Liquidity Constrained Markets versus Debt Constrained Markets[*]

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Abstract: This paper compares two different models in a common environment. The first model has liquidity constraints in that consumers save a single asset that they cannot sell short. The second model has debt constraints in that consumers cannot borrow so much that they would want to default, but is otherwise a standard complete markets model. Both models share the features that individuals are unable to completely insure against idiosyncratic shocks and that interest rates are lower than subjective discount rates. In a stochastic environment, the two models have quite different dynamic properties, with the debt constrained model exhibiting simple stochastic steady states, while the liquidity constrained model has greater persistence of shocks.

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1. Introduction

There is considerable empirical evidence that both individual consumers and larger entities such as countries bear more idiosyncratic risk than is consistent with complete and frictionless Arrow-Debreu markets. Evidence at the level of the individual consumer is discussed, for example, in Hayashi [1985] and Zeldes [1989], who show that individual consumption is poorly correlated with aggregate consumption. Evidence at the international level is discussed, for example, in Backus, Kehoe, and Kydland [1992], who point out the low correlation between consumption levels across countries.

That individuals bear idiosyncratic risk can be captured by many departures from the Arrow-Debreu framework. Three important examples of such models are incomplete market models, where there are not enough securities to insure against all events; models of liquidity constraints in which individual consumers are assumed unable to borrow as much as they would like in loan markets; and models of adverse selection and moral hazard. Incomplete market models are discussed by Radner [1972], Hart [1975], and Duffie and Shafer [1985], for example. Examples of models of liquidity constraints can be found in Bewley [1980], Dumas [1980], Townsend [1980], Scheinkman and Weiss [1986], Abel [1990], Kehoe, Levine, and Woodford [1992] and Heaton and Lucas [1997]. Models of liquidity constraints typically involve incomplete markets, as not only are there short sales constraints on securities, but securities are limited in number as well. These papers have largely focused on the computation of special types of equilibria in economies where the outside asset is a fiat money of no intrinsic value. In these equilibria shocks have long term consequences. We show that this is also the case in the incomplete market model considered in this paper.

Models of adverse selection and moral hazard, with the notable exception of Prescott and Townsend [1984], are not ordinarily general equilibrium models, so fall outside the scope of this paper, but the interested reader should consult Green [1987] who shows some of the links between asset market models and models of adverse selection.

Models with incomplete markets and/or liquidity constraints typically have the properties that in equilibrium individuals bear idiosyncratic risk, and interest rates are lower than subjective discount rates. There is also a fourth model that shares these properties: a model with individually rational debt constraints. Here the setup differs from that of Arrow-Debreu only in the assumption that a portion of the endowment is inalienable and cannot be seized if a consumer goes bankrupt. This model has been studied by Schechtman and Escurdero [1977], Manuelli [1986], Marcet and Marimon [1992], and Kehoe and Levine [1993]. Kocherlakota [1996], and Albuquerque and Hopenhyn [1999]. It has been applied to the study of existing asset markets by Kehoe and Perri [1998], Krueger and Perri [1998], and Alvarez and Jermann [2000]. It is worth noting that there are two distinct models of debt constraints: those in which traders can be excluded from spot markets, or those, as in Kehoe and Levine [1993] where they cannot. The latter possibility leads to a failure of the welfare theorems, and is conceptually more like the incomplete market model. In the single good model studied here, and widely used in applications, including the papers cited above, however, there is no spot market, and as a result the welfare theorems hold.

In contrast to the general equilibrium approach employed in this paper, models with debt constraints can also be analyzed using the tools of optimal contracts. See, for example, Kocherlakota [1996] and Albuquerque and Hopenhyn [1999].

This paper directly compares the debt constrained model to the incomplete markets/liquidity constrained model in the same physical environment in which consumers alternate either deterministically or randomly between having high and low endowments. The bottom line is that the debt constrained model, largely because it involves a much smaller departure from the Arrow-Debreu framework, leads to a vastly simpler and more tractable model of equilibrium in the stochastic case, but nevertheless incorporates the main features of equilibrium idiosyncratic risk bearing and interest rates lower than subject discount rates.

