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Computation and Multiplicity of Economic Equilibria

1. INTRODUCTION

The principal model in economic theory is the Walrasian model of general economic equilibrium. In it, consumers choose to demand and supply goods to maximize a utility function subject to the constraint that the value of what they demand must equal the value of what they supply. Producers choose to demand and supply goods to maximize profits subject to the restrictions of a production technology. An equilibrium of this model is a vector of prices, one for each good, which the agents all take as given in solving their maximization problems, such that demand is equal to supply for every good. Economists use this type of model to do comparative statics analysis: they first compute a benchmark equilibrium for the model; they then change a parameter of the model such as a tax rate; finally they compare the new equilibrium with the benchmark. Large-scale empirical models are often used to do policy analysis (see, for example, Kehoe and Serra-Puche, 1983, and Shoven and Whalley, 1984).

In this paper we study several simple, highly stylized versions of the Walrasian equilibrium model. (Debreu, 1959; Arrow and Hahn, 1971; and Mas-Colell, 1985 are good references on the mathematics of general equilbrium theory.) Our emphasis is on the formal properties of these models that make them different from many models in the physical sciences, rather than on the economics of the models. In particular, we discuss the problems involved in proving the existence of equilibria and of computing equilibria. We give special attention to the possibility of multiplicity of equilibria.

2. EXISTENCE OF EQUILIBRIUM AND BROUWER'S FIXED-POINT THEOREM

We begin with the simplest possible model, an *exchange economy* in which the only economic activity is exchange of goods among consumers. We discuss models with production later. There are m consumers and n goods. Consumer i, $i = 1, 2, \ldots, m$, is endowed with nonnegative amounts, $w^i = (w^i_1, w^i_2, \ldots, w^i_n)$, of goods. He also has a *utility function* $u_i : R^n_+ \rightarrow R$ that is strictly concave and monotonically increasing. When faced by the price vector $p \in R^n_+$, he chooses the consumption plan x^i to maximize $u_i(x)$ subject to the *budget constraint* $p \cdot x \leq p \cdot w^i$ and nonnegativity constraint $x \geq 0$.

The solution to this problem, the consumer's demand function $x^i(p)$, is continuous (at least for strictly positive prices); is homogeneous of degree zero, $x^i(\lambda p) = x^i(p)$ for all $\lambda > 0$ and all p; and satisfies the budget constraint, $p \cdot x^i(p) = p \cdot w^i$ for all p. The *aggregate excess demand function*,

$$f(p) = \sum_{i=1}^{m} (x^i(p) - w^i),$$

therefore is continuous, is homogeneous of degree zero, and satisfies $p \cdot f(p) = 0$ for all p. This final property is known as *Walras' law* and is as close as economics comes to having a law of conservation.

Our price domain is $R^n_+ \backslash \{0\}$, the set of all nonnegative price vectors except the origin. There is often a technical problem with continuity of excess demand when some prices approach zero. Some, but not necessarily all, of the corresponding excess demands might then approach infinity. There are simple ways to get around this problem, however (see, for example, Kehoe, 1982). Since nothing of conceptual significance is involved, we shall ignore any problems posed by discontinuity of demand due to zero prices.

An *equilibrium* is a price vector $\hat{p}$ for which $f(\hat{p}) \leq 0$. Notice that Walras' law implies that $f_j(\hat{p}) < 0$ only if $\hat{p}_j = 0$; in other words, we allow supply to exceed demand for a good only if it is free. Walras (1874) himself had two arguments for the existence of equilibrium: first, he counted equations and unknowns in the definition of equilibrium and verified that they are equal. Second, he proposed a dynamic adjustment process that would bring the system into equilibrium from an arbitrary starting price vector. Although each of these approaches provided important insights, neither is correct.

The system $f(\hat{p}) = 0$ involves n equations and n unknowns. Walras recognized that there are two offsetting complications: because of Walras' law, one equation is redundant, but because of homogeneity we can impose a price normalization. In other words, we can either eliminate a variable by choosing a numeraire, a good in whose terms all values are measured, by setting, for example, $\hat{p}_1 = 1$, or we can add an equation, such as $\sum_{i=1}^{n} \hat{p}_i = 1$. Consequently, we are left with a system with the same number of equations and unknowns. Although this clearly does not, as Walras seems to have thought it does, assure us of the existence of a solution, it does, as we shall see, tell us something about local uniqueness of equilibria.

The second approach to existence followed by Walras was a disequilibrium adjustment process that he called *tâtonnement*, or groping. In it, an auctioneer adjusts prices systematically by raising the prices of goods in excess demand and lowering those of goods in excess supply. Samuelson (1941, 1942) formalized this process as the system of differential equations

$$\frac{dp_j}{dt} = f_j(p(t)).$$

(Walras himself thought of tâtonnement more as a nonlinear Gauss-Seidel method.)

This approach was popular in economics for a time, and many economists searched for conditions under which it leads to convergence (see, for example, Arrow, Block, and Hurwicz, 1959). There are problems in giving the process a real-time interpretation, however, and, in any case, it became less popular after Scarf (1960) constructed an example in which, unless $p(0) = \hat{p}$, the unique equilibrium of his example, the solution converges to a limit cycle.

