the restriction that $v \in T_p(K_{B^*})$. The restrictions on u are less obvious. Assumption 9 implies that the vector $y \in R^k$ that satisfies $x = B(p)y$ varies smoothly with p and x; in fact, we can compute $y = (B'(p)B(p))^{-1}B(p)x$. The easiest way to compute Dy_p and Dy_x is to differentiate the identity $x \equiv B(p)y(p,x)$ and solve. The results are $Dy_p = -(B'(p)B(p))^{-1}B(p)H(p)$ and $Dy_x = (B'(p)B(p))^{-1}B(p)$. For a vector (v,u) to be in $T_{(p,x)}(L(B^*,B))$ it must satisfy $(I - B(p)D_x y)u - (H(p) + B(p)D_p y)v = 0$. Consequently,

$$T_{(p,x)}(L(B^*,B)) = \{(v,u) \in R^n \times R^n \mid v'B^*(p) = 0, v'p = 0,$$
$$(I - B(p))(B'(p)B(p))^{-1}B'(p)(u - H(p)v) = 0\}.$$

LEMMA 2: *Suppose that* (ξ, a) *satisfies Assumption* 9 *and that, for all possible combinations* B *and* B^*, graph(ξ) $\pitchfork L(B^*,B)$. *Then* (ξ, a) *satisfies Assumption* 10.

PROOF: If (ξ, a) violates Assumption 10, then the matrix of activities that earn zero profit at some equilibrium p, $B^*(p)$, has more columns than the matrix in use, $B(p)$. For this particular pair of matrix functions B^* and B, $\dim L(B^*,B) = n-1-(k^*-k) < n-1$. If, however, $L(B^*,B)$ and graph(ξ) are transversal, then $\dim L(B^*,B) + \dim \text{graph}(\xi) \geq \dim T(P)$, which implies $\dim L(B^*,B) \geq n-1$. *Q.E.D.*

Consider the function $f^B : \text{int}\, X \to R^n$ defined by the rule

$$f^B(p) = (I - B(p)(B'(p)B(p))^{-1}B'(p))\xi(p).$$

The advantage of our normalization $\|p\| = 1$ is that f^B is a tangent vector field on K_B since $p'f^B(p) = 0$ for all $p \in K_B$. In other words, $f^B(p) \in T_p(K_B)$ for all $p \in K_B$. Notice that $f^B(\hat{p}) = 0$ if $\hat{p}$ is an equilibrium of (ξ, a). We can differentiate $p'f^B(p) \equiv 0$ at $\hat{p}$ to obtain $\hat{p}'Df^B_{\hat{p}} + (f^B(\hat{p}))' = \hat{p}'Df^B_{\hat{p}} = 0$. Similarly differentiating $B'(p)f^B(p) \equiv 0$, we establish that $Df^B_{\hat{p}}$ maps $T_{\hat{p}}(K_B)$ into itself. In fact, using techniques for differentiating matrices found in any econometrics textbook we can establish that

$$Df^B_{\hat{p}} = (I - B(\hat{p})(B'(\hat{p})B(\hat{p}))^{-1}B'(\hat{p}))(D\xi_{\hat{p}} - H(\hat{p})).$$

LEMMA 3: *Suppose that* (ξ, a) *satisfies Assumption* 9 *and that, for all possible combinations* B *and* B^*, $\text{graph}(\xi) \pitchfork L(B^*, B)$. *Suppose further that* $\hat{p}$ *is an equilibrium of* (ξ, a) *and that* $B(\hat{p})$ *is the associated matrix of activities in use. Then the matrix*

$$\begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{p}} - H(\hat{p}) & B(\hat{p}) \\ 0 & B'(\hat{p}) & 0 \end{bmatrix}$$

is nonsingular.