2. The Environment

There are an infinite number of discrete time periods $t = 0,1,\ldots$. In each period there are two types of consumers $i = 1,2$, and a continuum of each type of consumer. There is a single consumption good x; the representative consumer of type i consumes x_t^i in period t. The infinite vector of consumption is $(x_0^i, x_1^i \ldots) \in \ell_\infty^{++}$, where ℓ_∞^{++} is the set of sequences that are bounded and positive. Both consumers have the common stationary additively separable utility $U(x_0^i, x_1^i \ldots) = (1-\delta)\sum_{t=0}^{\infty} \delta^t u(x_t^i)$. The period utility function is twice continuously differentiable with $Du(x) > 0$, satisfies the boundary condition

$Du(x) \to \infty$ as $x \to 0$, and has $D^2u(x) < 0$. The common discount factor δ satisfies $0 < \delta < 1$.

There are two types of capital: human capital (or labor) and physical capital (trees or land). The services of the (one unit of) human capital held by type *i* consumer in period *t* are denoted w_t^i. These services take on one of two values, ω^b and ω^g, with $\omega^b < \omega^g$, corresponding to bad and good productivity respectively. Moreover, if one consumer has good productivity, then the other consumer has bad productivity, so if $w_t^i = \omega^b$ then $w_t^{-i} = \omega^g$ (where $-i$ is the type of consumer who is not type *i*). We start by assuming that productivity alternates between good and bad, so if $w_t^i = \omega^g$ then $w_{t+1}^i = \omega^b$. Subsequently, we will allow for a more general process of randomly switching between the two productivity pairs $(w_t^1, w_t^2) = (\omega^g, \omega^b)$ and $(w_t^1, w_t^2) = (\omega^b, \omega^g)$.

There is one unit of physical capital in the economy. This capital is durable and returns $r > 0$ of the consumption good in every period. If $r = 0$, physical capital would be interpreted as fiat money, but we do not allow this case. Since $r > 0$ we may interpret physical capital as trees, with *r* being the amount of consumption good produced every period by the trees. A consumer of type *i* holds a share θ_t^i of the capital stock at the beginning of time *t*. Initial physical capital holdings are θ_0^i.

The total supply of the consumption good in this economy is the sum of the individuals' productivity, plus the return on the single unit of physical capital $\omega^g + \omega^b + r$. We denote this aggregate supply as ω. The social feasibility conditions for this economy in each period are

$$x_t^1 + x_t^2 \le \omega^g + \omega^b + r = \omega$$

$$\theta_t^1 + \theta_t^2 \le 1.$$

3. Market Arrangements

In this physical environment, we consider two different models of intertemporal trade. In the *liquidity constrained economy* consumers can only carry out intertemporal trade by exchanging real capital. The consumption good is taken to be numeraire, and the price of physical capital in period *t* is denoted by v_t. The objective of a consumer of type *i* is to solve the problem

$$\max\ (1-\delta)\sum_{t=0}^{\infty}\delta^t u(x_t^i)$$

subject to

$$x_t^i + v_t\theta_{t+1}^i \le w_t^i + (v_t + r)\theta_t^i$$
$$\theta_t^i \ge 0, \quad \theta_0^i \text{ given}$$
$$t = 0, 1, \ldots$$

The crucial feature of this model is that physical capital can be held only in nonnegative amounts, and that there are no securities or other assets that can be traded besides physical capital. To understand this better, it is useful to think of trade as taking place at different physical locations around the circle, as shown in Figure 1. Only consumers at the same location can trade; the measure of both types of consumers is the same. The type 1 consumers do not move, and type 2 consumers move counter-clockwise. The essential feature is that type 2 consumers move in such a fashion, say a single radian each period, that they never return to the same location. In this model, intertemporal trade can be carried on only by exchanging physical capital, and physical capital can not be held in negative quantities, so this explains both why there is only one security, and why it cannot be sold short. Later in the paper, we discuss the consequences of allowing physical capital to be borrowed.