In a series of papers of increasing generality, Sonnenschein (1973), Mantel (1974), and Debreu (1974) proved that the excess demand function f is arbitrary except for continuity, homogeneity, and Walras' law. Specifically, Debreu proved that for any function f that satisfies these properties, there are n consumers with strictly concave, monotonically increasing utility functions whose individual excess demands sum to f. To see that this implies that the tâtonnement process is arbitrary, let us use homogeneity to normalize prices to lie on the intersection of the unit sphere and the positive orthant, $\sum_{i=1}^{n} p_i^2 = 1$, $p_i \geq 0$. Walras' law implies that $f(p)$ defines a vector field on the sphere:

$$\frac{d}{dt}\left(\sum_{i=1}^{n} p_i(t)^2\right) = 2\sum p_i(t)\,\frac{dp_i}{dt} = 2\sum p_i(t) f_i(p(t)) = 0.$$

Consequently, continuity, homogeneity, and Walras' law imply no more than that the tâtonnement process defines a continuous vector field on the sphere. The solution for an initial condition $p(0) = p_0$ is fairly arbitrary. With three goods, for example, there can be a stable limit cycle, as in Scarf's (1960) example. See Figure 1. With four or more goods, the tâtonnement process can generate chaotic dynamics.

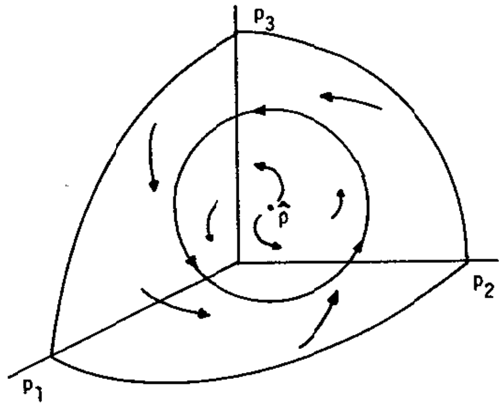

FIGURE 1 Scarf's example.

Although Wald (1936) had proved the existence of equilibria for two special models earlier, Arrow and Debreu (1954) and McKenzie (1954) realized that the existence of equilibria in more general models could be demonstrated using ***Brouwer's fixed-point theorem*** or some variant. Brouwer's theorem says that a continuous map $g : S \rightarrow S$ of some nonempty, compact, convex set S into itself leaves some point fixed, that is, $\hat{x} = g(\hat{x})$ for some $\hat{x} \in S$.

Renormalizing prices, we can use the simplex $S = \{p \in R^n \mid e \cdot p = 1, p \geq 0\}$ as our price domain. (Here $e = (1, \ldots, 1)$.) Walras' adjustment process suggests $g(p) = p + f(p)$ as a map whose fixed points are equilibrium. The problem is that $g(p)$ is not necessarily in S for all $p \in S$. For any $q \in R^n$, let $\pi(q)$ be that point in S that is closest to q in euclidean distance. Define

$$g(p) = \pi(p + f(p)).$$

Since S is convex, π and, therefore, g are continuous.

We claim that $f(\hat{p}) \leq 0$ if and only if $\hat{p} = g(\hat{p})$. To prove this, notice $g(p)$ solves the problem

$$\min (1/2)\, \|g - p - f(p)\|^2,$$
$$\text{s.t. } e \cdot g = 1$$
$$g \geq 0$$

if and only if

$$g - p - f(p) + \lambda e \geq 0$$
$$g \cdot (g - p - f(p) + \lambda e) = 0$$

for some Lagrange multiplier λ. Suppose that $\hat{p} = g(\hat{p})$. Walras' law implies that $0 = \hat{p} \cdot f(\hat{p}) = \hat{\lambda} \hat{p} \cdot e = \hat{\lambda}$. Consequently, $f(\hat{p}) \leq 0$. To prove the converse that $f(\hat{p}) \leq 0$ implies $g(\hat{p}) = \hat{p}$, merely requires setting $g = p$ and $\lambda = 0$ in conditions that define $g(p)$.

The relationship between the existence theorem and Brouwer's theorem is very close: as we have just argued, the existence of equilibrium follows quickly from Brouwer's theorem. Furthermore, as Uzawa (1962) first noticed, Brouwer's theorem follows quickly from the existence theorem. Suppose that $g : S \rightarrow S$ is continuous. We can construct an excess demand function f that satisfies continuity, homogeneity, and Walras' law and is such that $\hat{p} = g(\hat{p})$ if and only if $f(\hat{p}) \leq 0$. Since we know that aggregate excess demand is arbitrary except for these properties, it follows that, if we know that an equilibrium exists for every economy, then we know Brouwer's theorem. Consequently, except for special cases, Brouwer's theorem or its equivalent is necessary for proving the existence of equilibrium.