PROOF: Lemma 2 implies that we need only consider the case $B = B^*$. We begin by arguing that $Df_{\hat{p}}^B : T_{\hat{p}}(K_B) \to T_{\hat{p}}(K_B)$ is onto, that is, has rank $n - k - 1$, if $\text{graph}(\xi) L(B, B)$. Suppose that it does not. Then there exists $v \in T_{\hat{p}}(K_B)$, $v \neq 0$, such that $Df_{\hat{p}}^B v = (I - B(B'B)^{-1}B')(D\xi_{\hat{p}} - H(\hat{p}))v = 0$. This implies that the tangent spaces $T_{(\hat{p}, \xi(\hat{p}))}(\text{graph}(\xi))$ and $T_{(\hat{p}, \xi(\hat{p}))}(L(B, B))$ overlap; both include the nonzero vector $(v, D\xi_{\hat{p}} v)$. Since both tangent spaces have dimension $n - 1$, it is therefore impossible for their sum to be a space of dimension $2(n - 1)$. As a result, $\text{graph}(\xi)$ and $L(B, B)$ cannot be transversal, which is the desired contradiction.

The next step of the proof is to demonstrate that

$$\text{rank}\left[\left(I - B(B'B)^{-1}B' \right) \left(D\xi_{\hat{p}} - H(\hat{p}) \right) \right] = n - k - 1$$

implies our contention. This is a matter of simple, but tedious, algebraic arguments similar to those used by Kehoe **[10]** in the proofs of his Lemma 4 and Theorem 6. We omit it here. *Q.E.D.*

Suppose that we renormalize prices $\hat{\pi} = (1/e'\hat{p})\hat{p}$. Observe that

$$\det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{p}} - H(\hat{p}) & B(\hat{p}) \\ 0 & B'(\hat{p}) & 0 \end{bmatrix}$$

$$= \|\hat{\pi}\|^{n-k-1} \det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} - H(\hat{\pi}) & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix}.$$

Consequently, we can combine Lemmas 2 and 3 to obtain the following theorem.

THEOREM 4: *Suppose that* (ξ, a) *satisfies Assumption* 9 *and is such that* $\text{graph}(\xi) \pitchfork L(B^*, B)$ *for all possible combinations* B^* *and* B; *then* (ξ, a) *is a regular economy.*

The above arguments suggest that economies that are not regular are somehow pathological because they correspond to non-transversal intersections of certain manifolds. If we are able to perturb these manifolds in a sufficient number of directions, then the smallest perturbation results in the manifolds becoming transversal. We would therefore expect most economies to be regular. The following theorem formalizes this intuition (see Guillemin and Pollack [7, pp. 67–69]).

TRANSVERSALITY THEOREM: *Let M, V, and N be smooth manifolds where $\dim M = m$, $\dim N = n$, and $m \leq n$, and let Z be a smooth submanifold of N. Suppose that $F: M \times V \to N$ is a C^1 map transversal to Z. For any $v \in V$ let $f_v: M \to N$ be defined by the rule $f_v(x) = F(x, v)$. Then the set $U \subset V$ for which $f_v \pitchfork Z$, $v \in U$, has full Lebesgue measure.*

This theorem says that almost all maps are transversal to a given submanifold in the target space if the maps come from a rich enough family. To demonstrate that almost all economies are regular we must translate the measure-theoretic concept of genericity involved in the statement of this theorem into a topological concept. For the infinite dimensional space of economies a natural concept of a generic property is one that holds for an open dense set. We actually need the transversality theorem only to prove the density of regular economies; openness follows immediately from definitions. Nevertheless, it should be stressed that, if we are willing to restrict ourselves to some appropriately defined finite dimensional subset of E, we could prove that the set of regular economies has full Lebesgue measure. In fact, it is by doing just this that we are able to prove the density of regular economies.

We begin by arguing that the set of profit maps that satisfy Assumption 9 is an open dense subset of $\mathscr{A}$. Lemma 1 implies that $a \in \mathscr{A}$ satisfies Assumption 9 if it is transversal to a finite number of submanifolds of R^m. Standard arguments imply that the set of maps that satisfy this property is open in $\mathscr{A}$. We need to prove that this set is dense in $\mathscr{A}$. Consider the subset of profit maps that satisfy a stronger version of Assumption 8, that there exists some $\pi > 0$ such that $a(\pi) < 0$. Since it is easily verified that this subset is open and dense in $\mathscr{A}$, if we prove that the set of profit maps that satisfy Assumption 9 is dense in it, we have demonstrated our contention.