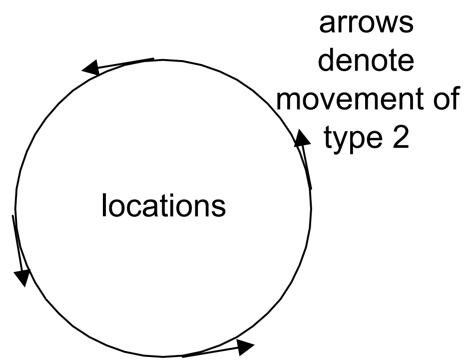

Figure 1

In the liquidity constrained economy, an *equilibrium* is an infinite sequence of consumption levels, capital holdings, and capital prices such that consumers maximize utility given their constraints, and such that the social feasibility conditions are satisfied.

The second model of intertemporal trade that we examine is the *debt constrained economy*. Here we allow borrowing and lending and, in the stochastic case that we discuss later, the sale and purchase of insurance contracts. There are, however, debt constraints. These come about because consumers have the option of going bankrupt, or otherwise opting out of intertemporal trade. If they choose to do this, they renege on all existing debts. They are excluded from all further participation in intertemporal trade, however, and their physical capital is seized. The endowment of human capital is assumed to be inalienable: it cannot be taken away, nor can consumers be prevented from consuming its returns. Notice that unlike the model of trading physical capital, which can be completely decentralized, this model requires a credit agency, a government, or some central authority to keep track of who has gone bankrupt and to assure that their capital is seized and that they do not continue to borrow and lend.

Formally, this is a model in which consumers face the individual rationality constraint

$$(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}u(x_\tau^i) \geq (1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}u(w_\tau^i).$$

This says that in every period, the value of continuing to participate in the economy is no less than the value of dropping out. In this setting, the absence of private information implies that no consumer actually goes bankrupt in equilibrium: the credit agency will never lend so much to consumers that they will choose bankruptcy. This is very unlike the incomplete markets bankruptcy models of Dubey, Geanakoplos, and Shubik [1988] and Zame [1993].

In this debt constrained economy, since markets are complete, consumers purchase consumption in period *t* for p_t and they sell the return on their capital $w_t^i + r\theta_0^i$ at the same price. The corresponding optimization problem is

$$\max\,(1-\delta)\sum_{t=0}^{\infty}\delta^t u(x_t^i)$$

subject to

$$\sum_{t=0}^{\infty} p_t x_t^i \le \sum_{t=0}^{\infty} p_t (w_t^i + \theta_0^i r)$$

$$(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}u(x_\tau^i) \ge (1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}u(w_\tau^i), \quad t = 0,1,\ldots$$

Notice that we have written the budget constraint in the Arrow-Debreu form. As is usual in this sort of model, and as we show formally in appendix, we can equally well formulate the budget constraint as a sequence of complete securities markets,

$$x_t^i + v_t\theta_{t+1}^i \le w_t^i + (v_t + r)\theta_t^i$$
$$\theta_t^i \ge -\Theta, \quad \theta_0^i \text{ given}$$
$$t = 0,1,\ldots$$

The constraint $\theta_t^i \ge -\Theta$ rules out Ponzi schemes, but unlike the liquidity constrained economy where $\Theta = 0$, here Θ is a positive constant chosen large enough not to constrain to borrowing.

An *equilibrium* of the debt constrained economy is an infinite sequence of consumption levels and consumption prices such that consumers maximize utility given their constraints and such that the social feasibility condition for consumption is satisfied.

We start by examining symmetric *steady states* of both the liquidity and the debt constrained economy. In a symmetric steady state

$$x_t^i = \begin{cases} x^g & \text{if } w_t^i = \omega^g \\ x^b & \text{if } w_t^i = \omega^b. \end{cases}$$

Because $x^g + x^b = \omega$, we can characterize consumption at a symmetric steady state by the single number x^g. As is usual in steady state analysis, to implement the steady state as an equilibrium, we must create a transfer payment between the consumers so that they satisfy their budget constraints. Later we extend the analysis of the debt constrained economy to more general dynamic equilibria.