An obvious candidate for the excess demand function based on g is $f(p) = g(p) - p$; the problems are that it does not satisfy homogeneity or Walras' law. Homogeneity is trivial, however: if we have a function $f : S \rightarrow R^n$, we can define f on all $R^n_+ \backslash \{0\}$ as $f(\pi(p))$, where, of course, $\pi(p) = (1/e \cdot p)p$. Let us, therefore, without loss of generality, restrict ourselves to the price domain S. To make f obey Walras' law, we define $\lambda(p) = p \cdot g(p)/p \cdot p$ and set

$$f(p) = g(p) - \lambda(p)p.$$

Suppose now that $f(\hat{p}) \leq 0$. This implies that, in fact, $f(\hat{p}) = 0$ since, by Walras' law, $f_j(\hat{p}) < 0$ would imply $\hat{p}_j = 0$, in which case $f_j(\hat{p}) = g_j(\hat{p}) \geq 0$. That $f(\hat{p}) = 0$, however, implies that $g(\hat{p}) = \lambda(\hat{p})\hat{p}$. Since $e \cdot g(\hat{p}) = e \cdot \hat{p} = 1$, this implies that $\lambda(\hat{p}) = 1$ and, therefore, $g(\hat{p}) = \hat{p}$. The converse, that $g(\hat{p}) = \hat{p}$ implies $f(\hat{p}) = 0$, follows from $\hat{p} \cdot g(\hat{p})/\hat{p} \cdot \hat{p} = 1$.

3. ECONOMIC EQUILIBRIUM AND OPTIMIZATION

Unlike many problems in the physical sciences, the economic equilibrium problem cannot usually be solved as an optimization problem: although an equilibrium is the solution to an optimization problem, finding the right optimization problem is, in general, as difficult as finding the equilibrium itself.

Pareto (1909) first realized that the allocation of goods $(\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^m)$ associated with an equilibrium $\hat{p}$ has a property now known as *Pareto efficiency*. There is no alternative allocation $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m)$ that is superior in the sense that is feasible, $\sum_{i=1}^m \bar{x}^i \leq \sum_{i=1}^m w^i$, and $u_i(\bar{x}^i) \geq u_i(\hat{x}^i)$, $i = 1, 2, \ldots, m$, with strict inequality some i. In other words, there is no way to reallocate goods to make some consumer better off that does not make another consumer worse off. The argument, due to Arrow (1951) and Debreu (1954), is simple: suppose to the contrary, that there is an allocation superior to the competitive allocation. Then $\hat{p} \cdot \bar{x}^i \geq \hat{p} \cdot \hat{x}^i = \hat{p} \cdot w^i$ with strict inequality wherever $u_i(\bar{x}^i) > u_i(\hat{x}^i)$; otherwise, $\hat{x}^i$ would not maximize utility subject to the budget constraint. Consequently, $\sum_{i=1}^m \hat{p} \cdot \bar{x}^i > \sum_{i=1}^m \hat{p} \cdot w^i$.

Multiplying the feasibility conditions by $\hat{p}$, however, yields $\hat{p}\cdot\sum_{i=1}^{m}\bar{x}^i \leq \hat{p}\cdot\sum_{i=1}^{m} w^i$, which can be rearranged as $\sum_{i=1}^{m}\hat{p}\cdot\bar{x}^i \leq \sum_{i=1}^{m}\hat{p}\cdot w^i$. This contradiction shows that there can be no allocation that is Pareto superior to the competitive allocation.

Since an equilibrium is Pareto efficient, the associated allocation solves

$$\max \sum_{i=1}^{m} \alpha_i u_i(x^i)$$
$$\text{s.t.} \sum_{i=1}^{m} x^i \leq \sum_{i=1}^{m} w^i$$
$$x^i \geq 0$$

for some nonnegative weights α_i. How do we find the right weights α_i? Associated with the feasibility conditions are nonnegative Lagrange multipliers $p(\alpha) = (p_1(\alpha), p_2(\alpha), \ldots, p_n(\alpha))$. It is easy to show that, for any vector α, the prices $p(\alpha)$ and allocation $(x^1(\alpha), x^2(\alpha), \ldots, x^m(\alpha))$ satisfy all of the conditions for an equilibrium except the individual budget constraints. In the case where u_i is continuously differentiable, for example, this is simply a matter of showing that the necessary and sufficient conditions for a solution to this problem can be rearranged into the conditions for utility maximization. If we give each consumer a net transfer given by the *transfer function*

$$t_i(\alpha) = p(\alpha)\cdot(x^i(\alpha) - w^i), \qquad i = 1, 2, \ldots, m,$$

then $p(\alpha)$ is an equilibrium of the economy with transfer payments. In this economy budget constraints have the form $p\cdot x^i \leq p\cdot w^i + t_i(\alpha)$.

To find an equilibrium, we must find a vector $\hat{\alpha}$ such that $t(\hat{\alpha}) = 0$. The transfer functions t are continuous, homogeneous of degree one, and sum to zero. The functions $f_i(\alpha) = -t_i(\alpha)/\alpha_i$, in fact, have the same formal properties as excess demand functions. Finding an equilibrium using this approach, which is due to Negishi (1960), involves solving a fixed-point problem in R^m rather than R^n. This is sometimes useful if $m < n$, for example, if m is finite and n is infinite.

4. COMPUTATION OF EQUILIBRIA

Scarf (1967, 1973, 1982) realized that any algorithm that could be guaranteed to compute economic equilibria would have to be able to compute fixed points of arbitrary maps $g : S \to S$. He developed such an algorithm. Numerous researchers have further improved algorithms of this type, now known as *simplicial algorithms* (see, for example, Eaves, 1972; Todd, 1976; and van der Laan and Talman, 1980).