Choose an $(m-n) \times (m-n)$ matrix G that is nonsingular. For any $v \in R^{(m-n)}$ define the function $\delta: (\operatorname{int} X) \times R^{(m-n)} \to R^m$ by the rule

$$\delta(p, v) = p'e\begin{bmatrix} 0 \\ G \end{bmatrix} v.$$

Here, of course, 0 is $n \times (m-n)$. (Recall that the n disposal activities are fixed.) For any fixed $v \in R^{(n-m)}$ and $a \in \mathscr{A}$ define $a_v(p) = a(p) + \delta(p, v)$. It is easy to check that $a_v \in \mathscr{A}$ for all v in some open set $V \subset R^{(m-n)}$ that contains the origin. Define $F: (\operatorname{int} X) \times V \to R^m$ by the rule $F(p, v) = a(p) + \delta(p, v)$. Notice that F is transversal to any submanifold of R^m since

$$DF_{(p,v)} = \begin{bmatrix} -I & 0 \\ * & p'eG \end{bmatrix}$$

is nonsingular and hence includes all R^m in its image. (Here the elements denoted $*$, the partial derivatives of the final $m-n$ components of F with respect to p, are of no consequence.) The transversality theorem implies that, for a set of full Lebesgue measure in V, a_v is transversal to any fixed submanifold in R^m. Recall that the intersection of a finite number of sets with full Lebesgue measure also has full Lebesgue measure and that a set with full Lebesgue measure is dense. Consequently, since the set V contains the origin, it is easy to argue that the set of profit maps that satisfy Assumption 9 is dense in $\mathscr{A}$.

We can now choose a fixed profit map $a \in \mathscr{A}$ that satisfies Assumption 9 and let perturbations of the excess demand function ξ do all the work. Any such vector of profit functions is associated with a finite number of matrices of net-output functions B^* and B. We prove that for any fixed combination B^* and B the set of excess demand functions for which graph(ξ) $\pitchfork L(B^*,B)$ is open and dense in $\mathscr{A}$. Since the intersection of a finite number of open dense sets is open dense, this implies the genericity of regular economies.

Our argument follows the same lines as above. Again openness follows immediately from the transversality condition. We redefine $\delta : (\operatorname{int} X) \times R^n \to R^n$ by the rule

$$\delta(p,v) = \frac{p'v}{p'e} - v.$$

For any fixed $v \in R^n$, δ satisfies Assumptions 1–3. Consequently, $\xi_v(p) = \xi(p) + \delta(p,v)$ is an element of $\mathscr{D}$ if ξ is. In addition, δ satisfies the condition that $D\delta_v(p,v)$ has rank $n-1$ for any $(p,v) \in (\operatorname{int} X) \times R^n$ since it can easily be verified that the only vectors $x \in R^n$ for which $x'D\delta_v(p,v) = 0$ are scalar multiples of p. Define $F: P \times R^n \to T(P)$ by the rule $F(p,v) = (p, \xi(p) + \delta(p, v))$. For a fixed $v \in R^n$ the image of F is, obviously, graph(ξ_v). We want to prove that $F \pitchfork L(B^*,B)$. Differentiating F, we obtain

$$DF_{(p,v)} = \begin{bmatrix} I & 0 \\ D\xi_p + D\delta_p & D\delta_v \end{bmatrix}.$$

Notice that the image of the linear map $DF_{(p,v)} : T_p(P) \times R^n \to T_p(P) \times T_p(P)$ has dimension $2(n-1)$ since $T_p(P)$ has dimension $n-1$ and $D\delta_v$ has rank $n-1$. Consequently, this image must fill up the tangent space to $T(P)$ at any point $(p,x) \in T_p(P) \times T_p(P)$, since it too has dimension $2(n-1)$. The transversality theorem therefore implies that graph(ξ_v) is transversal to any submanifold of $T(P)$ for all v in some set of full Lebesgue measure in R^n. It is now easy to prove the density of the transversality condition in $\mathscr{D}$.

Our arguments have yielded the following result:

THEOREM 5: *The set of regular economies $\mathscr{R}$ is open and dense in $\mathscr{E}$.*

Unfortunately, our demonstration of this result relies on perturbations of the excess demand function that may not be appropriate in economies where production plays an important role. In such economies there are likely to be primary commodities, which are inelastically supplied as inputs to the production process, and intermediate commodities, which are only produced in order to produce other commodities. Obviously, if we perturb the excess demand function of such an economy, we may destroy the primary and intermediate characteristics of these goods. Kehoe [12] resolves this problem for economies with activity analysis production technologies. His analysis can easily be extended to the more general model that we are using here.