4. Comparison of Liquidity and Debt Constrained Markets

We can now compare the steady state equilibria of the liquidity and debt constrained economies. Throughout the analysis we use first order Euler conditions to characterize the optimum of the consumer. It is well known that, together with a transversality condition, the Euler conditions are necessary and sufficient for a path to be an optimum. See Scheinkman [1976] and Araujo and Scheinkman [1977]. These same papers show that the transversality condition is satisfied if the path is bounded. In our analysis, the paths we study all converge to (or even begin at) a steady state, so they are bounded. As a result we focus our analysis on the first order conditions.

We begin by characterizing equilibria in the liquidity constrained economy. In this economy, x^g is determined by the fact that the consumer with good productivity is free to purchase as much physical capital as he wishes from the consumer with bad productivity. His marginal utility in the current period is $Du(x^g)$, while next period he will have bad productivity, and marginal utility $Du(x^b) = Du(\omega - x^g)$. Consequently, the first order condition for the consumer's maximization problem can be written as

$$\frac{Du(x^g)}{Du(\omega - x^g)} = \delta \frac{v_{t+1} + r}{v_t}.$$

In the appendix we show that any equilibrium prices $v_0, v_1 \ldots$ satisfying these equations must be bounded as well. Simple algebraic manipulation then implies that, if $v_t \geq 0$ for all *t*, then $v_t = v$ for all *t*.

The three conditions that must be satisfied are the budget constraints in the good and bad state, and the first order condition in the good state

$$\begin{aligned} &x^g + v\theta^g = \omega^g + (v+r)\theta^b \\ &x^b + v\theta^b = \omega^b + (v+r)\theta^g \\ &Du(x^g)v = \delta Du(\omega - x^g)(v+r). \end{aligned}$$

Multiplying the first equation by $Du(x^g)$ and the second by $\delta Du(\omega - x^g)$, we use $\theta^g + \theta^b = 1$ and $x^g + x^b = \omega$ to find

$$\begin{aligned} &Du(x^g)(x^g - \omega^g) + \delta Du(\omega - x^g)(\omega - x^g - \omega^g) = \\ &-Du(x^g)v + \delta Du(\omega - x^g)(v+r) \\ &+Du(x^g)(2v+r)\theta^b - \delta Du(\omega - x^g)(2v+r)\theta^b. \end{aligned}$$

Substituting the first order for the good state into the right hand side of this equation, we obtain

$$\begin{aligned} &Du(x^g)(x^g - \omega^g) + \delta Du(\omega - x^g)(\omega - x^g - \omega^g) = \\ &\left(Du(x^g) + \delta Du(\omega - x^g)\right) r\theta^b. \end{aligned}$$

It is convenient to define

$$f^L(x^g) = Du(x^g)(x^g - \omega^g) + \delta Du(\omega - x^g)(\omega - x^g - \omega^b).$$

There are two possibilities: either $\theta^b > 0$ at $x^g = \omega/2$ or $\theta^b = 0$ for $x^g \in [\omega/2, \omega^g]$. In the latter case, we have $f^L(x^g) = 0$. In the former case, $f^L(\omega/2) \geq 0$.

We have demonstrated the following result.

Proposition 1: *A symmetric steady state x^g of the liquidity constrained economy is characterized by*

$f^L(\omega/2) \geq 0$ *and* $x^g = \omega/2$ *or*

$\omega^g > \omega/2$, $f^L(x^g) = 0$ *and* $x^g \in [\omega/2, \omega^g]$.

We turn next to the debt constrained economy. We define the consumption set for each individual to be the set of nonnegative consumption plans that are individually rational. Given this definition, the model is a standard complete markets model with a finite number of consumer types. The standard argument implies that the equilibrium is Pareto efficient: Suppose, to the contrary that there exists an alternative allocation that is feasible, satisfies the individual rationality constraints, yields at least as much utility to both consumers, and yields strictly more utility to at least one consumer. Then this alternative allocation must assign to the consumer that is strictly better off a consumption bundle that costs strictly more than his endowment at the equilibrium prices,

$$\sum_{t=0}^{\infty} p_t \tilde{x}_t^i > \sum_{t=0}^{\infty} p_t (w_t^i + \theta_0^i r).$$

