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Gambling for Redemption and Self-Fulfilling Debt Crises*

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ABSTRACT

We develop a model for analyzing the sovereign debt crises of 2010–2013 in the Eurozone. The government sets its expenditure-debt policy optimally. The need to sell large quantities of bonds every period leaves the government vulnerable to self-fulfilling crises in which investors, anticipating a crisis, are unwilling to buy the bonds, thereby provoking the crisis. In this situation, the optimal policy of the government is to reduce its debt to a level where crises are not possible. If, however, the economy is in a recession where there is a positive probability of recovery in fiscal revenues, the government may optimally choose to "gamble for redemption," running deficits and increasing its debt, thereby increasing its vulnerability to crises.

Keywords: Debt crisis; Roll-over crisis; Recession; Eurozone
JEL classification:

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1. Introduction

This paper develops a model of the sort of sovereign debt crisis that we have witnessed during 2010–2013 in European countries such as Greece, Ireland, Italy, Portugal, and Spain. During this period, the yields on bonds issued by the governments of these countries had substantial spreads over the yields on German bonds, as seen in the data presented in figure 1. Arellano, Conesa, and Kehoe (2012) provide a timeline and detailed discussion of these events. In our model, we interpret the spread as capturing the probability that investors assign to a government defaulting on its debt. We model the government as setting its expenditure-debt policy optimally, given a probability of a recovery in fiscal revenues. In doing so, the government can optimally choose to "gamble for redemption," increasing its debt and exposing the economy to increasing vulnerability to speculative attacks. We provide a theory of sovereign debt crises in which both borrowers and lenders behave optimally, but where countries borrow so much, and lenders are willing to lend them that much, as to make a default unavoidable. Our theory contrasts alternative explanations based on misperceptions or other forms of irrationality.

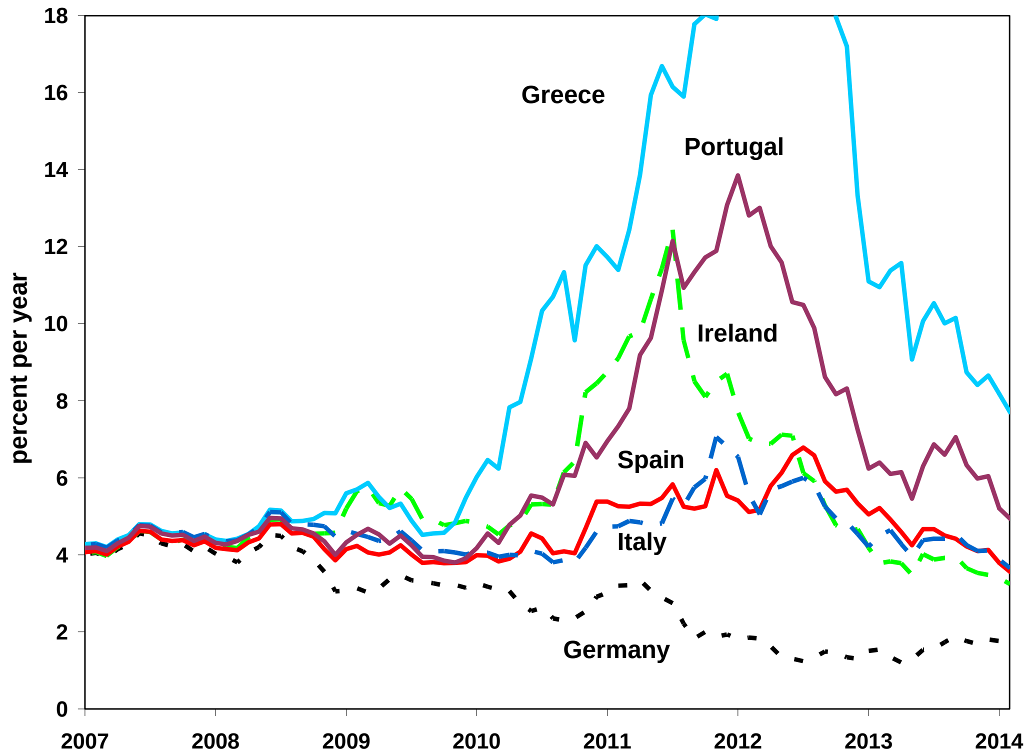

Figure 1: Harmonized long-term interest rates on government bonds in selected Eurozone countries

Note: Interest rate on Greek bonds was 29.24 percent per year in February 2012.

Our paper provides a theory for analyzing government behavior in the face of fiscal pressures and a tool for testing the implications of alternative policy responses. In our analysis, as in Cole and Kehoe (1996, 2000), we characterize in a simple Markov structure the time consistent policy of a strategic government that is faced with nonstrategic bond holders.

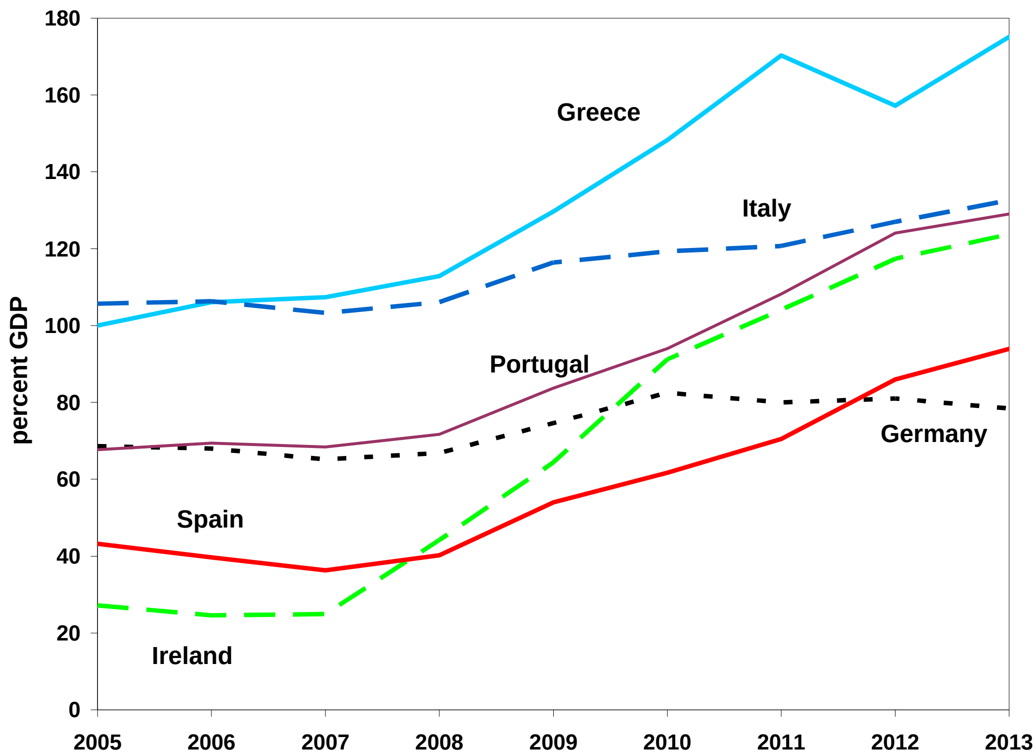

Figure 2: Government debt in selected European countries

The recent worldwide recession — which continues in some countries — and the policies intended to overcome it have generated very large government budget deficits and increases in government debt over the entire developed world. Figure 2 plots debt to GDP ratios for the most troubled European economies — the so-called PIIGS: Portugal, Ireland, Italy, Greece, Spain — as well as Germany. (We ignore the crisis in Cyprus, which has been a banking crisis, rather than a debt crisis of the sort we analyze here.) These data are at odds with the theory developed by Cole and Kehoe (1996, 2000), who argue that the optimal policy of a government that faces a positive probability of a self-fulfilling debt crisis is to pay down its debt.

In our model, the crucial element that drives a government to risk suffering a self-fulfilling debt crisis is the drop in government revenues that occurs as the result of a recession. Figure 3 shows that the worldwide recession that started in 2008 was still ongoing in Greece, Italy, Portugal, and Spain in 2013. Notice that the drops in real government revenues (deflated by the GDP deflator) — presented in figure 4 — are also very large.

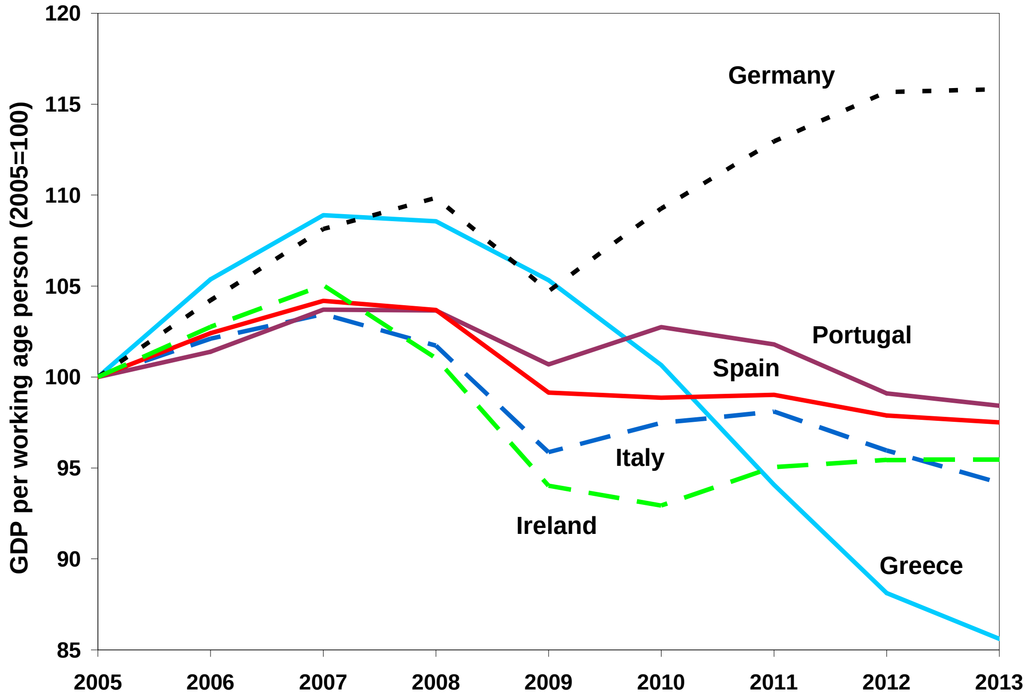

Figure 3: Real GDP in selected Eurozone countries

It is worth pointing out that according to the Kehoe-Prescott (2002) definition of a great recession — that real GDP per working-age person fall 20 percent below a balanced growth path of 2 percent per year — Greece was already in a great depression by 2011 and deeply in it by 2013 and by 2013 Ireland and Italy were on the edge of being in great depressions. Two or three more years without recovery could also push Portugal and Spain into great depressions.

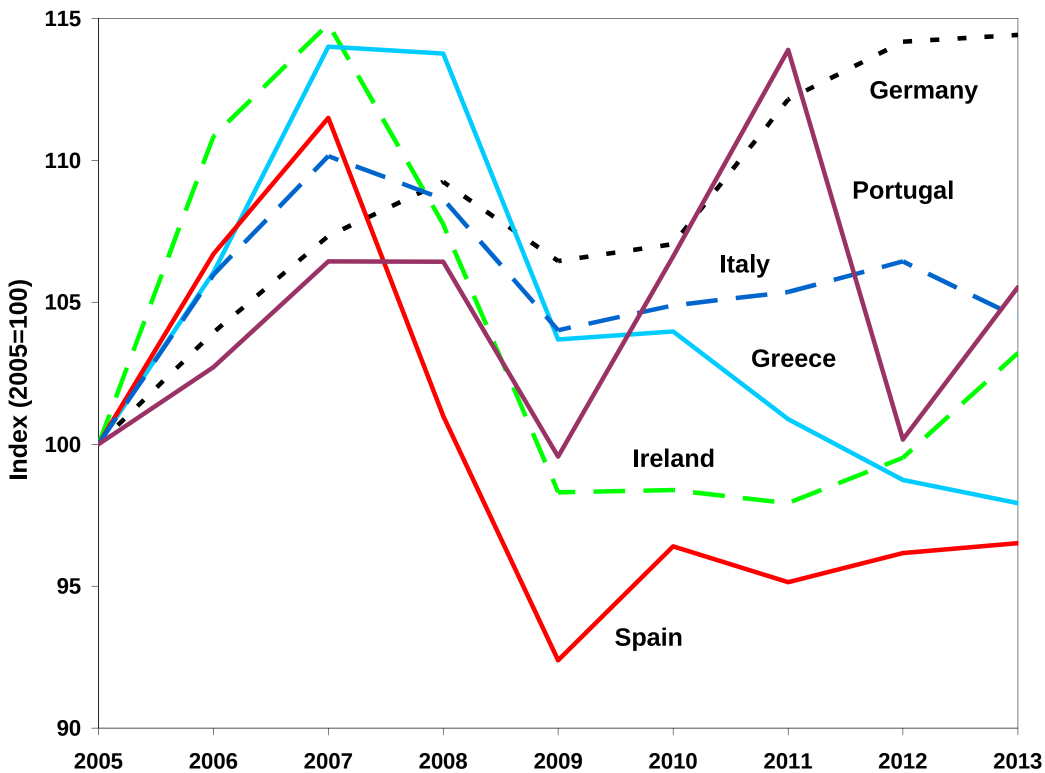

Figure 4: Real government tax revenues in selected Eurozone countries

Our analysis shows that — under certain conditions, which correspond to parameter values and the fundamentals of the economy — it is optimal for the government to "gamble for redemption." By this we mean that, hoping for a recovery of government revenues, the government does not undertake painful adjustments to reduce spending, and debt continues to increase. Indeed, the government strategy follows a martingale gambling strategy that sends the economy into the crisis zone if the recovery does not happen soon enough. Under other conditions, however, the government gradually reduces the level of debt to exit the crisis zone and avert the possibility of a liquidity crisis, as in Cole and Kehoe (1996, 2000). The data in figures 1 and 5 indicate that the governments of the PIIGS continued to borrow even as the spreads on their debt indicated the danger of self-fulfilling debt crises.