6. DECREASING RETURNS

In this section we sketch a procedure for extending our results to economies with decreasing-returns production technologies. The production side of such an economy is again specified by a vector of C^2 profit functions. The first n profit functions correspond to disposal activities. In addition there are k profit functions, $r_j : R^n_+ \setminus \{0\} \to R$, $j = 1, \ldots, k$, that are dual to production functions that exhibit strictly decreasing returns. These profit functions differ from the ones we defined for constant-returns production because there is now no restriction such as $\|x\| = 1$ imposed. Although Hotelling's lemma still holds, it is no longer true that every nonnegative scalar multiple of the gradient of the profit function is a feasible net-output combination. Notice that we are implicitly assuming in Assumption 4 that these profit functions have been smoothly bounded away from infinity. Notice too that, since shut-down is allowed, these profit functions are always nonnegative.

The positive profits that are earned by activities that are used in equilibrium must somehow be distributed to consumers. The easiest way to specify this process is to assign each consumer a fixed share of each profit function, which may be thought of as a firm. Consumer excess demand then depends on profits made on the production side of the model.

There are two ways to treat this type of model. The first is to specify an excess demand function that has both a consumption and a production component. We define $z : R^n_+ \setminus \{0\} \to R^n$ by the rule $z(\pi) = \xi(\pi, r(\pi)) - B(\pi)e$. Here B is the $n \times k$ matrix function whose columns are the gradients of the non-disposal profit functions. The excess demand function z naturally satisfies Assumptions 1–3. We focus our attention on the pure exchange economy specified by excess demand z and free disposal. An equilibrium of such an economy is a price vector $\hat{\pi}$ such that $z(\hat{\pi}) \leq 0$. Differentiating z, we obtain $Dz_\pi = D\xi_\pi + D\xi_r B'(\pi) - H(\pi)$ where H is defined as before and all activity levels are unity. H can be thought of as the Jacobian matrix of the aggregate supply function $B(\pi)e$. If there are no zero prices at equilibrium $\hat{\pi}$, then we can calculate the index as

$$\text{index}(\hat{\pi}) = (-1)^n \operatorname{sgn}\left(\det\begin{bmatrix} 0 & e' \\ e & D\xi_{\hat{\pi}} + D\xi_{\hat{r}} B'(\hat{\pi}) - H(\hat{\pi}) \end{bmatrix}\right).$$

The second way to treat a model with decreasing-returns production involves defining an additional good to represent the nonmarketed factors of production peculiar to each firm (see, for example, McKenzie [**16**]). There are then $n + k$ goods in the model. The first n profit functions, which allow free disposal of the first n goods, stay fixed. An additional k profit functions are defined as $a_{n+j}(\pi_1, \ldots, \pi_{n+k}) = r_j(\pi_1, \ldots, \pi_n) - \pi_{n+j}$, $j = 1, \ldots, k$. The vector of profit functions $a : R_+^{n+k}\backslash\{0\} \to R^{n+k}$ would satisfy Assumptions 4–8 except that it lacks components that correspond to free disposal activities for the final k goods. The assumption of strictly decreasing returns implies, however, that good $n + j$ has a zero price only if the corresponding firm j does not operate. In this case, $a_{n+j}(\pi_1, \ldots, \pi_{n+k}) = -\pi_{n+j}$ if $r_j(\pi_1, \ldots, \pi_n) = 0$.

The restrictiveness of the assumption that r_j is C^2 is clear in this context. We are, in fact, assuming that a firm's optimal net-output function is C^1 even at prices where it just becomes optimal to shut down. The problem is similar to the one that is encountered in smoothing a consumer's excess demand function when there are corner solutions to the utility maximization problem. We have chosen to ignore both of these minor technical problems. It would be possible to deal with them, however, by demonstrating that, even if such nondifferentiabilities existed in r and ξ, they would not occur at equilibria of almost all economies (see Mas-Colell [**14**, pp. 87–89]).