In R^n, a k-dimensional *simplex* is the convex hull of $k+1$ points, called *vertices*, $v^1, v^2, \ldots, v^{k+1}$, that have the property that the k vectors $v^1 - v^{k+1}, \ldots, v^k - v^{k+1}$ are linearly independent. The price simplex S, for example, has vertices e^i, $i = 1, 2, \ldots, n$, where $e^i_i = 1$, $e^i_j = 0$, $j \neq i$. A *face* of a simplex is a lower-dimensional simplex whose vertices are vertices of the large simplex. In R^3, for example, the point e^1 is a zero-dimensional face of S and the convex hull of e^1 and e^2 is a one-dimensional face. A *subdivision* of S divides S into smaller simplices so that every point in S is an element of some subsimplex and the intersection of any two subsimplices is either empty or a face of both.

Scarf's approach to computation of equilibria is based on a constructive proof of a version of *Sperner's lemma*: assign to every vertex of a simplicial subdivision of S a *label*, an integer from the set $1, 2, \ldots, n$, with the property that a vertex v on the boundary of S receives a label i for which $v_i = 0$. Then there exists a subsimplex whose vertices have all of the labels $1, 2, \ldots, n$.

Scarf's algorithm for finding this completely labeled subsimplex is to start in the corner of S where there is a subsimplex with boundary vertices with all of the labels $2, 3, \ldots, n$. See Figure 2. If the additional vertex of this subsimplex has the label 1, then the algorithm stops. Otherwise, it proceeds to a new subsimplex with all of that labels $2, 3, \ldots, n$: the original subsimplex has two faces that have all of these labels. One of them includes the interior vertex. The algorithm moves to the unique other subsimplex that shares this face. If the additional vertex of this subsimplex has the label 1, the algorithm stops. Otherwise, it proceeds, moving to

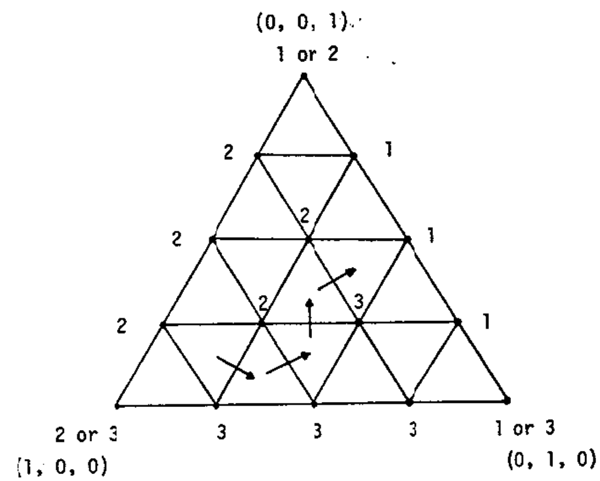

FIGURE 2 Scarf's algorithm for finding completely labeled subsimplex.

the unique subsimplex that shares the new face and has the labels $2, 3, \ldots, n$. The algorithm cannot try to exit through a boundary face. (Think of what labels the vertices of such a face must have.) Nor can it cycle. (To cycle there must be some subsimplex that is the first that the algorithm encounters for the second time, but the algorithm must have previously encountered both of the subsimplices that share the two faces of this subsimplex with the labels $2, 3, \ldots, n$.) Since the subdivision consists of a finite number of subsimplices, the algorithm must terminate with a completely labeled subsimplex.

To see the connection of this algorithm with Brouwer's theorem, we assign a vertex v with a label i for which $g_i(v) \geq v_i$. Since $e \cdot g(v) = e \cdot v = 1$, there must be such an i. Notice that, since $g_i(v) \geq 0$, i can be chosen such that the labeling convention on the boundary is satisfied. A completely labeled subsimplex has vertices $v^1, v^2, \ldots, v^n$ such that $g_i(v^i) \geq v^i_i$. To prove Brouwer's theorem, we consider a sequence of subdivisions whose *mesh*, the maximum distance between vertices in the same subsimplex, approaches zero. Associate each subdivision with a point in a completely labeled subsimplex. Since S is compact, this sequence of points has a convergent subsequence. Call the limit of this subsequence $\hat{p}$. Since g is continuous, we know $g_i(\hat{p}) \geq \hat{p}_i$, $i = 1, 2, \ldots, n$. Since $e \cdot g(\hat{p}) = e \cdot \hat{p} = 1$, $g(\hat{p}) = \hat{p}$.

Scarf did not consider an infinite sequence of subdivisions, which is the nonconstructive aspect of this proof. Instead, he worked with a subdivision with a small mesh. Any point in a completely labeled subsimplex serves as an approximate fixed point in the sense that $\|g(x) - x\| < \varepsilon$ where ε depends on the mesh and the modulus of continuity of g.

An alternative algorithm for computing fixed points was developed by Smale (1976), who called it the *global Newton's method*. It is based on Hirsch's (1963) proof of Brouwer's theorem. Let S now be the disk $\{x \in R^n \mid x \cdot x \leq 1\}$; like the simplex, it is a nonempty, compact, convex set. Smale developed an algorithm for computing fixed points of a continuously differentiable map $g : S \to S$ that has the property that $g(x) = 0$ for every x on the boundary of S, the sphere $\partial S = \{x \in R^n \mid x \cdot x = 1\}$. He also showed how to extend this algorithm to situations where g is an arbitrary continuous map and S is again the simplex.