In our model, not running down debt, or running it up until default is unavoidable, can be part of the optimal strategy under some circumstances. In contrast, Reinhart and Rogoff (2009) argue that some countries fail to adjust and are vulnerable to a potential crisis because both the governments and their lenders are fooling themselves into thinking that "this time is different." As such, a country's vulnerability to a crisis would be the result of self-delusion and lack of rationality. In contrast to this view, we provide a model in which such apparently irrational behavior can be an optimal response to fundamentals by both borrowing governments and lenders that perfectly understand the risks of a crisis.

This paper is most closely related to those of Cole and Kehoe (1996, 2000). Aguiar and Amador (2015) provide a general overview of the literature on debt crises. Similar frameworks have been used to analyze currency crises following Calvo (1988). Cole and Kehoe provide a dynamic stochastic general equilibrium model of a country subject to the possibility of a self-fulfilling debt crisis in every period. The substantial difference between their framework and ours is that, in their framework, debt crises are liquidity crises that are due solely to the inability to roll over debt. As such, a decisive action by a third party providing a loan or a bailout would be enough to avert the problem. Indeed, Cole and Kehoe (1996, 2000) intend their model as an analysis of the financial crisis in Mexico in 1994–1995, and, on that occasion, the decisive intervention of the Clinton administration on 31 January 1995 was enough to end the crisis. European Union rescue packages for Greece, Ireland, and Portugal have not had the same healing properties. In fact, quite the opposite seems to be the case, with the spreads on bonds, relative to Germany's, continuing to rise despite the announcements and initial implementations of the rescue packages. This result suggests more fundamental solvency problems than those present in a standard liquidity crisis of the type studied in Cole and Kehoe, as discussed by Chamley and Pinto (2011) for the Greek case. Our model accommodates this issue.

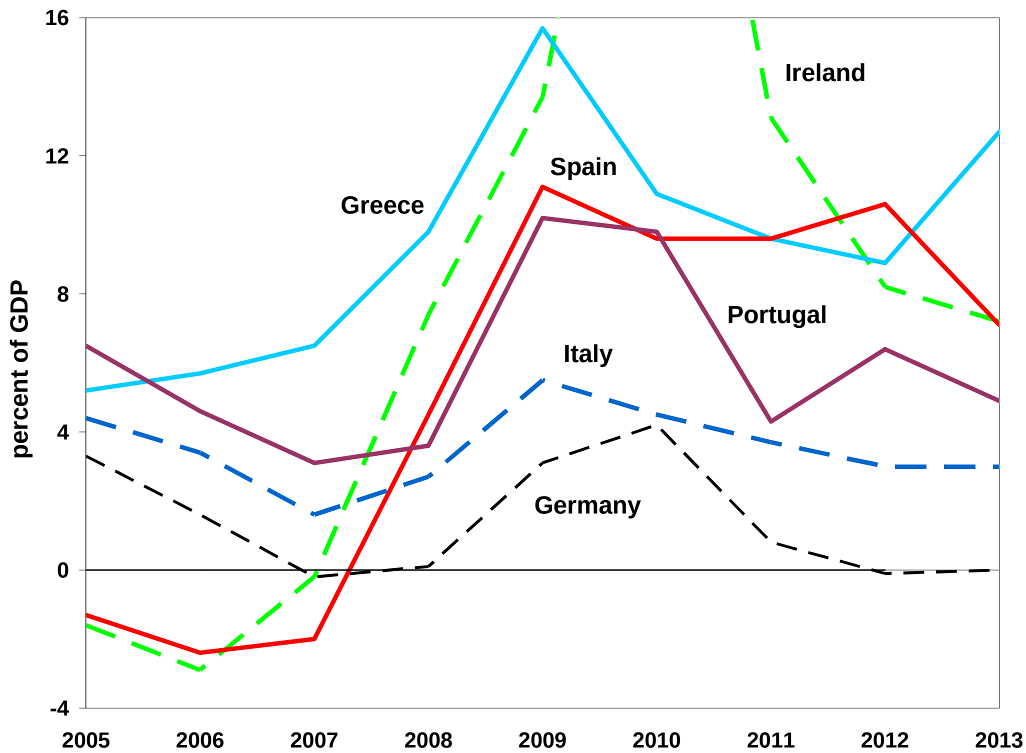

Figure 5: Net government borrowing in selected Eurozone countries
Note: Net government borrowing in Ireland in 2010 was 30.6 percent of GDP.

The model we propose extends the Cole-Kehoe analysis to incorporate a severe recession of uncertain recovery. By doing that, we are incorporating a motive for consumption smoothing as in Arellano (2008) and Aguiar and Gopinath (2006), who focus on default incentives on international borrowing over the cycle, but do not allow for self-fulfilling debt crises. For studies of individual rather than sovereign default of unsecured debt, see Chaterjee *et al*. (2007) and Livshits *et al*. (2007). In our model, there is a trade-off between the benefits of consumption smoothing and the increased vulnerability associated with increasing the level of debt. Our quantitative results relate this trade-off — and whether we should observe gambling for redemption as an optimal response — to fundamentals of the economy such as the severity of a recession, the likelihood of a recovery, the existing stock of debt, and so on.

Our model establishes conditions under which a debt crisis can occur, and how that possibility shapes the government's optimal behavior, but is silent about why at a particular point in time a crisis might or might not occur. Indeed, once the government is in the crisis zone — and we show under what conditions a government will find it optimal to enter it — a potential crisis is triggered by a nonfundamental random variable: a sunspot. A debt crisis is one of the two potential equilibria in the crisis zone, and it happens as a sudden event. In contrast, Lorenzoni and Werning (2013) propose an alternative theory with multiple equilibria where debt crises are slow moving events.

We first analyze theoretically a model with one-period bonds for special cases. Later we will solve computationally the whole model for long-lived bonds. Gambling for redemption is optimal because the recession is very severe, as seen in the data for the PIIGS, and we assume there is a nontrivial probability of a recovery. We also discuss the implications of the existence of bonds at different maturities. The basic implication is that, as maturity increases, the incentives to gamble increase, even for intermediate levels of debt.

2. General model

The model has a structure similar to that of Cole and Kehoe (1996, 2000). The major innovation is that output is stochastic, introducing a motive for consumption smoothing in the tradition of Aiyagari (1994) and Huggett (1993) — that is, there is uninsurable idiosyncratic risk — making it sometimes optimal for the government to gamble for redemption. With this added complication, we have chosen to simplify the model by eliminating the representative household's consumption-investment choice. Allowing for private investment would be conceptually straightforward, but tedious.

The state of the economy in every period $s=(B,a,z_{-1},\zeta)$ is the level of government debt B, whether or not the private sector is in normal conditions $a=1$ or in a recession $a=0$, whether default has occurred in the past $z_{-1}=0$ or not $z_{-1}=1$, and the value of the sunspot variable ζ. The country's GDP is

$$y(a,z)=A^{1-a}Z^{1-z}\bar{y} \qquad (1)$$

where $1>A,Z>0$. Before period 0, $a=1$, $z=1$. In period 0, a unexpectedly becomes $a_0=0$ and GDP drops from $y=\bar{y}$ to $y=A\bar{y}<\bar{y}$. In every period t, $t=1,2,\ldots$, a_t becomes 1 with probability p, $1>p>0$. Once $a=1$, it stays equal to 1 forever. The drop in productivity by the factor Z is the country's default penalty. Once $z=0$, it stays equal to 0 forever. Here the default penalty occurs in the same period as the crisis. Figure 6 illustrates a possible evolution of the country's GDP over time. In terms of the crises in the Eurozone, we can think of $t = 0$ as 2008.

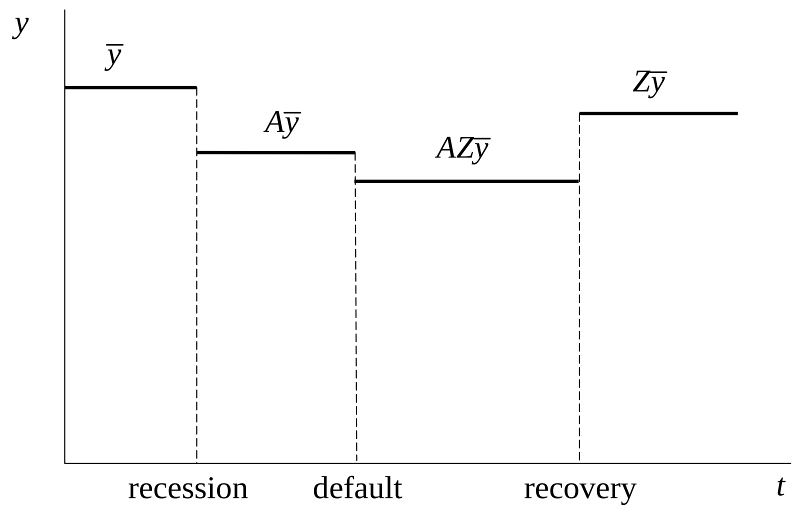

Figure 6: A possible time path for GDP

Government tax revenue is $\theta y(a,z)$ where we assume, as do Cole and Kehoe (1996, 2000) to keep things simple, that the tax rate θ is fixed. Given that there is no consumption-investment choice, the consumption of the representative household is

$$c(a,z) = (1-\theta)y(a,z). \tag{2}$$

The government offers B' in new bonds for sale and chooses whether or not to repay the debt becoming due, B. The government's budget constraint is

$$g + zB = \theta y(a,z) + q(B',s)B', \tag{3}$$

where $q(B',s)$ is the price that international bankers that pay for B', g is government expenditure, and $z \in \{0,1\}$ is a binary variable that denotes the government decision to default or repay.

In every period, ζ is drawn from the uniform distribution on $[0,1]$. If $\zeta > 1-\pi$, international bankers expect there a crisis to occur and do not lend to the government if such a crisis would be self-fulfilling. This allows us to set the probability of a self-fulfilling crisis at an arbitrary level π, $1 \geq \pi \geq 0$, if the level of debt is high enough for such a crisis to be possible. The timing within each period is like that in Cole and Kehoe (1996, 2000):

1. The shocks a and ζ are realized, the aggregate state is $s=(B,a,z_{-1},\zeta)$, and the government chooses how much debt B' to sell.

2. Each of a continuum of measure one of international bankers chooses how much debt b' to purchase. (In equilibrium, of course, $b'=B'$.)

3. The government makes its default decision z, which determines y, c, and g.

The three crucial elements of this timing are as follows. First, the government faces a time consistency problem because, when offering B' for sale, it cannot commit to repaying B. Second, since all uncertainty has been resolved at the beginning of the period, there is perfect foresight in equilibrium within the period and, in particular, international bankers do not lend if they know the government will default. Third, whether or not a crisis occurs during the period depends on B, whereas — if no crisis occurs — the price of new bonds depends only on B'.

Given this timing, we can reduce the government's problem choosing c,g,B',z to solve

$$\begin{aligned} V(s) = \max \ \ & u(c,g)+\beta EV(s') \\ \text{s.t. } & c=(1-\theta)y(a,z) \\ & g+zB=\theta y(a,z)+q(B',s)B' \\ & z=0 \text{ if } z_{-1}=0 . \end{aligned} \tag{4}$$

Here $z=1$ is the decision not to default, and $z=0$ is the decision to default.

In general, we assume that, for any B such that $A\bar{y}-B$ is an element of the feasible set of levels for government expenditures g,

$$u_g((1-\theta)A\bar{y},\theta A\bar{y}-B)>u_g((1-\theta)\bar{y},\theta\bar{y}-B). \tag{5}$$

In other words, the marginal social benefit of government spending is higher during a recession than it is in normal times. This assumption provides the government with the incentive to transfer resources into the current period during a recession from future periods in which the economy has recovered. It is satisfied by any concave utility function separable in c and g. It is also satisfied by functions such as $\log(c+g-\bar{c}-\bar{g})$.

2.1. Bond prices

International bankers are risk neutral with discount factor β so that the bond prices $q(B',s)$ are determined by the probability of default in the next period. There is a continuum of measure one of bankers. Each solves the dynamic programming problem

$$W(b,B',s) = \max x + \beta EW(b',B'',s')$$
$$x + q(B',s)b' = w + z(B',s,q(B',s))b \qquad (6)$$
$$x \geq 0,\ b \geq -A.$$

The constraint $b \geq -A$ eliminates Ponzi schemes, but A is large enough so that the constraint does not otherwise bind. We assume that the banker's endowment of consumption good w is large enough to rule out corner solutions in equilibrium. We refer to this assumption as the assumption that the banker has deep pockets.

There are four cutoff levels of debt: $\bar{b}(a)$, $\bar{B}(a)$, $a = 0,1$:

1. If $B \leq \bar{b}(0)$, the government does not default when the private sector is in a recession even if international bankers do not lend, and, if $B > \bar{b}(0)$, the government defaults when the private sector is in a recession if international bankers do not lend.

2. If $B \leq \bar{b}(1)$, the government does not default when the private sector is in normal conditions even if international bankers do not lend, and, if $B > \bar{b}(1)$, the government defaults when the private sector is in normal conditions if international bankers do not lend.

3. If $B \leq \bar{B}(0)$, the government does not default when the private sector is in a recession if international bankers lend, and, if $B > \bar{B}(0)$, the government defaults when the private sector is in a recession even if international bankers lend.