Furthermore, it must assign the other consumer a consumption bundle that costs at least as much as his endowment because, if it assigns a consumption bundle that costs strictly less, the consumer could spend the extra income, make himself better off, and not violate his individual rationality constraint,

$$\sum_{t=0}^{\infty} p_t \tilde{x}_t^i \geq \sum_{t=0}^{\infty} p_t (w_t^i + \theta_0^i r).$$

Together, these two conditions imply that the alternative allocation costs more than the aggregate endowment.

$$\sum_{t=0}^{\infty} p_t (\tilde{x}_t^1 + \tilde{x}_t^2) > \sum_{t=0}^{\infty} p_t \omega.$$

As in the model without debt constraints, this implies that the alternative allocation cannot be feasible, which contradicts the assumption that there is a Pareto superior allocation.

Proposition 2: *An equilibrium allocation in the debt constrained economy is Pareto efficient.*

In a symmetric steady state, the first best is to equalize consumption between the two consumers, $x^g = \omega / 2$. It may be impossible to reach this allocation without violating the individual rationality constraint, however. To achieve a Pareto improvement over autarky using a stationary allocation, consumption must be transferred from the consumer with good productivity to the consumer with bad productivity. Eventually, the individual rationality constraint for the consumer with good productivity may be violated: the consumer with good productivity would prefer to declare bankruptcy rather than to make the transfer. We conclude that, if consumption between the two consumers is not equalized, then the individual rationality constraint for the consumer with good productivity must bind exactly. The utility that the consumer with good productivity receives in equilibrium is proportional to $u(x^g) + \delta u(\omega - x^g)$; the utility he would have received from his endowment is proportional to $u(\omega^g) + \delta u(\omega^b)$. If we define

$$f^D(x^g) = u(x^g) - u(\omega^g) + \delta\big(u(\omega - x^g) - u(\omega^b)\big),$$

then the exact binding of the individual rationality constraint can be written $f^D(x^g) = 0$.

We can summarize this discussion.

Proposition 3: *A symmetric steady state* x^g *of the debt constrained economy is characterized by*

$f^D(\omega / 2) \geq 0$ *and* $x^g = \omega / 2$ *or*

$\omega^g > \omega / 2$, $f^D(x^g) = 0$ *and* $x^g \in [\omega / 2, \omega^g]$.

We can now compare steady states of the two models by studying the functions f^L and f^D: Concavity of the utility function implies that f^D is concave. Since f^L

replaces the utility differences in f^D with the slope of the utility function multiplied by the difference between the two consumption levels x^g and $x^b = \omega - x^g$, concavity of u also implies that $f^D(x^g) > f^L(x^g)$. Finally, $r > 0$ implies that $\omega = \omega^g + \omega^b + r > \omega^g + \omega^b$, and this means that $f^L(\omega^g) > 0$.

Figure 2 shows what f^L and f^D look like in the case where $f^D(\omega/2) < 0$. From this figure we can immediately see that steady states of both types exist: since each function f is continuous and positive at ω^g, either it is positive at $\omega/2$, in which case $\omega/2$ is a steady state, or it must be zero somewhere on the interval $[\omega/2, \omega^g]$, and that zero is a steady state. Moreover, since f^D is concave, it can be zero at most in this interval, so that in the debt constrained economy the steady state is unique. If we calculate Df^L and substitute in the interior steady state condition $f^L(x^g) = 0$, we find that at interior steady state $Df^L(x^g) > 0$. As can be seen in Figure 2, this together with the boundary condition $f^L(\omega^g) > 0$ implies that in the liquidity constrained economy the steady state is unique. We sum up our discussion with a proposition.