If S had no fixed points, we could define a map

$$h(x) = \lambda(x)\big(x - g(x)\big)$$

where $\lambda(x) = (x - g(x)) \cdot (x - g(x))^{-1/2}$. This map would be a *retraction* of S into its boundary: it would continuously map S into ∂S and be the identity on ∂S. Hirsh proved that no such map could exist, thereby proving Brouwer's theorem. Smale proposed starting with a regular value of $x - g(x)$, a point $\bar{x} \in \partial S$ such that $I - Dg(\bar{x})$ is nonsingular. *Sard's theorem* says that the set of regular values has full measure. The algorithm then follows the solution to

$$\lambda\big(x(t)\big)\Big(x(t) - g\big(x(t)\big)\Big) = \bar{x}.$$

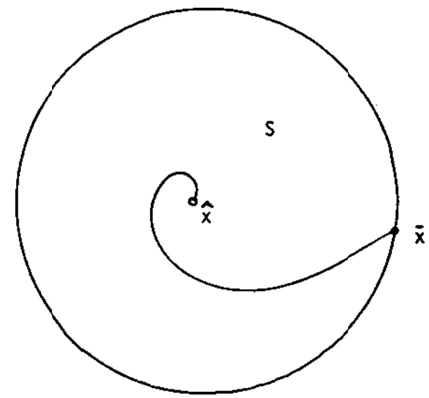

FIGURE 3 Illustration of Smale's algorithm.

Since the path $x(t)$ cannot return to any other boundary point, and since it cannot return to $\bar{x}$ because it is a regular value, it must terminate at a fixed point.

Smale shows that $x(t)$is the solution to the differential equation

$$\big[I - Dg\big(x(t)\big)\big]\frac{dx}{dt} = \mu\big(x(t)\big)\Big(x(t) - g\big(x(t)\big)\Big)$$

where $\mu(x) = \mathrm{sgn}(\det[I - Dg(x)])$. Except for the factor μ, this is a continuous version of Newton's method for solving $x - g(x) = 0$:

$$x_{t+1} = x_t - \big[I - Dg(x_t)\big]^{-1}\big(x_t - g(x_t)\big).$$

5. MODELS WITH PRODUCTION

We now generalize our model to include a simple production technology. It is specified by a $k \times n$ *activity analysis matrix* A. Columns of A represent feasible production plans: positive entries denote outputs, negative entries inputs. The set of technologically feasible production plans is $Y = \{x \in R^n \mid x = Ay,\ y \geq 0\}$. Here $y = (y_1, y_2, \ldots, y_k)$ is a vector of nonnegative *activity levels*.

This type of linear production technology, initially formalized by Koopmans (1951), but with many antecedents, is a fairly general example of a *constant returns* technology: $x \in Y$ if and only if $\lambda x \in Y$ for all $\lambda \geq 0$. Economists also sometimes work with *decreasing returns* technologies where $x \in Y$ implies $\lambda x \in Y$ for $0 \geq \lambda \geq 1$. In both cases they assume Y is a convex set. Decreasing returns are probably best thought of, both in terms of formal mathematics and of economic intuition, however, as constant returns in an economy with additional goods that are inputs to specific groups of production activities and are in fixed supply. *Increasing returns*, where Y is not convex, are another matter: they are not well handled by general equilibrium theory, since they would cause firms to become very large relative to the size of the economy, in which case the competitive assumption that all agents take prices as given is untenable. We assume that A is such that there is free disposal and that there is no output without any inputs. *Free disposal* means that, if $x \in Y$ and $x' \leq x$, then $x' \in Y$. This can be ensured by including vectors $-e^i$, $i = 1, 2, \ldots, n$ as columns in A. That $-e^i$ is a column of A means that good i can be thrown away without using other inputs. That there is no output is without any inputs means that $Y \cap R^n_+ = \{0\}$. This can be ensured by making sure that $Ay \geq 0$ and $y \geq 0$ imply $Ay = 0$. An *equilibrium* of this model is a price vector $\hat{p}$ such that, for some vector of activity levels $\hat{y} \geq 0$,

$$f(\hat{p}) = A\hat{y},$$
$$\hat{p}A \leq 0,$$
$$e \cdot \hat{p} = 1.$$

Notice that Walras' law and the first equilibrium condition imply that $\hat{p} \cdot f(\hat{p}) = \hat{p} \cdot (A\hat{y}) = 0$. In other words, the *economic profit* made by the production plan $\hat{p} \cdot (A\hat{y})$, revenue minus expenditures, is equal to zero in equilibrium. The second equilibrium condition implies that $\hat{p} \cdot (A\hat{y}) \leq 0$ for any $y \geq 0$. Consequently, the production plan

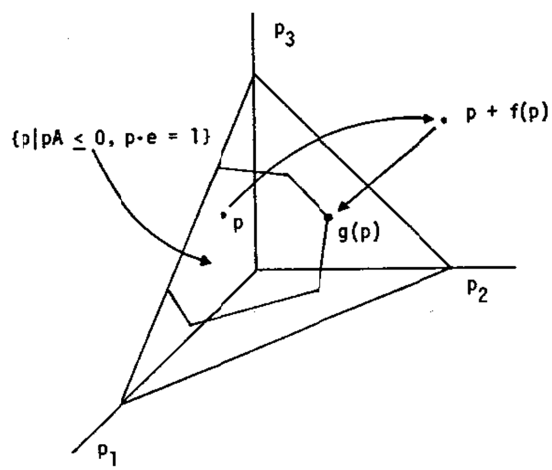

FIGURE 4 Continuous function $g : S \to S$.