4. If $B \leq \bar{B}(1)$, the government does not default when the private sector is in normal conditions if international bankers lend, and, if $B > \bar{B}(1)$, the government defaults when the private sector is in normal conditions even if international bankers lend.

The assumption that once $z=0$, it stays equal to 0 forever says that a country that defaults is permanently excluded from international borrowing or lending. This assumption can be modified at the cost of complicating the analysis. The assumption has two consequences for the relation of the bond price q to the current state s. First, once default has occurred, international bankers do not lend:

$$q(B',(B,a,0,\zeta))=0\,. \tag{7}$$

Second, during a crisis, international bankers do not lend:

$$q(B',(B,a,1,\zeta))=0 \tag{8}$$

whenever $B>\bar{b}(a)$ and $\zeta>1-\pi$. Otherwise, the bond price q only depends on the amount of bonds B' that the government offers for sale.

We focus on the case where

$$\bar{b}(0)<\bar{b}(1)<\bar{B}(0)<\bar{B}(1)\,. \tag{9}$$

The first-order condition for the international bankers' utility maximization problem implies that

$$q(B',s)=\beta Ez(B'(s'),s',q(B'(s'),s'))\,, \tag{10}$$

which implies that in recessions

$$q(B',(B,0,1,\zeta))=\begin{cases}\beta & \text{if } B'\le\bar{b}(0)\\ \beta\left(p+(1-p)(1-\pi)\right) & \text{if } \bar{b}(0)<B'\le\bar{b}(1)\\ \beta(1-\pi) & \text{if } \bar{b}(1)<B'\le\bar{B}(0)\\ \beta p(1-\pi) & \text{if } \bar{B}(0)<B'\le\bar{B}(1)\\ 0 & \text{if } \bar{B}(1)<B'\end{cases} \tag{11}$$

and in normal times

$$q(B',(B,1,1,\zeta))=\begin{cases}\beta & \text{if } B'\le\bar{b}(1)\\ \beta(1-\pi) & \text{if } \bar{b}(1)<B'\le\bar{B}(1)\,.\\ 0 & \text{if } \bar{B}(1)<B'\end{cases} \tag{12}$$

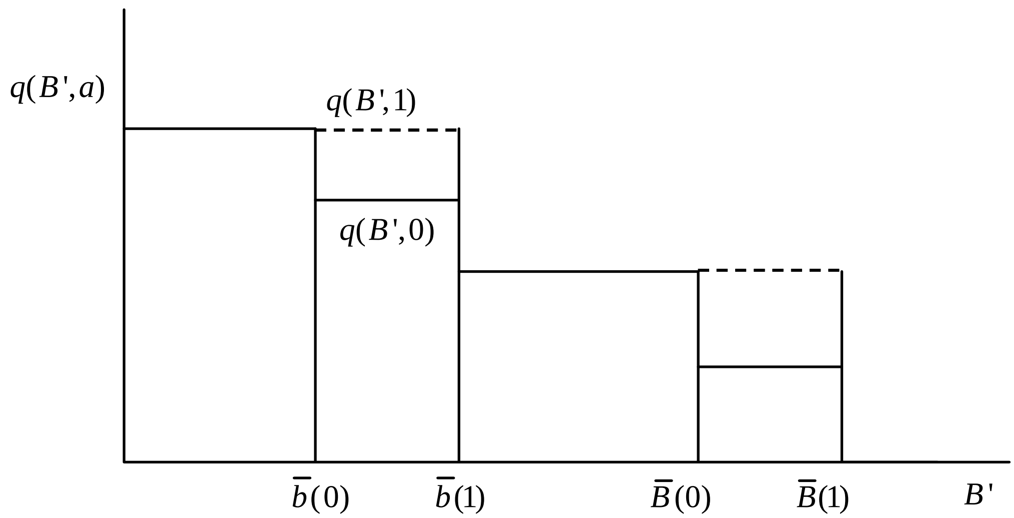

Figure 7: Bond prices as a function of new bonds offered and state of private sector

There are other possibilities. Suppose, for example, that $\bar{B}(0) < \bar{b}(1)$, that is,

$$\bar{b}(0) < \bar{B}(0) < \bar{b}(1) < \bar{B}(1)\,. \tag{13}$$

Here the solution to the international bankers' utility maximization problem implies that

$$q(B',(B,0,1,\zeta)) = \begin{cases} \beta & \text{if } B' \le \bar{b}(0) \\ \beta\left(p + (1-p)(1-\pi)\right) & \text{if } \bar{b}(0) < B' \le \bar{B}(0) \\ \beta p & \text{if } \bar{B}(0) < B' \le \bar{b}(1) \\ \beta p(1-\pi) & \text{if } \bar{b}(1) < B' \le \bar{B}(1) \\ 0 & \text{if } \bar{B}(1) < B' \end{cases}\,. \tag{14}$$

There is another possible case where

$$\bar{b}(0) < \bar{b}(1) = \bar{B}(0) < \bar{B}(1)\,. \tag{15}$$

Here

$$q(B',(B,0,1,\zeta)) = \begin{cases} \beta & \text{if } B' \le \bar{b}(0) \\ \beta\left(p + (1-p)(1-\pi)\right) & \text{if } \bar{b}(0) < B' \le \bar{b}(1) \\ \beta p(1-\pi) & \text{if } \bar{b}(1) < B' \le \bar{B}(1) \\ 0 & \text{if } \bar{B}(1) < B' \end{cases}\,. \tag{16}$$

Since the second and third cases, where $\bar{B}(0) \le \bar{b}(1)$, are only possible for very low values of A, catastrophic recessions, we focus on the first case. Furthermore, in these other cases the model is not informative about self-fulfilling crises because, after the recession hits, the country is either not vulnerable to a crisis or defaults immediately.

2.2. Definition of equilibrium

An equilibrium is a value function for government $V(s)$ and policy functions $B'(s)$ and $z(B',s,q)$ and $g(B',s,q)$, a value function for bankers $W(b,B',s)$ and policy correspondence $b'(b,B',s)$, and a bond price function $q(B',s)$ such that

1. Given the policy functions $z(B',s,q)$ and $g(B',s,q)$ and the price function $q(B',s)$, $V(s)$ and $B'(s)$ solve the government's problem at the beginning of the period:

$$\begin{gathered} V(B,a,z_{-1},\zeta) = \max \quad u(c,g) + \beta EV(B',a',z,\zeta') \\ \text{s.t. } c = (1-\theta)y(a,z(B',s,q(B',s))) \\ g(B',s,q(B',s)) + z(B',s,q(B',s))B = \theta y(a,z) + q(B',s)B'. \end{gathered} \tag{17}$$

2. $b'(b,B',s)$ solves the banker's problem and $q(B',s)$ is consistent with market clearing and rational expectations:

$$B'(s) \in b'(b,B',s) \tag{18}$$

$$q(B',s) = \beta Ez(B',s,q(B',s)) . \tag{19}$$

3. Given the value function $V(s)$, $z(B',s,q)$ and $g(B',s,q)$ solve the government's problem at the end of the period:

$$\begin{gathered} \max \quad u(c,g) + \beta EV(B',a',z,\zeta') \\ \text{s.t. } c = (1-\theta)y(a,z) \\ g + zB = \theta y(a,z) + qB' \\ z = 0 \text{ or } z = 1, \text{ but } z = 0 \text{ if } z_{-1} = 0 . \end{gathered} \tag{20}$$

Notice that, when the government solves its problem at the beginning of the period, it takes as given the optimal responses both of international bankers and of itself later in the period.

In particular, the government cannot commit to repaying its debt and not defaulting later in the period. Furthermore, since the occurrence of a crisis depends on the amount of debt to be repaid, B, not the amount of debt offered for sale, B', once a sunspot has occurred that signals that a self-fulfilling crisis will take place during that period, there is nothing that the government can do to avoid it.

Also worth noting is that this model has many equilibria. Our definition of equilibrium restricts our attention to equilibria with a simple Markov structure. Many other possibilities exist. If we include the date in the state $s=(B,a,z_{-1},\zeta,t)$, for example, we could allow crises to occur only in even periods t or in periods that are prime numbers, or we could allow the probability of a crisis π to be time varying in other ways. We could, for example, have π itself follow a Markov process so as to mimic the sort of time-varying spreads seen in the data in figure 1. The advantage of our simple Markov structure is that it makes it easy to characterize and compute equilibria.

3. Self-fulfilling liquidity crises

In the general model, we need to resort to numerical examples to illustrate the possibilities and do comparative statics analysis. Before turning to the results for the general model, we study two special cases, where we can provide analytical characterizations of the equilibria in which we are interested. The first is the case where $a=1$, that is, where the private sector has recovered and where there is no incentive for the government to gamble for redemption. This is a simplified version of the Cole-Kehoe model (1996, 2000) without private capital. To keep our discussion simple, we omit the details of proofs that can be found in Cole and Kehoe (2000).

Notice that we can easily modify the analysis of this case to study the limiting case where $a=0$ and $p=0$, that is, where there is a recession but no possibility for recovery, simply by replacing $\bar{y}$ with $A\bar{y}$ in what follows. In this case, where self-fulfilling crises are possible, but where there is no incentive for the government to gamble for redemption, the optimal strategies of the government involve either leaving debt constant or running it down to eliminate the possibility of a crisis. In the next section, we consider the other extreme case, where recovery is possible but self-fulfilling crises are not.

We start by assuming that $\pi = 0$. Notice that, since a recovery has already occurred in the private sector, p is irrelevant. To derive the optimal government policy, we solve

$$\begin{aligned} &\max \;\; \textstyle\sum_{t'=t}^{\infty} \beta^{t'} u(c_{t'}, g_{t'}) \\ &\text{s.t. } c_{t'} = (1-\theta)\bar{y} \qquad (21) \\ &g_{t'} + B_{t'} = \theta\bar{y} + \beta B_{t'+1} \\ &B_{t'} = B \\ &B_{t'} \leq \bar{B}(1). \end{aligned}$$

The first-order conditions are

$$\beta^{t'} u_g((1-\theta)\bar{y}, g_{t'}) = \lambda_{t'} \qquad (22)$$

$$\lambda_{t'+1} = \beta \lambda_{t'} \qquad (23)$$

The transversality condition is

$$\lim_{t' \to \infty} \lambda_{t'} B_{t'+1} \geq 0. \qquad (24)$$

The first-order conditions imply that

$$u_g((1-\theta)\bar{y}, g_{t'}) = u_g((1-\theta)\bar{y}, g_{t'-1}) \qquad (25)$$

$$g_{t'+1} = g_{t'} = \hat{g}. \qquad (26)$$

Consequently,

$$B_{t'+1} = \frac{1}{\beta}\left(\hat{g} + B_{t'} - \theta\bar{y}\right). \qquad (27)$$

Suppose that

$$\hat{g} = \theta\bar{y} - (1-\beta)B. \qquad (28)$$

Then $B_s = B$. Otherwise, since $\beta < 1$, $B_{t'}$ is explosive. Too low a $\hat{g}$ results in a path for $B_{t'}$ that violates the transversality condition. Too high a $\hat{g}$ results in a path for $B_{t'}$ that hits $\bar{B}(1)$. Neither can be optimal.

We can calculate the value of being in state $s = (B, a, z_{-1}, \zeta) = (B, 1, 1, \zeta)$ as

$$V(B,1,1,\zeta)=\frac{u((1-\theta)\bar{y},\theta\bar{y}-(1-\beta)B)}{1-\beta}. \tag{29}$$

The calculation of utility when default has occurred, when $z=0$, is mechanical. In that case $B=0$ and

$$c=(1-\theta)Z\bar{y} \tag{30}$$

$$g=\theta Z\bar{y}. \tag{31}$$

Notice that, once a default has occurred, ζ and π are irrelevant. Consequently, when $s=(B,a,z_{-1},\zeta)=(B,1,0,\zeta)$,

$$V(B,1,0,\zeta)=V_d(1)=\frac{u((1-\theta)Z\bar{y},\theta Z\bar{y})}{1-\beta}. \tag{32}$$

Let us calculate $\bar{b}(1)$. Let $V_n(B,a,q)$ be the value of not defaulting when the price of new debt is q. The utility of repaying B even if the international bankers do not lend is

$$V_n(B,1,0)=u((1-\theta)\bar{y},\theta\bar{y}-B)+\frac{\beta u((1-\theta)\bar{y},\theta\bar{y})}{1-\beta}, \tag{33}$$

whereas the utility of defaulting $V_d(a)$ is

$$V_d(1)=\frac{u((1-\theta)Z\bar{y},\theta Z\bar{y})}{1-\beta}. \tag{34}$$

Consequently, $\bar{b}(1)$ is determined by the equation

$$V_n(\bar{b}(1),1,0)=V_d(1) \tag{35}$$

$$u((1-\theta)\bar{y},\theta\bar{y}-\bar{b}(1))+\frac{\beta u((1-\theta)\bar{y},\theta\bar{y})}{1-\beta}=\frac{u((1-\theta)Z\bar{y},\theta Z\bar{y})}{1-\beta}. \tag{36}$$

Determining $\bar{B}(1)$ is more complicated because it depends on the optimal debt policy in the crisis zone. Suppose that $B_0>\bar{b}(1)$ and the government decides to reduce B to $\bar{b}(1)$ in T periods, $T=1,2,...,\infty$. The first-order conditions for the government's problem imply that

$$g_t=g^T(B_0), \tag{37}$$

that is, government spending is constant while the government is reducing its debt. The government's budget constraints are

$$
\begin{aligned}
g^T(B_0) + B_0 &= \theta\bar{y} + \beta(1-\pi)B_1 \\
g^T(B_0) + B_1 &= \theta\bar{y} + \beta(1-\pi)B_2 \\
&\vdots \\
g^T(B_0) + B_{T-2} &= \theta\bar{y} + \beta(1-\pi)B_{T-1} \\
g^T(B_0) + B_{T-1} &= \theta\bar{y} + \beta\bar{b}(1).
\end{aligned} \tag{38}
$$