Each consumer receives an initial endowment of good $n + j$ equal to his share of the profits of firm j. Since the sum of profit shares for each firm is unity, the aggregate initial endowment of each good $n + 1, \ldots, n + k$ is also unity. Each of these goods is considered a primary good in the sense that $\xi_{n+j}(\pi) \equiv -1$, $j = 1, \ldots, k$. To facilitate the comparison of the calculations of the index for this formulation to the previous one, let us abuse notation a bit by partitioning $\pi \in R^{n+k}$ into

$$\begin{bmatrix} \pi \\ r \end{bmatrix}$$

where $\pi \in R^n$ and $r \in R^k$. We similarly partition $\xi(\pi, r)$ into

$$\begin{bmatrix} \xi(\pi, r) \\ -e \end{bmatrix}$$

and $B(\pi)$ into

$$\begin{bmatrix} B(\pi) \\ -I \end{bmatrix}.$$

Again assuming that there are no zero prices at equilibrium (π, r), we can write

$$\operatorname{index}(\hat{\pi}, \hat{r}) = (-1)^{n+k} \operatorname{sgn}\left[\det\begin{bmatrix} 0 & e' & e' & 0 \\ e & D\xi_{\hat{\pi}} - H(\hat{\pi}) & D\xi_{\hat{r}} & B(\hat{\pi}) \\ e & 0 & 0 & -I \\ 0 & B'(\hat{\pi}) & -I & 0 \end{bmatrix}\right]$$

where e has the appropriate dimension. Performing elementary row and column operations that do not change this expression, we obtain

$$\operatorname{index}(\hat{\pi},\hat{r}) = (-1)^n \operatorname{sgn}\left(\det\begin{bmatrix} 0 & e' + e'B'(\hat{\pi}) \\ e + B(\hat{\pi})e & D\xi_{\hat{\pi}} + D\xi_{\hat{r}}B'(\hat{\pi}) - H(\hat{\pi}) \end{bmatrix}\right).$$

This is, of course, the same as the expression that we derived previously. The only difference is that now we have rescaled so that $e'\hat{\pi} + e'\hat{r} = e'\hat{\pi} + e'B'(\hat{\pi})\hat{\pi} = 1$ rather than $e'\hat{\pi} = 1$.

Equilibria that have zero prices have the same index in either of the above formulations. If some price is zero, then Assumption 10 implies that the corresponding disposal activity is used in equilibrium. By expanding the determinantal expression for the index along both the column and the row that contain this activity it can easily be shown that the index is the same as that for the economy where the free good does not appear.

7. UNIQUENESS OF EQUILIBRIUM

The most significant consequence of our results is that they permit us to establish conditions sufficient for uniqueness of equilibria. If the parameters of a regular economy (ξ, a) are such that $\operatorname{index}(\hat{\pi}) = +1$ at every equilibrium $\hat{\pi} \in \Pi(\xi, a)$, then the set of equilibrium prices consists of a single point. A partial converse to this observation is also valid: If an economy (ξ, a) has a unique equilibrium $\hat{\pi}$, then it cannot be the case that $\operatorname{index}(\hat{\pi}) = -1$. The condition that $\operatorname{index}(\hat{\pi}) = +1$ at every equilibrium is, therefore, necessary as well as sufficient for uniqueness in almost all cases.

Kehoe [11] has studied the implications of the index theorem for uniqueness of equilibrium in economies with activity analysis production technologies. His two principal results are that an economy has a unique equilibrium if its excess demand function satisfies the weak axiom of revealed preference or if there are $n-1$ activities in use at every equilibrium. An economic interpretation of the first condition is that the aggregate excess demand function behaves like that of a single consumer. An interpretation of the second condition is that the economy is an input-output system; that is, there is no joint production, and consumers hold initial endowments of a single good, which cannot be produced.