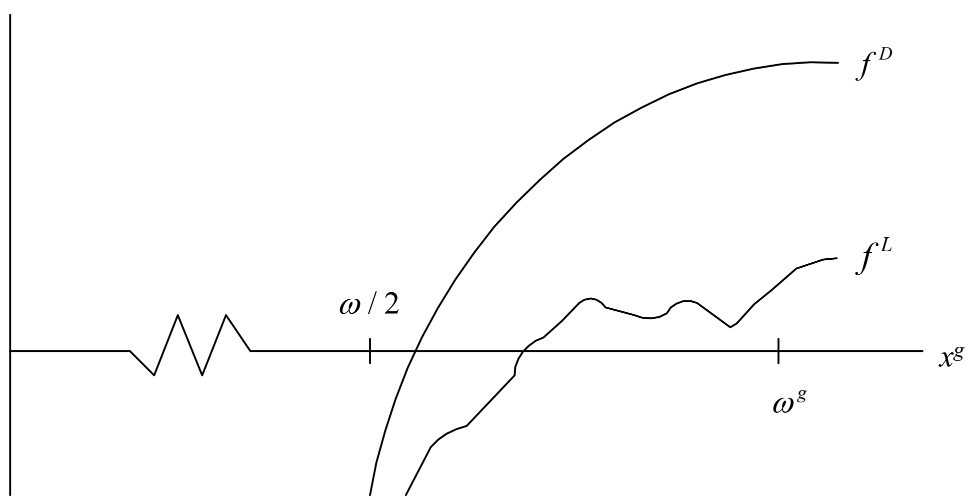

Figure 2

Proposition 4: *A symmetric steady state exists both in the liquidity constrained and in the debt constrained economy. In each case there is only one symmetric steady state.*

In the limiting case where $\delta = 1$ in the liquidity constrained economy, we can calculate $f^L(\omega/2) = Du(\omega/2)r > 0$. For δ sufficiently close to 1, $f^L(\omega/2) > 0$, and so the only liquidity constrained symmetric steady state will be the symmetric first best

$x^g = \omega / 2$. Since $f^D(x^g) > f^L(x^g)$ the same statement is true in the debt constrained economy case: in both cases we reach full efficiency when consumers are sufficiently patient.

In a similar vein, we see that

$$\operatorname{sgn} f^L(\omega/2) = \operatorname{sgn}\left[(\omega/2 - \omega^g) + \delta(\omega/2 - \omega^b)\right].$$

Increasing *r*, holding ω^g and ω^b fixed, has the effect of increasing $\omega = \omega^g + \omega^b + r$. When *r* is sufficiently large, $\omega/2 \geq \omega^g$ and $f^L(\omega/2) \geq 0$ again imply that both liquidity constrained and debt constrained symmetric steady states are first best. In other words, if the gross return to the stock of physical capital is sufficiently large relative to the productivity of human capital, then markets are fully efficient.

The intuition for these results is simple: In the liquidity constrained economy increasing δ increases the steady state price of physical capital *v*, thus increasing $v + r$. Increasing *r* does this directly. The larger is $v + r$, the easier it is for consumers to smooth consumption using trades in physical capital. In the debt constrained economy increasing δ increases the penalty for bankruptcy that a consumer suffers from being excluded from intertemporal trade. Increasing *r* increases the penalty that he suffers from losing his collateral, his endowment of physical capital. The larger are these penalties, the easier it is to satisfy the individual rationality constraints.

The interest rate at the symmetric steady state can be calculated from a ratio of marginal utilities

$$i = \frac{Du(x^g)}{\delta Du(x^b)} - 1.$$

In the symmetric first best this gives the usual complete market interest rate equal to the subjective discount rate $1/\delta - 1$. When the symmetric first best is not reached, $x^g > x^b$, so the interest rate will be lower than the subjective discount rate. The intuition is simple: Borrowers are constrained, lenders are not. To keep the level of loans from lenders as low as is required in equilibrium, the market must have a low rate of interest.

The general features of both the liquidity constrained and the debt constrained economy can be illustrated by a simple numerical example. Suppose that utility is given by $u(x) = \log x$, and that the endowments and discount factors are $\omega^g = 24$, $\omega^b = 9$, $r = 1$, $\delta = 1/2$. Here the consumers are quite impatient, and their productivity fluctuates substantially. In addition, human capital is much more important than physical capital. In the liquidity constrained economy we compute

$$f^L(x^g) = (x^g - 24) / x^g + \frac{1}{2}(25 - x^g) / (34 - x^g) = 0 ,$$

from which it follows that $x^g = 20.63, \quad x^b = 13.37$. By way of contrast, in the debt constrained economy,

$$f^D(x^g) = \log x^g - \log 24 + \frac{1}{2}\left(\log(34 - x^g) - \log 9\right) = 0 ,$$

resulting in $x^g = 18, \quad x^b = 16$. As can be seen, the liquidity constrained economy has less consumption smoothing, and indeed, the debt constrained economy exhibits a large degree of consumption smoothing. As we shall see below, if the shock is more persistent, the degree of consumption smoothing is significantly reduced.