Multiplying each equation by $(\beta(1-\pi))^t$ and adding, we obtain

$$
\sum\nolimits_{t=0}^{T-1}(\beta(1-\pi))^t g^T(B_0) + B_0 = \sum\nolimits_{t=0}^{T-1}(\beta(1-\pi))^t \theta\bar{y} + (\beta(1-\pi))^{T-1}\beta\bar{b}(1) \tag{39}
$$

$$
g^T(B_0) = \theta\bar{y} - \frac{1-\beta(1-\pi)}{1-(\beta(1-\pi))^T}\left(B_0 - (\beta(1-\pi))^{T-1}\beta\bar{b}(1)\right). \tag{40}
$$

Notice that

$$
g^\infty(B_0) = \lim\nolimits_{T\to\infty} g^T(B_0) = \theta\bar{y} - (1-\beta(1-\pi))B_0. \tag{41}
$$

We can compute the value $V^T(B_0)$ of each of the policies of running down the debt in T periods, $T = 1,\ldots,\infty$. Letting $V_t^T(B_0)$ be the value of the policy where there are still t periods to go in running debt, we can write

$$
\begin{aligned}
V_T^T(B_0) &= u((1-\theta)\bar{y}, g^T(B_0)) + \beta(1-\pi)V_{T-1}^T(B_0) + \frac{\beta\pi u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} \\
V_{T-1}^T(B_0) &= u((1-\theta)\bar{y}, g^T(B_0)) + \beta(1-\pi)V_{T-2}^T(B_0) + \frac{\beta\pi u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} \\
&\vdots \\
V_2^T(B_0) &= u((1-\theta)\bar{y}, g^T(B_0)) + \beta(1-\pi)V_1^T(B_0) + \frac{\beta\pi u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} \\
V_1^T(B_0) &= u((1-\theta)\bar{y}, g^T(B_0)) + \frac{\beta u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{b}(1))}{1-\beta}.
\end{aligned} \tag{42}
$$

Notice that g increases from $g^T(B_0)$ to $\theta\bar{y}-(1-\beta)\bar{b}(1)$ in period T. To calculate $V^T(B_0)$, we use backward induction:

$$
\begin{aligned}
V_2^T(B_0) &= (1+\beta(1-\pi))u((1-\theta)\bar{y}, g^T(B_0)) \\
&+ \frac{\beta\pi u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} \\
&+ \beta(1-\pi)\frac{\beta u((1-\theta)\bar{y}, \theta\bar{y}-(1-\beta)\bar{b}(1))}{1-\beta} \\
V_3^T(B_0) &= (1+\beta(1-\pi)+(\beta(1-\pi))^2)u((1-\theta)\bar{y}, g^T(B_0)) \\
&+ (1+\beta(1-\pi))\frac{\beta\pi u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} \\
&+ (\beta(1-\pi))^2\frac{\beta u((1-\theta)\bar{y}, \theta\bar{y}-(1-\beta)\bar{b}(1))}{1-\beta} \\
&\vdots \\
V_T^T(B_0) &= (1+\beta(1-\pi)+(\beta(1-\pi))^2+...+(\beta(1-\pi))^{T-1})u((1-\theta)\bar{y}, g^T(B_0)) \\
&+ (1+\beta(1-\pi)+(\beta(1-\pi))^2+...+(\beta(1-\pi))^{T-2})\frac{\beta\pi u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} \\
&+ (\beta(1-\pi))^{T-2}\frac{\beta u((1-\theta)\bar{y}, \theta\bar{y}-(1-\beta)\bar{b}(1))}{1-\beta},
\end{aligned} \tag{43}
$$

and, of course, $V^T(B_0) = V_T^T(B_0)$:

$$
\begin{aligned}
V^T(B_0) &= \frac{1-(\beta(1-\pi))^T}{1+\beta(1-\pi)}u((1-\theta)\bar{y}, g^T(B_0)) + \frac{1-(\beta(1-\pi))^{T-1}}{1+\beta(1-\pi)}\frac{\beta\pi u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta} \\
&+ (\beta(1-\pi))^{T-2}\frac{\beta u((1-\theta)\bar{y}, \theta\bar{y}-(1-\beta)\bar{b}(1))}{1-\beta}.
\end{aligned} \tag{44}
$$

Notice that

$$
V^\infty(B_0) = \frac{u((1-\theta)\bar{y}, \theta\bar{y}-(1-\beta(1-\pi))B_0)}{1+\beta(1-\pi)} + \frac{\beta\pi u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{(1-\beta)(1+\beta(1-\pi))}. \tag{45}
$$

To find $\bar{B}(1)$, we solve

$$
\begin{aligned}
&\max\left[V^1(\bar{B}(1)), V^2(\bar{B}(1)), ..., V^\infty(\bar{B}(1))\right] \\
&= u((1-\theta)Z\bar{y}, \theta Z\bar{y}+\beta(1-\pi)\bar{B}(1)) + \frac{\beta u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{1-\beta}\cdot
\end{aligned} \tag{46}
$$

Our arguments have produced the following analytical characterization of $V(B,1,1,\zeta)$:

$$V(B,1,1,\zeta)=\begin{cases}\dfrac{u((1-\theta)\overline{y},\theta\overline{y}-(1-\beta)\overline{b}(1))}{1-\beta} & \text{if } B\le\overline{b}(1)\\ \max\left[V^1(B),V^2(B),...,V^\infty(B)\right] & \text{if } \overline{b}(1)<B\le\overline{B}(1),\ \zeta\le 1-\pi\\ \dfrac{u((1-\theta)Z\overline{y},\theta Z\overline{y})}{1-\beta} & \text{if } \overline{b}(1)<B\le\overline{B}(1),\ 1-\pi<\zeta\\ \dfrac{u((1-\theta)Z\overline{y},\theta Z\overline{y})}{1-\beta} & \text{if } \overline{B}(1)<B\end{cases}. \quad (47)$$

Some of the different possibilities for optimal government strategies — which vary with the initial debt — are illustrated in figure 8.

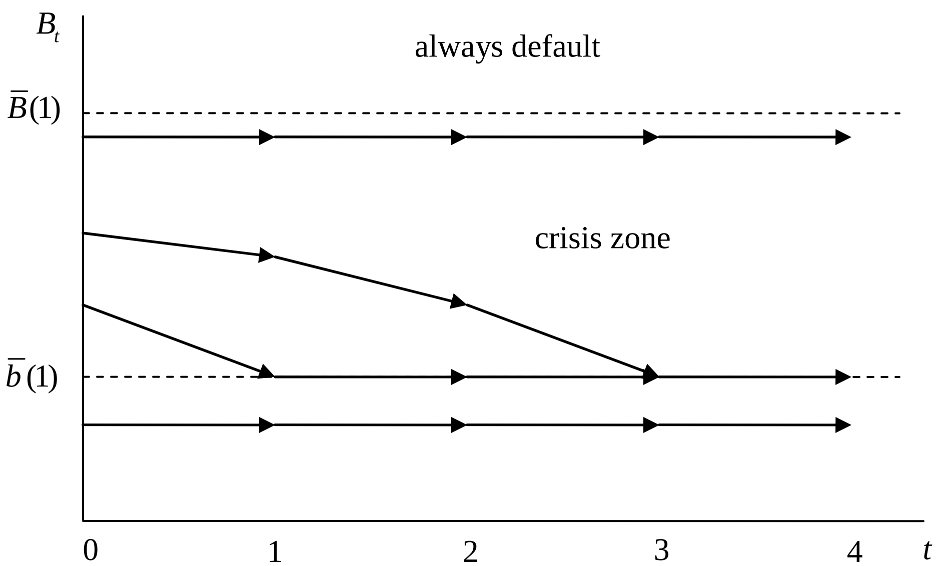

Figure 8: Optimal debt policy with self-fulfilling crises

4. Consumption smoothing without self-fulfilling crises

Suppose now that $a=0$ and $\pi=0$. That is, no self-fulfilling crises are possible, but the private sector is in a recession and faces the probability p, $1>p>0$, of recovering in every period, as depicted in figure 9. We can also interpret this as the limiting case in which crises can occur, but the government and the international bankers assign probability $\pi=0$ to them.

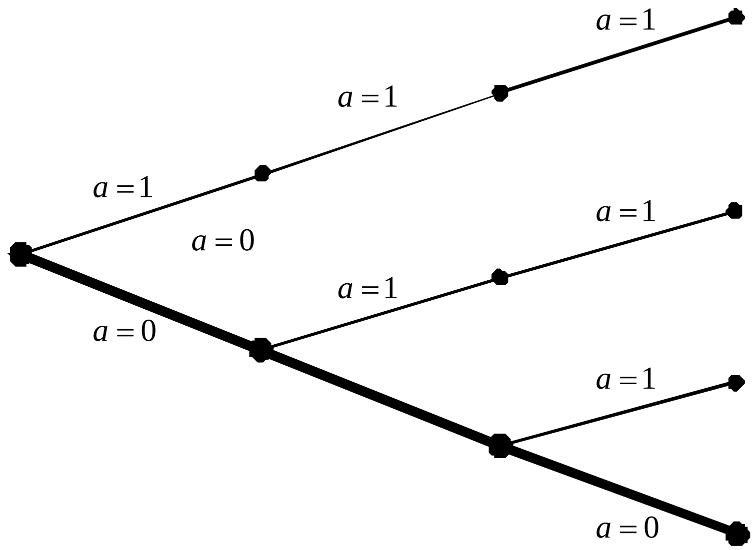

Figure 9: Uncertainty tree with recession path highlighted

In this section, we argue that the optimal government policy is to increase its debt as long as $a=0$. In fact, if the country is unlucky in the sense that $a=0$ long enough, the government may choose to eventually default. Consequently, the upper limits on the debt, $\bar{B}(0)$ and $\bar{B}(1)$, are crucial for our analysis. Because $\pi=0$, the optimal policy for debt after a recovery has occurred is to keep debt constant. Consequently, the condition that determines $\bar{B}(1)$ is similar to condition (36) for determining $\bar{b}(1)$ in the previous section:

$$\begin{aligned} &u((1-\theta)Z\bar{y},\theta Z\bar{y}+\beta\bar{B}(1))-u((1-\theta)\bar{y},\theta\bar{y}-(1-\beta)\bar{B}(1)) \\ &\quad=\frac{\beta}{1-\beta}\Big(u((1-\theta)\bar{y},\theta\bar{y}-(1-\beta)\bar{B}(1))-u((1-\theta)Z\bar{y},\theta Z\bar{y})\Big). \end{aligned} \tag{48}$$

To determine $\bar{B}(0)$, we suppose that, when the government has debt $B\le\bar{B}(0)$, it borrows B', where $\bar{B}(0)<B'\le\bar{B}(1)$, at price βp, then repays the next period if the private sector recovers and defaults otherwise. The value of borrowing $\bar{B}(1)$ at price βp, repaying the current debt, and then repaying in the next period if the private sector recovers and defaulting otherwise is

$$
\begin{aligned}
V_n(B) &= u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta p\bar{B}(1) - B) \\
&+ \beta(1-p)\left(\frac{u((1-\theta)AZ\bar{y}, \theta AZ\bar{y})}{1-\beta(1-p)} + \frac{\beta pu((1-\theta)Z\bar{y}, \theta Z\bar{y})}{(1-\beta)(1-\beta(1-p))}\right) \\
&+ \frac{\beta p}{1-\beta}u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{B}(1)).
\end{aligned} \tag{49}
$$

The value of borrowing $\bar{B}(1)$ at price βp and then defaulting is

$$
\begin{aligned}
V_d(B) &= u((1-\theta)AZ\bar{y}, \theta AZ\bar{y} + \beta p\bar{B}(1)) \\
&+ \beta(1-p)\left(\frac{u((1-\theta)AZ\bar{y}, \theta AZ\bar{y})}{1-\beta(1-p)} + \frac{\beta pu((1-\theta)Z\bar{y}, \theta Z\bar{y})}{(1-\beta)(1-\beta(1-p))}\right) \\
&+ \frac{\beta p}{1-\beta}u((1-\theta)Z\bar{y}, \theta Z\bar{y}).
\end{aligned} \tag{50}
$$

The equation that determines $\bar{B}(0)$ is, therefore,

$$
V_n(\bar{B}(0)) = V_d(\bar{B}(0)) \tag{51}
$$

$$
\begin{aligned}
&u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta p\bar{B}(1) - \bar{B}(0)) + \frac{\beta p}{1-\beta}u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{B}(1)) \\
&= u((1-\theta)AZ\bar{y}, \theta AZ\bar{y} + \beta p\bar{B}(1)) + \frac{\beta p}{1-\beta}u((1-\theta)Z\bar{y}, \theta Z\bar{y}).
\end{aligned} \tag{52}
$$

The government may, in fact, choose a lower level of the debt than $\bar{B}(1)$, but $\bar{B}(0)$ is the highest level of the debt B' at which the government can borrow at price $q(B', s) = \beta$ where there is no possibility of a self-fulfilling crisis, $\pi = 0$, and the economy is in a recession, $a = 0$. If the constraint $B' \le \bar{B}(1)$ does not bind, we can calculate the optimal B' by solving

$$
\begin{aligned}
&\max u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta pB' - B) \\
&+ \beta(1-p)\left(\frac{u((1-\theta)AZ\bar{y}, \theta AZ\bar{y})}{1-\beta(1-p)} + \frac{\beta pu((1-\theta)Z\bar{y}, \theta Z\bar{y})}{(1-\beta)(1-\beta(1-p))}\right) \\
&+ \frac{\beta p}{1-\beta}u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)B').
\end{aligned} \tag{53}
$$

The first-order condition is

$$
u_g((1-\theta)A\bar{y}, \theta A\bar{y} + \beta pB' - B) = u_g((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)B'). \tag{54}
$$

Letting $\hat{B}'(B)$ be the solution to this problem,

$$B'(B) = \min[\hat{B}'(B), \overline{B}(1)]. \tag{55}$$

There are two cases:

1. The government chooses to never violate the constraint $B \leq \overline{B}(0)$, and the optimal debt converges to $\overline{B}(0)$ if $a = 0$ for a sufficiently large number of periods.