Both of these conditions imply uniqueness of equilibrium in the more general model that we are considering here. The weak axiom of revealed preference, for example, implies that at any equilibrium $\hat{\pi}$ the Jacobian matrix $D\xi_{\hat{\pi}}$ is negative semi-definite (not necessarily symmetric) on the null space of the $n \times k$ matrix of activities in use, $B(\hat{\pi})$. Recall that

$$\operatorname{index}(\hat{\pi}) = \operatorname{sgn}\left(\det\left[V'(E + H(\hat{\pi}) - D\xi_{\hat{\pi}})V\right]\right)$$

where V is any $n \times (n-k)$ matrix whose columns span the null space of the columns of $B(\pi)$. The matrices E and $H(\hat{\pi})$ are both positive semi-definite. If ξ satisfies the weak axiom of revealed preference, then $-V'D\xi_{\hat{\pi}}V$ and $V'(E +$

$H(\hat{\pi}) - D\xi_{\hat{\pi}})V$ are also positive semi-definite. Consequently, if (ξ, a) is a regular economy, then index $(\hat{\pi}) = +1$ at every equilibrium.

An alternative expression for the index is

$$\operatorname{index}(\hat{\pi}) = \operatorname{sgn}\left(\det\begin{bmatrix} -\bar{J} + \bar{H} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix}\right).$$

If there are always $n-1$ activities in use at equilibrium, then $\bar{B}$ is an $(n-1) \times (n-1)$ square matrix. This implies that $\operatorname{index}(\hat{\pi}) = \operatorname{sgn}(\det[\bar{B}'\bar{B}]) = +1$. Therefore, an economy with $n-1$ activities in use at every equilibrium has a unique equilibrium.

Unfortunately, it seems that these two sets of conditions, which are extremely restrictive, are the only conditions that imply uniqueness of equilibrium in economies with production. For example, if the excess demand function $\xi \in \mathscr{D}$ does not satisfy the weak axiom, then it is possible to choose a vector of profit functions $a \in \mathscr{A}$ such that the economy (ξ, a) has multiple equilibria. On the production side of the economy the situation is even worse: If there is more than one $\pi \in S$ for which the profit map $a \in \mathscr{A}$ satisfies $a(\pi) \leq 0$, then it is easy to find an excess demand function $\xi \in \mathscr{D}$ such that the economy (ξ, a) has multiple equilibria. Obviously, general conditions that imply uniqueness of equilibrium would have to combine restrictions on the demand side with restrictions on the production side. An example of such a combination is the input-output condition mentioned earlier.

One direction to look in would seem to be combinations of restrictions on ξ and a that imply $-D\xi_{\hat{\pi}} + H(\hat{\pi})$ is positive semi-definite on the null space of $B(\hat{\pi})$ at every equilibrium $\hat{\pi}$. We already know that $H(\hat{\pi})$ satisfies this condition. What we want is that $H(\hat{\pi})$ somehow dominates $-D\xi_{\hat{\pi}}$ so that their sum is positive semi-definite. $H(\hat{\pi})$ measures the responsiveness of production techniques to price changes. $D\xi_{\hat{\pi}}$ measures the responsiveness of demand to price changes.

To get some idea of the relationship between these two, consider an economy with three goods and one profit function, $a : R^3_+ \setminus \{0\} \to R$, besides the three free disposal profit functions. If all prices are strictly positive at equilibrium, then $H(\hat{\pi})$ is just the 3×3 matrix of second partial derivatives of a weighted by a scalar activity level. Conditions on $D^2 a_{\hat{\pi}}$ are conditions on the curvature of the boundary of the intersection of the dual cone and the simplex, S_a, at $\hat{\pi}$. Given an excess demand function ξ and a price vector $\pi \in \operatorname{int} S$ that satisfies $\xi(\pi) \neq 0$, we can easily choose a so that $\hat{\pi}$ is an equilibrium with $\operatorname{index}(\hat{\pi}) = +1$. We set $Da_{\hat{\pi}} = \xi(\hat{\pi})$ and twist the boundary of S_a until $D^2 a_{\hat{\pi}}$ is large enough so that $\operatorname{index}(\hat{\pi}) = +1$. The condition that $\operatorname{index}(\hat{\pi}) = +1$ is equivalent in this case to π being a sink of the vector field $g - I$. In Figure 4 $\hat{\pi}$ goes from being a saddle point in (a), to a degenerate equilibrium in (b) to a sink in (c) and (d). In (a) $\operatorname{index}(\hat{\pi}) = -1$, in (c) and (d) $\operatorname{index}(\hat{\pi}) = +1$, while $\hat{\pi}$ is a critical point of $g - I$ in (b). Notice that (d) is the limiting case where the curvature of S_a is infinite at