It is also of interest to compute the interest rate. The subjective discount rate corresponding to a discount factor of 1/2 is 100 percent. In the liquidity constrained economy however, the interest rate is 29.6 percent, considerably lower. In the debt constrained economy it is 77.8 percent.

This example is also useful because it illustrates how the symmetric steady states of the two models can be implemented as equilibria. The problem that we must get around is that discounting puts the two consumers in asymmetric positions: the type of consumer who first has good productivity has a permanent advantage over the other type. The easiest way to compensate for this advantage and arrive at the steady state is to impose a transfer payment from one consumer type to the other. In the liquidity constrained model, we need the budget constraint for the consumer type who first has high productivity, say type 1, to hold in the first period,

$$x^g + v \le \omega^g + \theta_0^1 (v + r) - \tau .$$

This constraint holds with equality when $v = \omega^g - x^g$ if $\tau = \theta_0^1 (v + r)$.

In the debt constrained model, we need to transfer enough income so that the present discounted value of lifetime incomes are equal

$$\sum_{t=0}^{\infty} p_t (w_t^1 + \theta_0^1 r) - \tau = \sum_{t=0}^{\infty} p_t (w_t^2 + \theta_0^2 r) + \tau .$$

Alternatively, in the debt constrained model, we could introduce uncertainty before the first period, giving both consumer types equal chances of having the high productivity first, and allowing them to write contingent contracts against this initial uncertainty.

5. Short Sales

So far we have assumed that capital must be held in nonnegative amounts. This is an immediate consequence of the locational story given above. In the deterministic case, that there is only one asset, physical capital, means that only the inability to borrow prevents asset markets from being complete. In the next section, we consider a stochastic economy. In the stochastic case, that there is only one asset forces asset markets to be incomplete. In the stochastic setting, the locational story plays a more significant role, because it gives an economically sensible story of why asset markets are incomplete.

It is easy to work out what happens in the deterministic case when borrowing of physical capital is allowed, even though such sales are not compatible with our locational story. We assume that the constraint on short sales of capital takes the form $\theta_t^i \geq -d$. The only change in the previous analysis is that the consumer with bad productivity can now spend up to $1+d$ units of physical capital to purchase $\omega^g - x^g$ units of consumption. If we redefine

$$f^L(x^g) = Du(x^g)(x^g - \omega^g + rd) + \delta Du(\omega - x^g)(\omega - x^g - \omega^b + rd),$$

then the characterization of equilibrium in Proposition 3 continues to hold.

It is obvious that, if d is sufficiently large, $f^L(\omega/2) > 0$ and the symmetric first best is the unique symmetric steady state. Since a single asset is all that is needed for market completeness in the deterministic case, this should come as no surprise. There is also a unique level of debt $\hat{d}$ so that $f^L(\hat{x}^g) = 0$, where $\hat{x}^g$ is the unique solution of $f^D(\hat{x}^g) = 0$. In the numerical example in the previous section, setting $\hat{d} = 1.32$ results in a solution where $\hat{x}^g = 18$ in the modified liquidity constrained model, just as in the debt constrained model.

If our welfare criterion places equal weights on the two types utility and if the debt limit $d > \hat{d}$ (and $\hat{x}^g < \omega/2$), then the liquidity constrained equilibrium provides a higher welfare level than the debt constrained equilibrium. (If the discount factor is close enough to one, higher welfare in this sense will also imply Pareto dominance.) The implication is that to enforce the repayment of debt in the incomplete markets model when $d > \hat{d}$, it will be necessary to seize human as well as physical capital.

6. A Stochastic Environment

With deterministic alternation between productivities, the liquidity constrained economy and debt constrained economy are quite similar: the major difference is that the debt constraints allow greater trade. We now show that when we allow for random productivities, equilibrium with debt constraints continues to be described by a stochastic version of a steady state, but the liquidity constrained economy does not permit this type of simple equilibrium.