2. The government chooses to default in T periods if $a = 0$ for a sufficiently large number of periods.

The crucial parameter in determining which of these two cases the economy is in is the default penalty factor Z. If Z is sufficiently low, the government will choose to never default. If Z is close to 1, it will optimally choose to default after a sufficiently long number of periods in which $a = 0$.

4.1. Equilibrium with no default

Let us first consider the case where the government chooses to never violate the constraint $B \leq \overline{B}(0)$. For this to be an equilibrium, two things must be true:

1. The expected discounted value of steady state utility at $B = \overline{B}(0)$ must be higher than that of defaulting after bankers have purchased $B' = \overline{B}(0)$ at price β,

$$\begin{aligned}
&u((1-\theta)A\overline{y}, \theta A\overline{y} - (1-\beta)\overline{B}(0)) \\
&\quad + \beta(1-p)\left(\frac{u((1-\theta)A\overline{y}, \theta A\overline{y} - (1-\beta)\overline{B}(0))}{1-\beta(1-p)} + \frac{\beta p u((1-\theta)\overline{y}, \theta\overline{y} - (1-\beta)\overline{B}(0))}{(1-\beta)(1-\beta(1-p))}\right) \\
&\quad + \frac{\beta p}{1-\beta} u((1-\theta)\overline{y}, \theta\overline{y} - (1-\beta)\overline{B}(0)) \\
&\quad \geq u((1-\theta)AZ\overline{y}, \theta AZ\overline{y} + \beta\overline{B}(0)) \\
&\quad + \beta(1-p)\left(\frac{u((1-\theta)AZ\overline{y}, \theta AZ\overline{y})}{1-\beta(1-p)} + \frac{\beta p u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{(1-\beta)(1-\beta(1-p))}\right) \\
&\quad + \frac{\beta p}{1-\beta} u((1-\theta)Z\overline{y}, \theta Z\overline{y}).
\end{aligned} \tag{56}$$

2. The expected discounted value of steady state utility at $B=\bar{B}(0)$ must be higher than that of running up the debt one more time at price βp, repaying if the private sector recovers, and defaulting otherwise,

$$\begin{aligned}
&u((1-\theta)A\bar{y},\theta A\bar{y}-(1-\beta)\bar{B}(0)) \\
&\quad+\beta(1-p)\left(\frac{u((1-\theta)A\bar{y},\theta A\bar{y}-(1-\beta)\bar{B}(0))}{1-\beta(1-p)}+\frac{\beta pu((1-\theta)\bar{y},\theta\bar{y}-(1-\beta)\bar{B}(0))}{(1-\beta)(1-\beta(1-p))}\right) \\
&\quad+\frac{\beta p}{1-\beta}u((1-\theta)\bar{y},\theta\bar{y}-(1-\beta)\bar{B}(0)) \\
&\quad\geq u((1-\theta)A\bar{y},\theta A\bar{y}+\beta pB'(\bar{B}(0))-\bar{B}(0)) \\
&\quad+\beta(1-p)\left(\frac{u((1-\theta)AZ\bar{y},\theta AZ\bar{y})}{1-\beta(1-p)}+\frac{\beta pu((1-\theta)Z\bar{y},\theta Z\bar{y})}{(1-\beta)(1-\beta(1-p))}\right) \\
&\quad+\frac{\beta p}{1-\beta}u((1-\theta)\bar{y},\theta\bar{y}-(1-\beta)B'(\bar{B}(0))),
\end{aligned} \tag{57}$$

where $B'(B)=\min[\hat{B}'(B),\bar{B}(1)]$.

If these two conditions are satisfied, the optimal government policy is the solution to the dynamic programming problem:

$$V(B,a)=\max u((1-\theta)A^{1-a}\bar{y},\theta A^{1-a}\bar{y}+\beta B'-B)+\beta EV(B',a') \tag{58}$$

$$\text{s.t. } B\leq\bar{B}(0).$$

We write the Bellman's equation explicitly as

$$V(B,0)=\max u((1-\theta)A\bar{y},\theta A\bar{y}+\beta B'-B)+\beta(1-p)V(B',0)+\beta pV(B',1) \tag{59}$$

$$V(B,1)=\max u((1-\theta)\bar{y},\theta\bar{y}+\beta B'-B)+\beta V(B',1). \tag{60}$$

The first-order condition is

$$\beta u_g((1-\theta)A^{1-a}\bar{y},\theta A^{1-a}\bar{y}+\beta B'-B)=\beta EV_B(B',a'), \tag{61}$$

and the envelope condition is

$$V_B(B,a)=-u_g((1-\theta)A^{1-a}\bar{y},\theta A^{1-a}\bar{y}+\beta B'-B). \tag{62}$$

The envelope condition implies that $V(B,a)$ is decreasing in B. A standard argument — that the operator on the space of functions defined by Bellman's equations maps concave value functions into concave value functions — implies that $V(B,a)$ is concave in B.

The first-order condition (61) implies that the policy function for debt $B'(B,a)$ is increasing in B while the policy function for government spending $g(B,a)$ is decreasing in B. Our assumption (5) on the utility function u implies that $B'(0,0) > 0$ and that it is impossible for $B'(B,0) = B$ unless the constraint $B' \le \bar{B}(0)$ binds, which implies that $B'(B,0) > B$. We know that $B'(B,1) = B$. Figure 10 illustrates some optimal government strategies as functions of the initial debt.

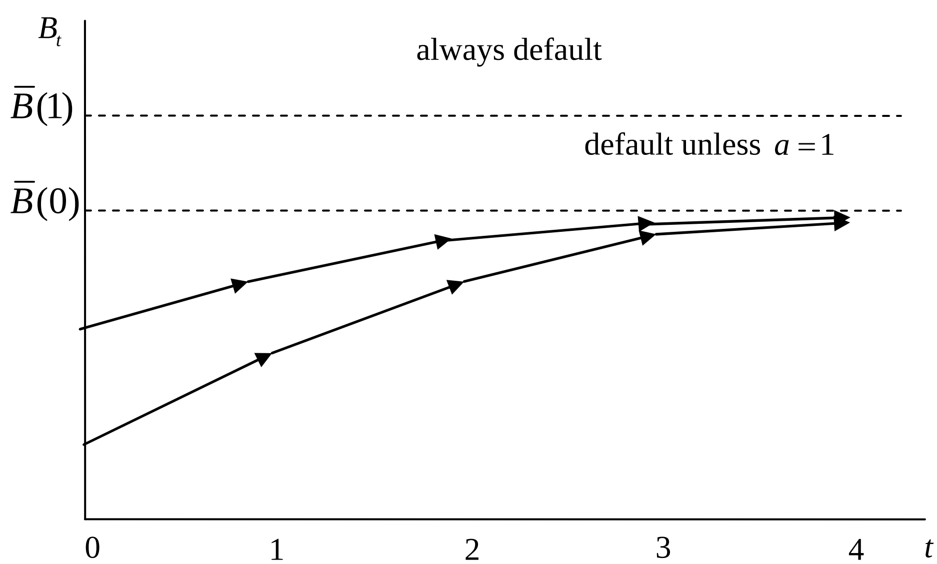

Figure 10: Optimal debt policy gambling for redemption when $B' \le \bar{B}(0)$ binds

4.2. Equilibrium with eventual default

Let us now consider the case where the government chooses to violate the constraint $B \le \bar{B}(0)$ with its sale of debt in period T, defaulting in period $T+1$ unless the private sector recovers. The optimal government policy along the branch of the uncertainty tree in figure 9 along which $a_t = 0$ is the solution to the finite horizon dynamic programming problem:

$$
\begin{aligned}
V_t(B_t) = \max u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta B_{t+1} - B_t) \\
+ \beta(1-p)V_{t+1}(B_{t+1}) + \beta p \frac{u((1-\theta)\bar{y}, \theta\bar{y} + (1-\beta)B_{t+1}))}{1-\beta}
\end{aligned}
\tag{63}
$$

$$
\text{s.t. } B_t \leq \bar{B}(0).
$$

We solve this by backward induction with the terminal value function:

$$
\begin{aligned}
V_T(B_T) = \max u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta p B_{T+1} - B_T) \\
+ \beta(1-p)\frac{u((1-\theta)Z\bar{y}, \theta Z\bar{y}))}{1-\beta} + \beta p \frac{u((1-\theta)\bar{y}, \theta\bar{y} + (1-\beta)B_{T+1}))}{1-\beta}
\end{aligned}
\tag{64}
$$

$$
\text{s.t. } \bar{B}(1) \geq B_{T+1} \geq \bar{B}(0).
$$

We then choose the value of T for which $V_0(B_0)$ is maximal. As long as the constraint $B_{T+1} \geq \bar{B}(0)$ binds, we can increase the value of $V_0(B_0)$ by increasing T.

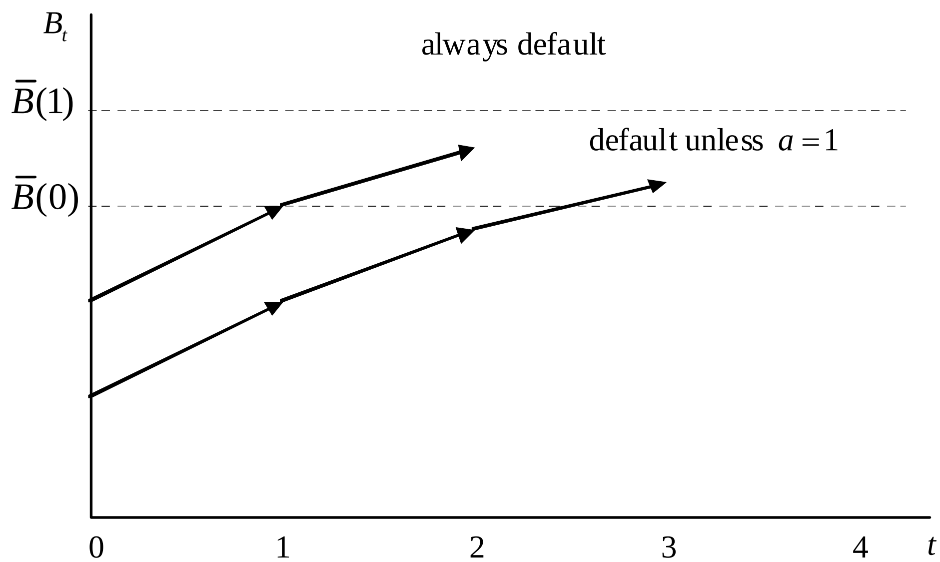

Figure 11: Optimal debt policy gambling for redemption when $B \leq \bar{B}(0)$ does not bind

In figure 11, we illustrate two possibilities, which depend on B_0. In one $T=1$, and in the other $T=2$.

The algorithm for calculating the optimal policy function is a straightforward application of backward induction. We work backward from the period in which the government borrows at price βp and defaults in the next period unless a recovery of the private sector occurs. Define

$$V_T(B) = \max u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta p B' - B) + \beta(1-p)\frac{u((1-\theta)Z\bar{y}, \theta Z\bar{y}))}{1-\beta} + \beta p \frac{u((1-\theta)\bar{y}, \theta\bar{y} + (1-\beta)B'))}{1-\beta} \tag{65}$$

$$\text{s.t. } \bar{B}(0) \le B' \le \bar{B}(1).$$

The steps of the algorithm are as follows:

1. Solve for the value function $V_0(B)$ and the policy function $B_0{}'(B)$ on a grid of bonds B on the interval $[\underline{B}, \bar{B}(0)]$. We can set the lower limit $\underline{B}$ equal to any value, including a negative value. In an application with a given initial stock of debt, we could set $\underline{B} = B_0$. We have already solved this problem analytically. The solution is $B'(B) = \min[\hat{B}'(B), \bar{B}(1)]$ unless $B'(B) < \bar{B}(0)$, in which case $B_0'(B) = \bar{B}(0)$. Consequently,

$$B_0{}'(B) = \max\left[\bar{B}(0), \min[\hat{B}'(B), \bar{B}(1)]\right]. \tag{66}$$

The values of B for which $B'(B) < \bar{B}(0)$ are those for which it is not optimal to set $T = 0$.

2. Let $t = 0$, and set $\tilde{B}_0 = \bar{B}(0)$.

3. Solve for the value function $V_{t+1}(B,0)$ and the policy function $B_{t+1}{}'(B)$ in Bellman's equation

$$V_{t+1}(B,0) = \max u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta B' - B) + \beta(1-p)V_t(B',0) + \beta p \frac{u((1-\theta)\bar{y}, \theta\bar{y} + (1-\beta)B))}{1-\beta}. \tag{67}$$

Let $\tilde{B}_t$ be the largest value of B for which $V_{t+1}(B,0) \ge V_t(B,0)$.