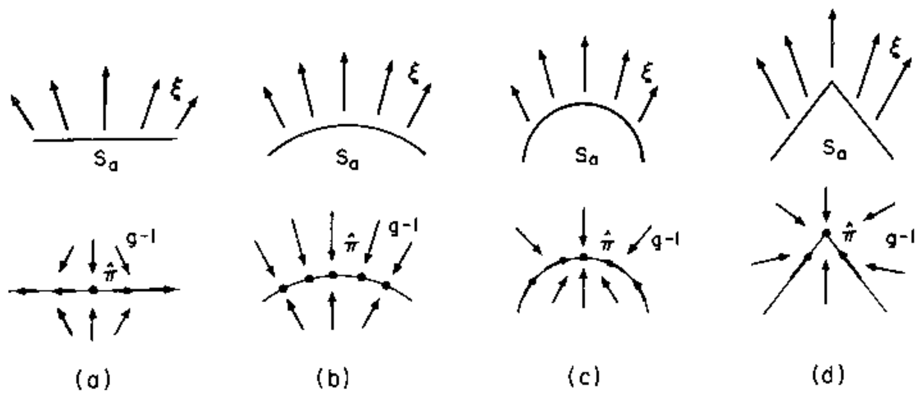

FIGURE 4.

$\hat{\pi}$. Here there are actually $2 = n - 1$ activities in use at equilibrium; we already know that index$(\hat{\pi}) = +1$ in such cases.

To make statements about uniqueness of equilibrium we would want to impose global restrictions on $D^2a_{\hat{\pi}}$ on S. To make $D^2a_{\hat{\pi}}$ a matrix of constants, for example, we would choose $a \in \mathscr{A}$ so that S_a is a sphere. To increase the curvature of the boundary of S_a at every point we would have to shrink the size of this sphere. The limiting case, of course, is where S_a shrinks to a single point. This situation is one where the production set has a hyperplane as its upper boundary and there is a complete reversibility of production in every direction. For a given ξ it may be necessary to choose $a \in \mathscr{A}$ so that S_a consists of a single point in order to ensure uniqueness of equilibrium. Consider $\xi(\pi) \equiv 0$ for example. Nevertheless, it is possible to prove that for almost all $\xi \in \mathscr{D}$ we can find an $a \in A$ so that (ξ, a) has a unique equilibrium and that S_a has an interior.

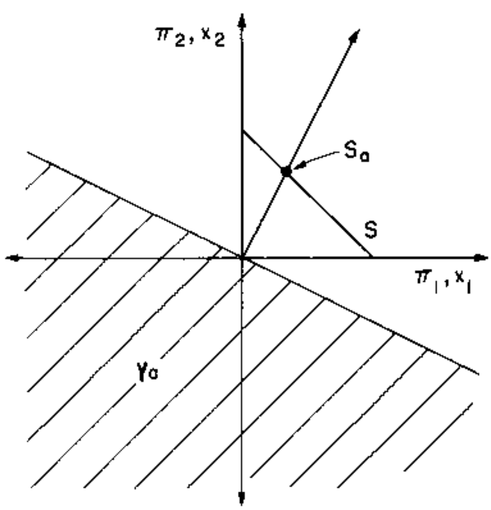

FIGURE 5.

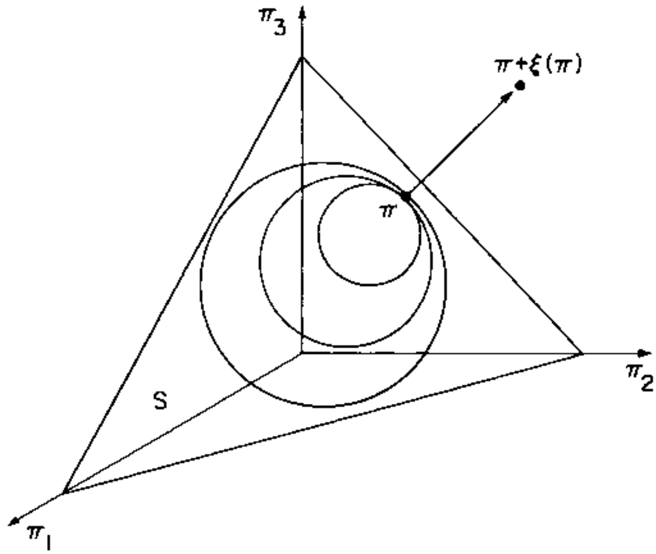

FIGURE 6.