$A\hat{y}$ maximizes profits at prices $\hat{p}$. (Activities in decreasing returns technology typically earn positive economic profits; these profits are best thought of as payments to sector specific fixed inputs.)

The analysis of the previous sections extends easily to models with production. To prove the existence of equilibrium, we again construct a continuous function $g : S \to S$. For any $p \in S$, we define $g(p)$ as the solution to

$$\min (1/2)\|g - p - f(p)\|^2$$
$$\text{s.t.} \quad gA \leq 0$$
$$e \cdot g = 1.$$

Notice that, because of free disposal, $gA \leq 0$ implies $g \geq 0$. g solves this problem if and only if

$$g - p - f(p) + Ay + \lambda e = 0$$
$$g \cdot (Ay) = 0$$

for some scalar λ and vector $y \geq 0$ of Lagrange multipliers. Once again Walras' law implies that $\hat{p}$ is an equilibrium if and only if $\hat{p} = g(\hat{p})$.

Equilibria of the model with production are Pareto efficient: the associated allocation and production plan solves

$$\max \sum \alpha_i u_i(x^i)$$
$$\text{s.t.} \sum_{i=1}^{m} x^i = Ay$$
$$x^i \geq 0, \quad y \geq 0.$$

Once again there are transfer functions $t_i(\alpha)$ such that $t(\hat{\alpha}) = 0$ is an alternative system of equilibrium conditions.

6. MULTIPLICITY OF EQUILIBRIA

Are equilibria unique? If not, are they locally unique? Do they vary continuously with the parameters of the economy? In recent years, economists have used the tools differential topology to investigate these questions.

Debreu (1970) first investigated the questions of local uniqueness and continuity of equilibrium in exchange economies with continuously differentiable excess demand functions. He defined a *regular economy* to be one for which the Jacobian matrix of excess demands $Df(\hat{p})$ with the first row and column deleted, the $(n-1) \times (n-1)$ matrix J, is nonsingular at every equilibrium. The first row is deleted because of Walras' law, the first column because of homogeneity; we are left with a square matrix because, as Walras had pointed out, the number of equations equals the number of unknowns in the equilibrium conditions. The inverse function theorem implies that every equilibrium of a regular economy is locally unique. Since the set S is compact and the equilibrium conditions involve continuous functions, this implies that a regular economy has a finite number of equilibria.

Let us rewrite the equilibrium conditions as $f(p,b) = 0$ where $b \in B$ and B is a topological space of parameters. If f and its partial derivatives with respect to p are continuous in both p and b, then the implicit function theorem implies that equilibria vary continuously at regular economies. Furthermore, in the case where B is the set of possible endowment vectors w^i, Debreu used Sard's theorem to prove that, for every b in an open set of full measure in B, $f(\cdot, b)$ is a regular economy. When B is the function space of excess demand functions with the uniform C^1 topology, an open dense set of B consists of regular economies. Consequently, if we are willing to restrict attention to continuously differentiable excess demand functions, a restriction that Debreu (1972) and Mas-Colell (1974) have shown is fairly innocuous, almost all economies, in a very precise mathematical sense, are regular.

Dierker (1972) noticed that a fixed-point index theorem could be used to count the number of equilibria of a regular economy. Let us define the fixed-point *index* of a regular equilibrium $\hat{p}$ as $\text{sgn}(\det[I - Dg(\hat{p})])$ whenever this expression is nonzero. Dierker showed that the index can also be written as $\text{sgn}(\det[-J])$. The *index theorem* says that $\sum \text{index}(\hat{p}) = +1$ where the sum is over equilibria of a regular economy. This result is depicted in Figure 5 where $n = 2$, $p_1 = 1 - p_2$, and $g_1(p_1, p_2) = 1 - g_2(p_1, p_2)$. Here $\text{index}(\hat{p}) = \text{sgn}(1 - \partial g_2/\partial p_2)$ and a regular economy is one where the graph of g does not become tangent to the diagonal.

Mas-Colell (1977) showed that any compact subset of S can be the equilibrium set of some economy f. If we restrict ourselves to regular economies and $n \geq 3$,

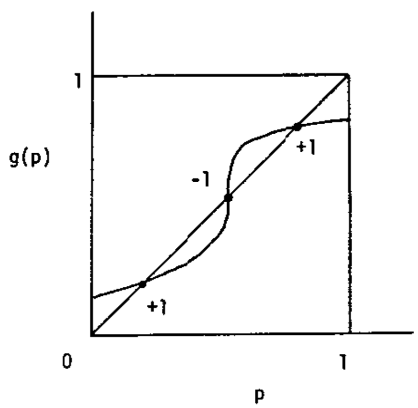

FIGURE 5 Illustration of the index theorem.

then the only restrictions placed on the number of equilibria are those given by the index theorem. (An equilibrium with index -1 must lie between two with index $+1$ if $n = 2$.) This implies that the number of equilibria is odd, and that there is a unique equilibrium if and only if $\text{index}(\hat{p}) = +1$ at every equilibrium.