4. Repeat step 3 until $\tilde{B}_t = \underline{B}$.

Let T be such that $\tilde{B}_T = \underline{B}$. We can prove that $\underline{B} < \tilde{B}_{T-1} < \tilde{B}_{T-2} < \cdots < \tilde{B}_1 < \bar{B}(0)$. Our algorithm divides the interval $[\underline{B}, \bar{B}(0)]$ into subintervals $[\underline{B}, \tilde{B}_{T-1})$, $[\tilde{B}_{T-1}, \tilde{B}_{T-2}), \ldots,$ $[\tilde{B}_1, \bar{B}(0; p, 0)]$. If the initial capital stock B_0 is in the subinterval $[\tilde{B}_t, \tilde{B}_{t-1}]$, then the optimal government policy is to increase B, selling debt B, $\bar{B}(0) < B \le \bar{B}(1)$, in period $t-1$, and defaulting in period t unless the private sector recovers. The optimal sequence of debt is B_0, $B_{t-1}'(B_0)$, $B_{t-2}'(B_{t-1}'(B_0))$ …, $B_0(B_1(\cdots(B_{t-1}'(B_0))))$.

5. Numerical results for the general model

We now solve the full model numerically to evaluate when gambling for redemption might happen in equilibrium. To solve for the equilibrium, we need to choose a functional form for the utility function,

$$u(c,g) = \log(c) + \gamma \log(g - \bar{g}). \tag{68}$$

The period length is one year, and the parameters we choose for our benchmark scenario are displayed in table 1.

Parameter	Value	Target
A	0.90	average government revenue loss (see figure 3)
Z	0.95	default penalty in Cole and Kehoe (1996)
p	0.20	expected recovery in five years
β	0.98	yield on safe bonds 2 percent annual
π	0.03	real interest rate in crisis zone 5 percent annual
γ	0.50	consumers value c twice as much as g
θ	0.40	government revenues as a share of output
$\bar{g}$	30.0	minimum government expenditure

Table 1: Parameter values in the benchmark scenario

Although our parameter values are not derived from a careful calibration of the model, they are intended to generate results that illustrate the sorts of possibilities that exist for reasonable parameter values. We choose a default penalty $1-Z$ of 5 percent, as do Cole and Kehoe (1996). Mendoza and Yue (2012) and Sosa-Padilla (2014) build models in which sovereign default endogenously generates an output loss. Their quantitative exercises suggest larger output losses (between 6 and 12 percent). Our benchmark cost of default is smaller, but we assume that the default cost is permanent, whereas theirs is transitory. Sensitivity analysis shows that increasing the default penalty, that is, decreasing Z, increases the upper debt limits $\bar{B}(0)$ and $\bar{B}(1)$.

We have chosen the probability of a recovery p of 20 percent, which implies that the expected waiting time for a recovery is $1/p$ or five years. This parameter is crucial to generating gambling for redemption. Higher probabilities of recovery generate more gambling; lower probabilities of recovery generate less.

The probability of a panic π is arbitrary, and we fix it at 3 percent, generating spreads of approximately 3 percent over the safe rate in the crisis zone. This magnitude is consistent with the risk premia we observe in the data in figure 1. As we have explained, we could have π itself follow a Markov process generating the sorts of time varying spreads observed in the data. Higher probabilities of self-fulfilling crises generate less gambling; lower probabilities of self-fulfilling crises generate more.

We have chosen A so that a recession results in a drop in GDP of 10 percent. More severe recessions generate more gambling; less severe recessions generate less.

We have set a miminum government expenditure level $\bar{g}$ at 30 percent of GDP in normal times. In normal times the government collects tax revenue of 40, but sees this number fall to 36 in the recession. The minimum expenditure, which can be interpreted as entitlements that leave less room for discretionary spending, implies that the curvature of utility is high once the recession hits.

In the theory that we have developed so far, we have restricted our analysis to one-period bonds. When we set the period length equal to one year, this assumption is very restrictive for the economies in the Eurozone that we consider, although it makes more sense for emerging market economies such as Mexico in the 1990s (see Cole and Kehoe, 1996). Table 2 shows the average maturity of debt the PIIGS and Germany.

	2008	2009	2010
Germany	6.3	5.9	5.9
Greece	8.4	7.9	7.1
Ireland	4.3	5.6	5.9
Italy	6.8	7.1	7.2
Portugal	6.2	6.1	5.8
Spain	6.6	6.4	6.6

Table 2: Weighted average term to maturity for total debt (years)

It is important to note that the average maturity data in table 2 are only suggestive of maturity structure of debt that is relevant for our model. What matters for us is how much debt becomes due every period. Consider, for example, a government that has half of its debt evenly distributed across maturity dates in 10-year bonds and half in 1-year bonds. The simple average maturity of debt is 3.25 (= (1/2)×(1 + (10 + 1)/2)) years, but 55 percent of the debt becomes due every year, more than if the debt were evenly distributed across maturities with 2-year bonds. In data like that in table 2 future payments are discounted, so that, with a 5 percent annual interest rate, the weighted average maturity of the debt in our example becomes 2.84 years, but 60.7 percent of the present discounted value of the debt becomes due every year.

Our theory is silent about the maturity structure of debt, which we take as exogenous. In our model, longer maturities are always better, since it is the need to roll over debt that generates the risk of default. For more sophisticated theories of endogenous maturity, see Aguiar and Amador (2015), Arellano and Ramanarayanan (2012), and Hatchondo *et al*. (2014).

To understand the crucial role of debt maturity for our results, we extend the model to introduce the feature that a given fraction δ of the existing stock of debt becomes due every period. As in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012), this feature renders the maturing of debt "memoryless," so that it is not necessary to keep track of the entire distribution of maturities of the debt.

Now the government's problem is to choose c, g, B', z to solve

$$\begin{gathered} V(s) = \max \; u(c,g) + \beta EV(s') \\ \text{s.t. } c = (1-\theta)y(a,z) \\ g + z\delta B = \theta y(a,z) + q(B',s)(B' - (1-\delta)B) . \end{gathered} \tag{69}$$

In general, the entire distribution of maturities of debt would determine the amount of debt due at any given period, δB, and this is not necessarily related to average maturity. Notice that problem (69) reduces to the one-period debt case when $\delta = 1$, or to infinitely lived debt as δ tends to 0.

With multiperiod debt, we need to change the price functions (11) and (12). In the benchmark scenario, where $\bar{b}(0) < \bar{b}(1) < \bar{B}(0) < \bar{B}(1)$, we can define prices recursively when the sunspot ζ is such that there is no self-fulfilling crisis, for normal and recession times, respectively:

$$q(B',1)=\begin{cases}\beta\left[\delta+(1-\delta)q'(\cdot)\right] & \text{if } B'\le \bar{b}(1)\\ \beta(1-\pi)\left[\delta+(1-\delta)q'(\cdot)\right] & \text{if } \bar{b}(1)<B'\le \bar{B}(1)\\ 0 & \text{if } \bar{B}(1)<B'\end{cases} \tag{70}$$

$$q(B',0)=\begin{cases}\beta\left[\delta+(1-\delta)q'(\cdot)\right] & \text{if } B'\le \bar{b}(0)\\ \beta\left(p+(1-p)(1-\pi)\right)\left[\delta+(1-\delta)q'(\cdot)\right] & \text{if } \bar{b}(0)<B'\le \bar{b}(1)\\ \beta(1-\pi)\left[\delta+(1-\delta)q'(\cdot)\right] & \text{if } \bar{b}(1)<B'\le \bar{B}(0)\\ \beta p(1-\pi)\left[\delta+(1-\delta)q'(\cdot)\right] & \text{if } \bar{B}(0)<B'\le \bar{B}(1)\\ 0 & \text{if } \bar{B}(1)<B'\end{cases}\quad . \tag{71}$$

Our benchmark scenario is a model where

$$\delta = 1/6\,, \tag{72}$$

which is the value implied by debt being evenly distributed across maturities with 6-year bonds, consistent with the empirical evidence in table 2, subject to the reservations that we have noted.

To calculate the thresholds for the safe zone $\bar{b}(0)$ and $\bar{b}(1)$, we need to make an assumption about what it means for international bankers to panic even if the government repays. To keep things simple, we assume that the panic lasts one period and then, if the country repays, international bankers resume lending. This is a somewhat arbitrary assumption about out-of-equilibrium behavior. The calculation (36) of $\bar{b}(1)$, for example, becomes

$$\begin{aligned}&u((1-\theta)\bar{y},\theta\bar{y}-\delta\bar{b}(1))+\frac{\beta u((1-\theta)\bar{y},\theta\bar{y}-(1-\beta)\delta(1-\delta)\bar{b}(1))}{1-\beta}\\ &=\frac{u((1-\theta)Z\bar{y},\theta Z\bar{y})}{1-\beta}.\end{aligned} \tag{73}$$

In figure 12 we plot the policy functions, together with the debt thresholds.

First, notice the value of the thresholds. The lower threshold is about 60 percent of GDP. Below this level, the government would not find it optimal to default even if it could not roll over its stock of debt. Any debt above this level makes the economy vulnerable to a self-fulfilling crisis. The upper threshold is about 104 percent of GDP. Above this level, interest payments are so large that the government chooses to default even if investors were willing to refinance the stock of debt.

If the economy is in normal times, then the optimal policy is to keep debt constant when the economy is in the safe region and decrease debt step-by-step when the economy is in the crisis zone. This is exactly the policy prescription of Cole and Kehoe (2000).

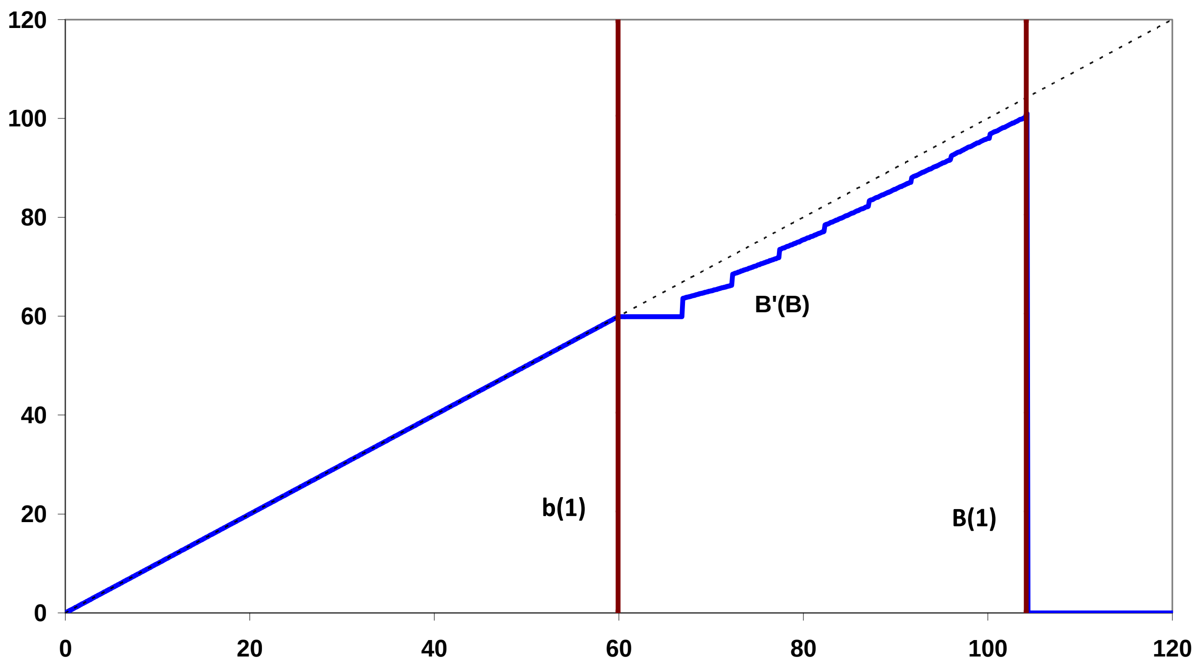

Figure 12: Policy function in normal times

Consider now that the economy unexpectedly falls into a recession. Figure 13 plots the impact of such a change. Notice that, as soon as the economy enters into the recession, both the lower and upper thresholds decrease. In our numerical example, governments of countries with a very low level of debt — below about 35 percent of the original GDP, about 39 percent of GDP when the recession hits — are still safe, and, as a result, they do consumption smoothing, gambling for redemption and increasing the level of debt until it reaches $\bar{b}(0)$.

Debt in the region between $\bar{b}(0)$ and $\bar{b}(1)$ displays interesting dynamics. These are debt levels where the economy was not vulnerable to a panic before the recession. When the recession hits, the economy becomes vulnerable and interest rates jump up. In terms of debt dynamics, this region is split. Governments in countries with initial debt that is close to the safe threshold in the recession, $\bar{b}(0)$, choose to lower debt to avoid paying the spread and to avoid the probability of a self-fulfilling crisis. The government in a country with a larger initial level of debt gambles for redemption until debt reaches the safe threshold in normal times, $\bar{b}(1)$, and stays there waiting for a recovery. Recall that above $\bar{b}(1)$ the government would have to pay an even higher spread and would face an even higher probability of a self-fulfilling crisis than it would if it were at or below $\bar{b}(1)$.