We just keep shrinking S_a. Such a result does not seem to be of much interest, however, because it involves almost complete reversibility of production, a very unpalatable assumption.

An insight into the nature of an activity analysis approximation to a smooth production technology is provided by considering a situation where

$$\det\begin{bmatrix} -\bar{J}+\bar{H} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix} > 0$$

at some equilibrium $\hat{\pi}$, but

$$\det\begin{bmatrix} -\bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix} < 0.$$

Such a situation is depicted in Figure 7. There are three equilibria in (a) that

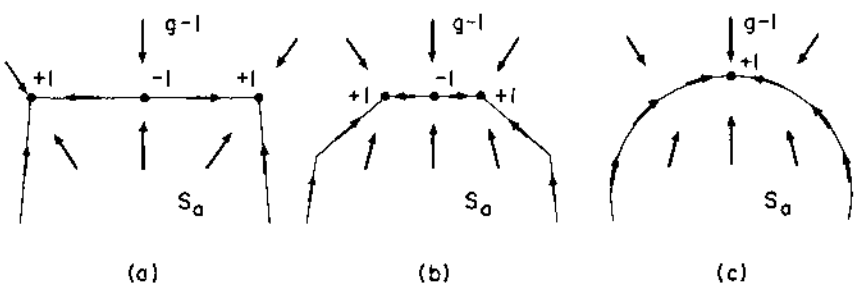

FIGURE 7.

coalesce into a single equilibrium as the approximation to the smooth production set becomes more accurate. Let $a : R^n_+ \setminus \{0\} \to R$ be a profit function and let $\pi^1, \ldots, \pi^k$ be a finite number of points in S. Then $\tilde{a}(\pi) = \max[\pi' A(\pi^1), \ldots, \pi' A(\pi^k)]$ provides an approximation to a that can be made arbitrarily accurate in the uniform C^0 metric by proper choice of the set $\pi^1, \ldots, \pi^k$. Unfortunately, such an approximation is not accurate enough in the C^2 topology we have defined on A. The curvature of the dual cone is an essential local characteristic of any equilibrium. The unique equilibrium in (c) can be mistaken for isolated, multiple equilibria if we use an activity analysis approximation to the underlying production technology.

Let us turn our attention to economies with decreasing-returns production technologies. Here index$(\hat{\pi}) = \text{sgn}(\det[-\bar{J}_1 - \bar{J}_2\bar{B}' + \bar{H}])$ where $\bar{J}_1$ is $D\xi_{\hat{\pi}}$ with one row and column deleted and $\bar{J}_2$ is $D\xi_{\hat{r}}$ with the same row deleted. It is possible to use Walras's law and homogeneity to prove that, if $D\xi_{\hat{\pi}} + D\xi_{\hat{r}}B'(\hat{\pi})$ has all its off diagonal elements positive and on diagonal elements negative, then index$(\pi) = +1$. This result was originally discovered by Rader [17], who did not use an index theorem. The interpretation that he gave it was that gross substitutability in demand implies uniqueness of equilibrium regardless of what the production technology looks like. The problem with this interpretation is that the term $D\xi_{\hat{r}}B'(\hat{\pi})$ involves a complex interaction between income effects from the demand side of the model and activities from the production side. It seems impossible to develop easily checked conditions to guarantee that $D\xi_{\hat{\pi}} + D\xi_{\hat{r}}B'(\hat{\pi})$ has the required sign pattern.

Our results shed light on the applicability of the comparative statics method to general equilibrium models. The assumptions that the equilibria of an economy are locally unique and vary continuously with its parameters are not at all restrictive. Almost all economies satisfy these conditions. Unfortunately, uniqueness of equilibrium is a more elusive property. The conditions that imply uniqueness seem to be too restrictive for most applications. There is obviously a need for more discussion on the relationship between comparative statics and uniqueness of equilibrium.

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