Mas-Colell (1975, 1985) and Kehoe (1980, 1983) have extended the concepts of regularity and fixed-point index to economies with production. They prove that regular production economies have the same desirable properties as regular exchange economies and that, in a precise sense, almost all economies are regular. Kehoe (1980) further calculates the index of a regular equilibrium as

$$\text{index}(\hat{p}) = \text{sgn}\left(\det\begin{bmatrix} -J & B \\ -B^T & 0 \end{bmatrix}\right).$$

Here B is the submatrix of A formed by deleting the first row from A and any column for which the corresponding activity level is zero. A regular economy, of course, is one for which this expression is nonzero.

Using this formula, Kehoe (1985) has constructed a simple example of a production economy with three equilibria. (Exchange economies with $n = m = 2$ that have multiple equilibria are also easy to construct; see, for example, Shapley and Shubik, 1977.) This example has $n = m = 4$. Consumer i solves

$$\max \sum_{j=1}^{4} \gamma_j^i \log x_j^i$$
$$\text{s.t.} \sum_{j=1}^{4} p_j x_j^i \leq \sum_{j=1}^{4} p_j w_j^i$$
$$x_j^i \geq 0.$$

The utility parameters γ_j^i are given in Table 1.

TABLE 1 Utility Parameters γ_j^i

	Consumer			
Commodity	1	2	3	4
1	0.52	0.86	0.5	0.06
2	0.4	0.1	0.2	0.25
3	0.04	0.02	0.2975	0.0025
4	0.04	0.02	0.0025	0.6875

TABLE 2 Endowment Parameters w_j^i

	Consumer			
Commodity	1	2	3	4
1	50	0	0	0
2	0	50	0	0
3	0	0	400	0
4	0	0	0	400

The endowment parameters w_j^i are given in Table 2.

The aggregate excess demand function f has the form

$$f_j(p_1, p_2, p_3, p_4) = \frac{\sum_{i=1}^4 \gamma_j^i \sum_{\ell=1}^4 p_\ell w_\ell^i}{p_j} - \sum_{i=1}^4 w_j^i, \quad i = 1, 2, 3, 4.$$

As can easily be verified, this function satisfies continuity, homogeneity, and Walras' law. The production side of the economy is given by a 4×6 activity analysis matrix A:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 6 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 & -4 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}.$$

TABLE 3 Equilibrium 1

	Consumer			
Commodity	1	2	3	4
1	26.000	43.000	200.000	24.000
2	20.000	5.000	80.000	100.000
3	2.000	1.000	119.000	1.000
4	2.000	1.000	1.000	275.000
u_i	2.948	3.396	4.947	5.204

TABLE 4 Equilibrium 2

	Consumer			
Commodity	1	2	3	4
1	26.000	67.431	48.490	83.089
2	12.754	5.000	12.368	220.771
3	8.249	6.468	119.000	14.280
4	0.578	0.453	0.070	275.000
u_i	2.775	3.804	3.858	5.483

The parameters of this example have been chosen so that $\hat{p}_1 = \hat{p}_2 = \hat{p}_3 = \hat{p}_4 = 1/4$ is an equilibrium with index$(\hat{p}) = -1$. Consequently, we know that there are multiple equilibria. Two other equilibria, both with index +1, have been found using an exhaustive computer search.

EQUILIBRIUM 1

$$p^1 = (0.25000, 0.25000, 0.25000, 0.25000)$$
$$y^1 = (0, 0, 0, 0, 52.000, 69.000)$$
$$\alpha^1 = (0.05556, 0.05556, 0.44444, 0.44444)$$

See Table 3.

EQUILIBRIUM 2

$$p^2 = (0.15942, 0.25000, 0.03865, 0.55193)$$
$$y^2 = (0, 0, 0, 0, 42.701, 81.198)$$
$$\alpha^2 = (0.03105, 0.04869, 0.06023, 0.86003)$$

See Table 4.

TABLE 5 Equilibrium 3

	Consumer			
Commodity	1	2	3	4
1	26.000	39.072	224.362	14.499
2	22.001	5.000	98.768	66.485
3	1.783	0.810	119.000	0.539
4	3.311	1.504	1.857	275.000
u_i	3.002	3.317	5.049	5.070

EQUILIBRIUM 3

$$p^3 = (0.27514, 0.25000, 0.30865, 0.16621)$$
$$y^3 = (0, 0, 0, 0, 53.180, 65.148)$$
$$\alpha^3 = (0.06363, 0.05782, 0.57104, 0.30751)$$

See Table 5.

Each of the equilibria of this example is Pareto efficient and solves a problem of maximizing a weighted sum of utilities subject to feasibility constraints. The weights α_i associated with each of the equilibria are listed above. We emphasize that the multiplicity of equilibria is not due to any sort of nonconvexity in the consumers' or producers' maximization problems. Indeed, for any vector of weights α_i, the social planning problem that produces Pareto efficient allocations always has a unique solution.

7. INTERTEMPORAL MODELS

Let us now consider models in which goods are distinguished by date. Time is discrete, and at each date $t = 1, 2, \ldots$, there are n goods. Although our models have infinite time horizons, our results have strong implications for models with long but finite horizons. For simplicity, we restrict attention to exchange economies: goods cannot be stored between periods because storage is best thought of as a kind of production. Moreover, the models are stationary: their parameters do not depend on the date.