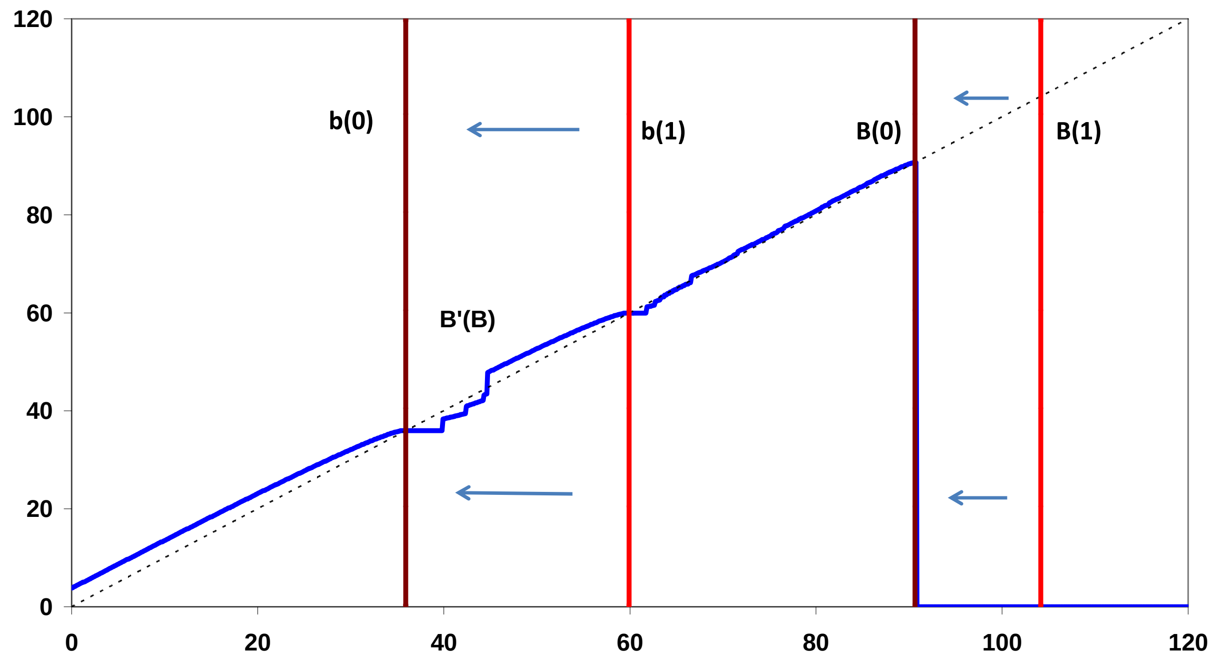

Figure 13: Policy function in recession

Initial debt in the region between $\bar{b}(1)$ and $\bar{B}(0)$ are debt levels for economies that were vulnerable before the crisis. The interest rate stays the same as before. Some economies with levels of debt above but close to 60 find it optimal to reduce their level of debt and wait for a recovery. For levels of debt above a level equal to about 70, the optimal policy is gambling for redemption all the way to the upper threshold. Notice that a default cost of 5 percent is large enough, and the probability of a recovery small enough, to deter the governments in these countries from gambling one more period and risking default if a recovery does not happen.

Finally, in the region between $\bar{B}(0)$ and $\bar{B}(1)$, we find economies that in good times could refinance their debt by paying a high enough interest rate as long as investors were willing to roll over their debt, but that, as soon as the economy hits a recession, become insolvent and default immediately.

Notice that the model accommodates all types of behavior: economies increasing their levels of debt at high, intermediate, and zero spreads, whereas other economies choose to reduce their levels of debt to reduce their interest payments as in Cole and Kehoe (2000).

The maximum level of sustainable debt is small relative to the magnitudes we are observing in some of the economies we consider, however. As discussed above, a key parameter in our exercise is the cost of default. Higher costs of default allow the economy to sustain higher levels of debt. Figure 14 displays the policy function in a recession with a default cost of 10 percent, $Z = 0.90$. In this case, the lower debt thresholds increase by a barely perceptible amount, but the upper threshold increases from 104 to 149 in normal times and from 91 to 132 in recessions. The main conclusions are unaffected, but the upper thresholds are significantly higher. In contrast, lowering the discount factor makes governments less patient, and therefore the upper thresholds shift to the left, making the amount of sustainable debt smaller.

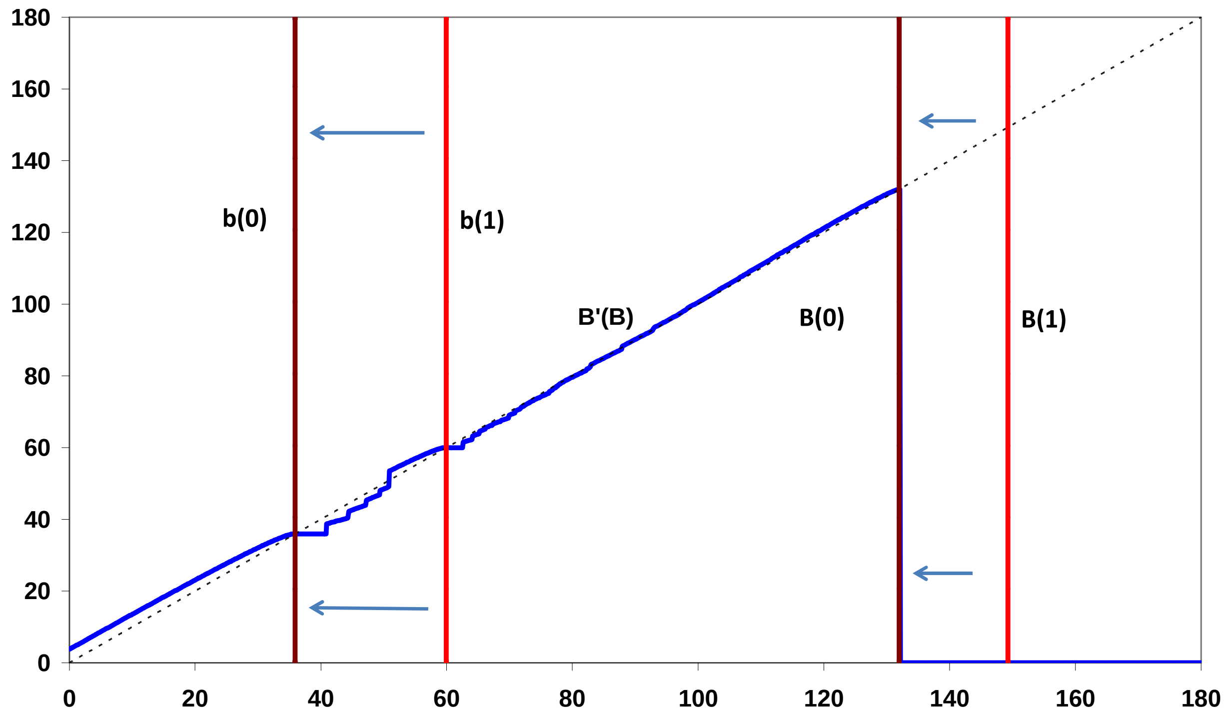

Figure 14: Policy function in recession with higher default penalty, $Z = 0.90$

5.1. The role of the maturity structure of debt

To understand the role of debt maturity, we can solve the model for different values of δ. As δ is made smaller, the lower and the upper thresholds converge because the government does not need to roll over debt, only pay debt service, as discussed in Cole and Kehoe (1996). In figures 15 and 16, we present the results for $\delta = 1.0$ and $\delta = 0.5$.

As δ becomes progressively larger, that is, the maturity of debt is lower, the thresholds decrease. In fact, the levels of the thresholds become more in line with the experience of emerging economies that borrow short term, where only low levels of debt are sustainable and rollover crises can emerge for low levels of debt. In contrast, as δ becomes smaller, the lower and upper thresholds converge as rollover risk disappears and very large levels of debt can be sustained — larger than 200 percent of GDP when $\delta = 0.05$.

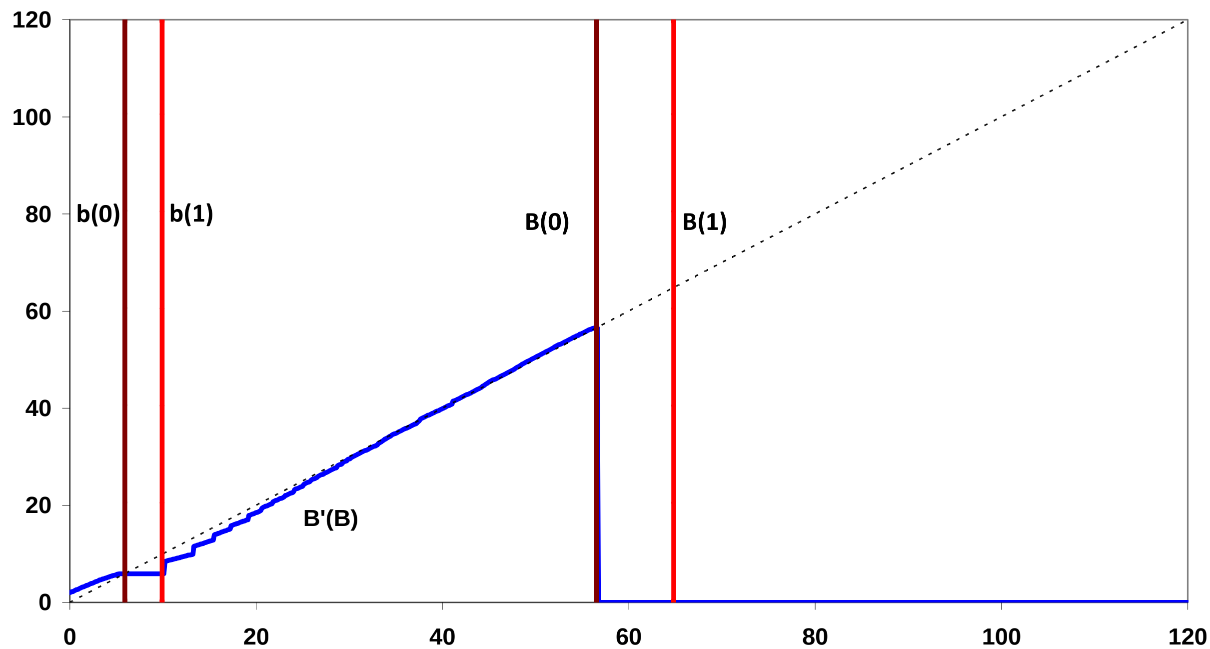

Figure 15: Policy function in recession for short maturity bonds, $\delta = 1.0$

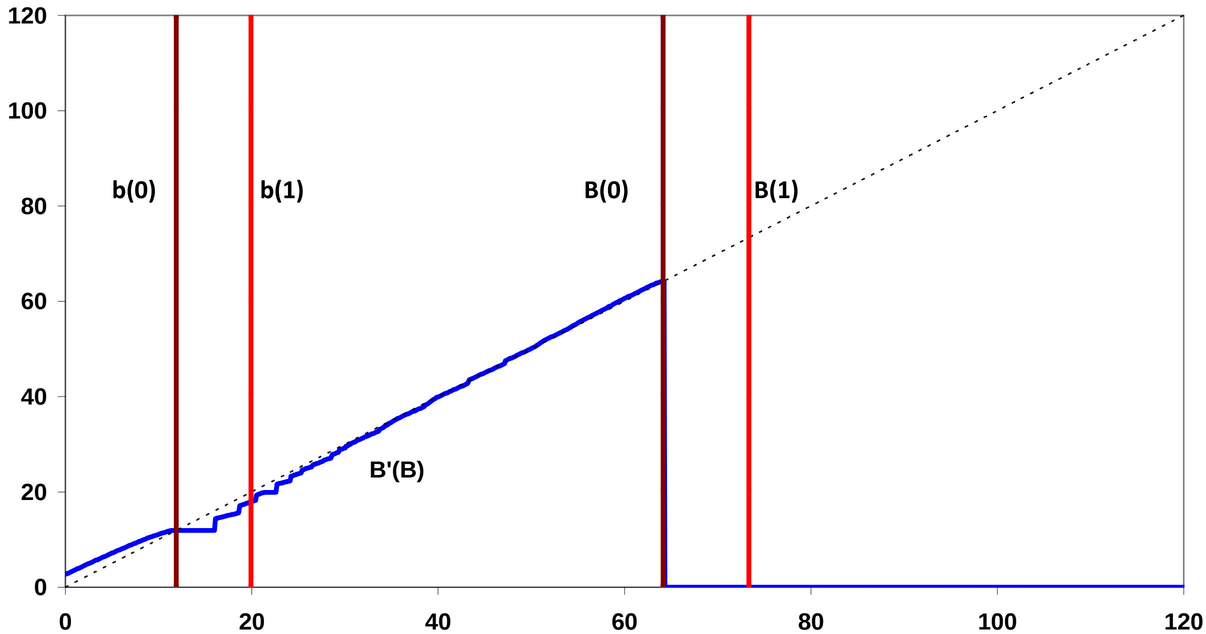

Figure 16: Policy function in recession for short maturity bonds, $\delta = 0.5$

6. Conclusions

We provide a theory that accounts for governments optimally increasing their levels of debt even in situations in which increasing debt levels makes them more vulnerable to a sovereign debt crisis. The key trade-off is between, on the one hand, decreasing debt to avoid the risk premium and eliminate the possibility of a self-fulfilling crisis and, on the other hand, increasing debt to smooth consumption when faced with a recession of uncertain recovery. If the consumption smoothing effect dominates, we call this optimal policy gambling for redemption, and our model can relate this feature of the equilibrium to the fundamentals of the economy and the maturity structure of debt.

We have restricted our analysis in this paper to a game between a government and atomistic international investors given the set of fundamentals. We have left two obvious extensions out of the paper. First, we could introduce into the game a third party with deep pockets that is able to bail out the government if a panic occurs. Roch and Uhlig (2012) and Conesa and Kehoe (2014) analyze the optimal debt policy in a model with bailouts or the possibility of bailouts. Second, we could extend our analysis to study a game in which the government can endogenously choose to engage in costly reforms to change the fundamentals.

References

Aguiar, M., and M. Amador (2014), "Take the Short Route: How to Repay and Restructure Sovereign Debt with Multiple Maturities," unpublished manuscript, Princeton and Federal Reserve Bank of Minneapolis.

Aguiar, M., and M. Amador (2015), "Sovereign Debt," in *Handbook of International Economics*, Vol. 4, ed. E. Helpman, K. Rogoff, and G. Gopinath, 647–687, Amsterdam: North-Holland.

Aguiar, M., and G. Gopinath (2006), "Defaultable Debt, Interest Rates and the Current Account," *Journal of International Economics*, 69, 64–83.

Aiyagari, S. R. (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, 109, 659–684.

Arellano, C. (2008), "Default Risk and Income Fluctuations in Emerging Economies," *American Economic Review*, 98, 690–712.

Arellano, C., J. C. Conesa, and T.J. Kehoe (2012), "Chronic Sovereign Debt Crises in the Eurozone, 2010–2012," *Federal Reserve Bank of Minneapolis Economic Policy Paper*, 12-4.

Arellano, C., and A. Ramanarayanan (2012), "Default and the Maturity Structure in Sovereign Bonds," *Journal of Political Economy*, 120, 187–232.

Calvo, G. A. (1988), "Servicing the Public Debt: The Role of Expectations," *American Economic Review*, 78, 647–661.

Chamley, C. P., and B. Pinto (2011), "Why Bailouts Tend Not to Work: An Example Motivated by Greece 2010," *The Economists' Voice*, 8, article 3.

Chatterjee, S., D. Corbae, M. Nakajima, and J. V. Ríos-Rull (2007), "A Quantitative Theory of Unsecured Consumer Credit with Risk of Default," *Econometrica*, 75, 1525–1589.

Chatterjee, S., and B. Eyigungor (2012), "Maturity, Indebtedness, and Default Risk," *American Economic Review*, 102, 2674–2699.

Cole, H. L., and T. J. Kehoe (1996), "A Self-Fulfilling Model of Mexico's 1994–1995 Debt Crisis," *Journal of International Economics*, 41, 309–330.

Cole, H. L., and T. J. Kehoe (2000), "Self-Fulfilling Debt Crises," *Review of Economic Studies*, 67, 91–116.

Conesa, J.C. and T. J. Kehoe (2014), "Is It Too Late to Bail Out the Troubled Countries in the Eurozone?" *American Economic Review: Papers and Proceedings*, 104, 88–93.

Hatchondo, J. C., and L. Martinez (2009), "Long-Duration Bonds and Sovereign Defaults," *Journal of International Economics*, 79, 117–125.

Hatchondo, J. C., L. Martinez, and C. Sosa-Padilla (2014), "Debt Dilution and Sovereign Default Risk," unpublished manuscript, Indiana University, International Monetary Fund, and McMaster University.

Huggett, M. (1993), "The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control*, 17, 953–969.

Kehoe, T. J., and E. C. Prescott (2002), “Great Depressions of the 20th Century,” *Review of Economic Dynamics*, 5, 1–18.

Livshits, I., J. MacGee, and M. Tertilt (2007), “Consumer Bankruptcy: A Fresh Start,” *American Economic Review*, 97, 402–418.

Lorenzoni, G., and I. Werning (2013), “Slow Moving Debt Crises,” unpublished manuscript, Northwestern University and Massachusetts Institute of Technology.

Mendoza, E. G., and V. Z. Yue (2012), “A General Equilibrium Model of Sovereign Default and Business Cycles,” *Quarterly Journal of Economics*, 127, 889–946.

Reinhart, C. M., and K. S. Rogoff (2009), *This Time Is Different: Eight Centuries of Financial Folly*, Princeton, NJ: Princeton University Press.

Roch, F., and H. Uhlig (2012), “The Dynamics of Sovereign Debt Crises and Bailouts,” unpublished manuscript, University of Chicago.

Sosa-Padilla, C. (2014), “Sovereign Defaults and Banking Crises,” unpublished manuscript, McMaster University.

Appendix: The algorithm for computing an equilibrium in the general model

The algorithm computes the four debt thresholds, the value functions, and the policy functions.

1. Compute the value function $V\left(B,a,z_{-1},\zeta\right)$ of being in the default state, where $B=0$ and $z_{-1}=0$. To simplify notation we denote it $V_d\left(a\right)$. Notice that these values are independent of the sunspot ζ, which becomes irrelevant after a default has occurred. The value function of defaulting in normal times, where $a=1$, is

$$V_d(1)=u(\left(1-\theta\right)Zy,\theta Zy)+\beta V_d(1), \qquad \text{(A.1)}$$

which is just a constant:

$$V_d(1)=\frac{1}{1-\beta}u((1-\theta)Zy,\theta Zy)\,. \qquad \text{(A.2)}$$

Similarly, in a recession, where $a=0$,

$$V_d(0)=u((1-\theta)AZy,\theta AZy)+\beta pV_d(1)+\beta(1-p)V_d(0) \qquad \text{(A.3)}$$

which is also a constant:

$$\begin{aligned}V_d(0)=&\frac{1}{1-\beta\left(1-p\right)}u((1-\theta)AZy,\theta AZy)\\&+\frac{\beta p}{\left(1-\beta\left(1-p\right)\right)\left(1-\beta\right)}u((1-\theta)Zy,\theta Zy)\end{aligned}\,. \qquad \text{(A.4)}$$

Notice that the value functions become those obtained above whenever the state of the economy determines that a self-fulfilling debt crisis happens or has happened in the past. To simplify notation, from this point on, we describe how to compute the value functions in the case of no default and drop the variable z_{-1} that determines whether a government has defaulted in the past and the sunspot ζ as arguments of the value functions. That is, from this point on, $V(B,a)$ is the value function if default has not happened today or anytime in the past.

2. Guess initial values for the thresholds $\bar{b}(0)$, $\bar{b}(1)$, $\bar{B}(0)$, $\bar{B}(1)$, where $\bar{b}(0) < \bar{b}(1) < \bar{B}(0) < \bar{B}(1)$, and the associated prices. (We could also modify the algorithm to calculate an equilibrium in the case where $\bar{b}(0) < \bar{B}(0) < \bar{b}(1) < \bar{B}(1)$.)

3. Perform value function iteration on a finite grid of values of debt to compute the value function in normal times, $a = 1$. Guess an initial value function in normal times if default has not happened in the past, $\tilde{V}(B,1)$, and an optimal debt policy, which is needed to recursively compute the prices, $\tilde{q}(B',1)$, defined as in equation (70), in the case with multiperiod debt. Then:

3.1. For values of initial debt $B \leq \bar{B}(1)$, the value function is

$$V(B,1) = \max[V_1(B,1), V_2(B,1)], \tag{A.5}$$

where V_1, V_2 are the values functions if next period bonds are in the regions $B' \leq \bar{b}(1)$ or $\bar{b}(1) < B' \leq \bar{B}(1)$, respectively:

$$\begin{aligned} V_1(B,1) &= \max u((1-\theta)y, \theta y + \tilde{q}(B',1)(B' - (1-\delta)B) - \delta B) + \beta \tilde{V}(B',1) \\ &\quad \text{s.t. } B' \leq \bar{b}(1) \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} V_2(B,1) &= \max u((1-\theta)y, \theta y + \tilde{q}(B',1)(B' - (1-\delta)B) - \delta B) + \beta(1-\pi)\tilde{V}(B',1) + \beta\pi V_d(1) \\ &\quad \text{s.t. } \bar{b}(1) < B' \leq \bar{B}(1). \end{aligned} \tag{A.7}$$

3.2. For high values of initial debt, $B > \bar{B}(1)$, set $V(B,1) = V_d(1)$.

3.3. If there is multiperiod debt, compute the pricing function, $q(B,1)$, recursively, as in equation (70), using the optimal policy function. If there is one-period debt, simply use equation (12).

3.4. If $\max_B \left| V(B,1) - \tilde{V}(B,1) \right| > \varepsilon$ and $\max_B \left| q(B,1) - \tilde{q}(B,1) \right| > \varepsilon$, where ε is a preset convergence criterion, then $\tilde{V}(B,1) = V(B,1)$ and $\tilde{q}(B,1) = q(B,1)$ and go to 3.1. Else, go to 4.

4. Perform value function iteration on a finite grid of values of debt to compute the value function in a recession, $a = 0$. Guess an initial value function if we are in a recession and the

government has not defaulted in the past: $\tilde{V}(B,0)$, and an optimal debt policy, which is needed to recursively compute the prices, $\tilde{q}(B',0)$, defined as in equation (71). Remember the value function in normal times, $V(B,1)$, is already known from step 3. Then:

4.1. For values of initial debt where $B \le \bar{B}(0)$, the value function is

$$V(B,0) = \max[V_1(B,0), V_2(B,0), V_3(B,0), V_4(B,0)], \tag{A.8}$$

where V_1, V_2, V_3, V_4 are the associated value functions if next period bonds are in the regions $B' \le \bar{b}(0)$, $\bar{b}(0) < B' \le \bar{b}(1)$, $\bar{b}(1) < B' \le \bar{B}(0)$, $\bar{B}(0) < B' \le \bar{B}(1)$, respectively:

$$\begin{aligned} V_1(B,0) = \max u((1-\theta)Ay, \theta Ay + \tilde{q}(B',0)(B' - (1-\delta)B) - \delta B) \\ + \beta p V(B',1) + \beta(1-p)\tilde{V}(B',0) \\ \text{s.t. } B' \le \bar{b}(0) \end{aligned} \tag{A.9}$$

$$\begin{aligned} V_2(B,0) = \max u((1-\theta)Ay, \theta Ay + \tilde{q}(B',0)(B' - (1-\delta)B) - \delta B) \\ + \beta p V(B',1) + \beta(1-p)\pi V_d(0) + \beta(1-p)(1-\pi)\tilde{V}(B',0) \\ \text{s.t. } \bar{b}(0) < B' \le \bar{b}(1) \end{aligned} \tag{A.10}$$

$$\begin{aligned} V_3(B,0) = \max u((1-\theta)Ay, \theta Ay + \tilde{q}(B',0)(B' - (1-\delta)B) - \delta B) \\ + \beta p \pi V_d(1) + \beta p(1-\pi)V(B',1) \\ + \beta(1-p)\pi V_d(0) + \beta(1-p)(1-\pi)\tilde{V}(B',0) \\ \text{s.t. } \bar{b}(1) < B' \le \bar{B}(0) \end{aligned} \tag{A.11}$$

$$\begin{aligned} V_4(B,0) = \max u((1-\theta)Ay, \theta Ay + \tilde{q}(B',0)(B' - (1-\delta)B) - \delta B) \\ + \beta p \pi V_d(1) + \beta p(1-\pi)V(B',1) + \beta(1-p)V_d(0) \\ \text{s.t. } \bar{B}(0) < B' \le \bar{B}(1). \end{aligned} \tag{A.12}$$

4.2. For high values of initial debt, $B > \bar{B}(0)$, set $V(B,0) = V_d(0)$.

4.3. If there is multiperiod debt, compute the pricing function, $q(B,0)$, recursively, as in equation (71), using the optimal policy function. If there is one-period debt, simply use equation (11).

4.4. If $\max_B \left|V\left(B,0\right)-\tilde{V}\left(B,0\right)\right| > \varepsilon$ and $\max_B \left|q\left(B,0\right)-\tilde{q}\left(B,0\right)\right| > \varepsilon$, then $\tilde{V}\left(B,0\right)=V\left(B,0\right)$ and $\tilde{q}\left(B,0\right)=q\left(B,0\right)$ and go to 4.1. Else, go to 5.

5. Update the threshold values:

5.1. Choose $\bar{b}_{new}(0)$ to be the highest point in the grid for B for which

$$u((1-\theta)Ay,\theta Ay-\delta B)+\beta pV((1-\delta)B,1)+\beta(1-p)V((1-\delta)B,0)\geq V_d(0)\,. \quad \text{(A.13)}$$

5.2. Choose $\bar{b}_{new}(1)$ to be the highest point in the grid for which

$$u((1-\theta)y,\theta y-\delta B)+\beta V((1-\delta)B,1)\geq V_d(1)\,. \quad \text{(A.14)}$$

5.3. Choose $\bar{B}_{new}(0)$ to be the highest point in the grid for which

$$\begin{aligned} V\left(B,0\right) &\geq u((1-\theta)ZAy,\theta ZAy+q(B',0)(B'(B,0)-(1-\delta)B)) \\ &\quad +\beta pV_d(1)+\beta(1,p)V_d(0), \end{aligned} \quad \text{(A.15)}$$

where $q(B',0)$ are the prices computed in step 5.

5.4. Choose $\bar{B}_{new}(1)$ to be the highest point in the grid for which

$$V(B,1)\geq u((1-\theta)Zy,\theta Zy+q(B',1)(B'(B,1)-(1-\delta)B))+\beta V_d(1)\,. \quad \text{(A.16)}$$

5.5. If $\left|\bar{b}_{new}(0)-\bar{b}(0)\right|>\varepsilon$ or $\left|\bar{b}_{new}(1)-\bar{b}(1)\right|>\varepsilon$ or $\left|\bar{B}_{new}(0)-\bar{B}(0)\right|>\varepsilon$ or $\left|\bar{B}_{new}(1)-\bar{B}(1)\right|>\varepsilon$, then $\bar{b}(0)=\bar{b}_{new}(0),\bar{b}(1)=\bar{b}_{new}(1),\bar{B}(0)=\bar{B}_{new}(0),\bar{B}(1)=\bar{B}_{new}(1)$ and go to 3. Else, exit.

Notice that the lower threshold in normal times, $\bar{b}(1)$, can be computed directly using condition (36) if there is one-period debt or condition (73) if there is multiperiod debt, since no information about policy functions is required. Hence, the iterative procedure would not be necessary, but we choose to do it this way to be consistent with the computation of the upper thresholds and the lower threshold in recession, which do depend on the policy function and hence require an iterative procedure.