Consider first a model with a finite number of infinitely lived consumers. Consumer i, $i = 1, 2, \ldots, m$, chooses a sequence of consumption vectors $x_1^i, x_2^i, \ldots$ to maximize a utility function, such as $\sum_{t=1}^{\infty} \beta_i^t u_i(x_t)$, where $0 < \beta_i < 1$, subject to $\sum_{t=1}^{\infty} p_t \cdot x_t \leq \sum_{t=1}^{\infty} p_t \cdot w^i$ and $x_t \geq 0$. One way to formulate the equilibrium conditions is to set supply equal to demand; this results in a system of an infinite number of equations and unknowns. Equilibria of this model are Pareto efficient, however, and we can use the approach using transfer functions to reduce this to a system of $m-1$ equations and $m-1$ unknowns. Using regularity analysis, we could argue that almost all such economies have a finite number of equilibria (see Kehoe and Levine, 1985).

Consider instead a model with an infinite number of consumers, an overlapping generations model of the type first considered by Samuelson (1958). This model has a number of features not shared by the model with a finite number of consumers: it may, for example, have equilibria that are not Pareto efficient. There may also be robust examples with a continuum of equilibria.

Consider an example with a single consumer in each generation born in period t, who lives for three periods. He has utility function $\sum_{\tau=t}^{t+2} \alpha^{\tau-t}(x_\tau^b - 1)/b$, where $b < 1$, and endowment stream (w_1, w_2, w_3). The equilibrium condition in periods $t = 3, 4, \ldots$, has the form

$$f(p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2}) = 0$$

since it involves demands by consumers born in $t-2$, $t-1$, and t. In addition to these consumers, there is an old consumer, who lives only one period, and a middle-aged consumer, who lives two, alive in the first period. The equilibrium conditions in the first two periods have the form

$$f_1(p_1, p_2, p_3) = 0$$
$$f_2(p_1, p_2, p_3, p_4) = 0.$$

This model, like the one previously described, implicitly assumes *perfect foresight*: consumers know, possible by solving the model themselves, what prices prevail in the future.

The standard approach to proving the existence of, and computing, equilibria involves truncating the time horizon at some date T. (See, for example, Balasko, Cass, and Shell, 1980; Auerbach, Kotlikoff, and Skinner, 1983.) Equilibria then depend on the anticipated values of $p_{T=1}$ and p_{T+2}. To prove existence, we consider a sequence of price sequences $(p_1^k, p_2^k, \ldots)$ that satisfy the equilibrium conditions in the first T_k periods. An equilibrium is the limit of a convergent subsequence in the product topology as $T_k \to \infty$. A computational procedure must stop at a finite T, however. In computing equilibria, a standard assumption is that the equilibrium converges to a *steady state*, a solution to the equilibrium condition in

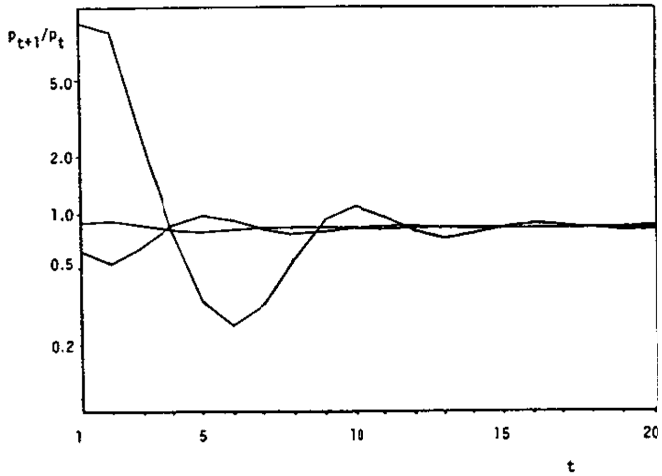

FIGURE 6 Three equilibria converging to the same steady state.

periods $t = 3, 4, \ldots$ of the form $p_t = \beta^{t-1} p_1$. (Physical scientists are often surprised that economists use the word equilibrium to describe something besides a steady state.) In this case, the expectations would be $p_{T+1} = \beta p_T$ and $p_{T+2} = \beta^2 p_T$. Equilibria need not, of course, converge to a steady state. Some may exhibit chaotic trajectories (see, for example, Benhabib and Day, 1982).

Unfortunately, this type of model can have a vast multiplicity of equilibria. Suppose, for example, we choose $a = 0.5$, $b = -3$, and $(w_1, w_2, w_3) = (3, 12, 1)$. Then, choosing the initial old and middle-aged consumers to ensure that the steady state satisfies the equilibrium conditions in periods 1 and 2, we can demonstrate that this economy has a continuum of equilibria that all converge to the steady state where $\beta = 0.7925$ (see Kehoe and Levine, 1987, for details).

Figure 6 depicts three of these equilibria. Notice that, if we slightly perturb the terminal conditions for p_{T+1}, p_{T+2} in, say, $T = 20$, then we produce drastically different equilibria. This illustrates the point that a continuum of equilibria in an infinite horizon model is typically symptomatic of sensitivity to terminal conditions in truncated versions of the model.

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