We modify the physical environment so that the consumer with good productivity is chosen randomly. Let $\eta_t \in \{1,2\}$ denote the consumer who has good productivity at time *t*. This random variable is assumed to follow a Markov process, which is characterized by a single number $0 < \pi < 1$, the probability of a *reversal,* that is, a transition from the state where type 1 has good productivity to the state where type 2 has good productivity, or vice versa. When $\pi = 1$ we are in the deterministic case.

The economy now takes place on a tree rather than over time. The root of the tree is denoted by η_0. A *state history* is a finite list $s = (\eta_1, \ldots, \eta_t)$ of events that have taken place through time $t(s)$, where $t(s)$ is the length of the vector *s,* the time at which *s* occurs. The history immediately prior to *s* is denoted $s-1$, and if the node σ follows s on the tree, we write $\sigma > s$. The countable set of all state histories is denoted *S.* The probability of a state history is computed from the Markov transition probabilities

$$\pi_s = pr(\eta_{t(s)} | \eta_{t(s)-1}) pr(\eta_{t(s)-1} | \eta_{t(s)-2}) \cdots pr(\eta_1 | \eta_0) .$$

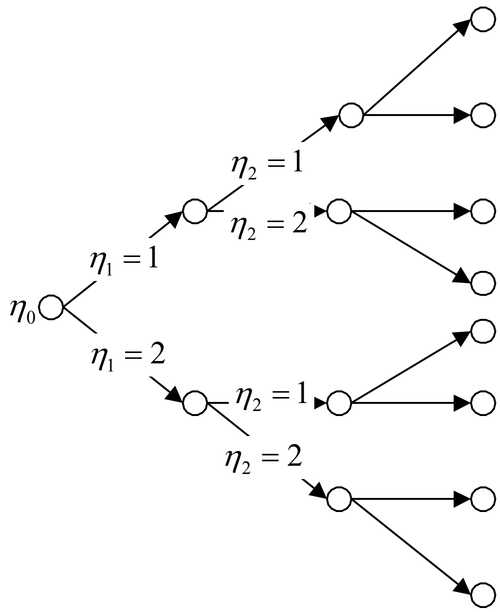

Figure 3

Consumption and endowments are now subscripted by state history, rather than by time. Utility for consumer *i* is the expected utility $(1-\delta)\sum_{s\in S}\delta^{t(s)}\pi_s u(x_s^i)$. Define θ_s^i to be the holding of capital at the end of state *s*. The optimization problem in the liquidity constrained case now becomes

$$\max\,(1-\delta)\sum_{s\in S}\delta^{t(s)}\pi_s u(x_s^i)$$

subject to

$$x_s^i + v_s\theta_{(s,\eta)}^i \le w_s^i + (v_s + r)\theta_s^i$$
$$\theta_s^i \ge 0, \quad \theta_0^i \text{ fixed.}$$

In the debt constrained economy the optimization problem of the consumer is

$$\max\,(1-\delta)\sum_{s\in S}\delta^{t(s)}\pi_s u(x_s^i)$$

subject to

$$\sum_{s\in S} p_s x_s^i \le \sum_{s\in S} p_s(w_s^i + \theta_0^i r)$$
$$(1-\delta)\sum_{\sigma\ge s}\delta^{t(\sigma)-t(s)}(\pi_\sigma/\pi_s)u(x_\sigma^i) \ge (1-\delta)\sum_{\sigma\ge s}\delta^{t(\sigma)-t(s)}(\pi_\sigma/\pi_s)u(w_\sigma^i).$$

As in the deterministic case, a proposition appendix shows that this Arrow-Debreu formulation of the budget constraint has an equivalent sequential markets formulation.

$$x_s^i + q_{(s,1)}\theta_{(s,1)}^i + q_{(s,2)}\theta_{(s,2)}^i \le w_s^i + (v_s + r)\theta_s^i$$
$$\theta_s^i \ge -\Theta, \quad \theta_0^i \text{ fixed,}$$

where $q_{(s,\eta)}$ is the price of the Arrow security traded in state s that promises a unit of physical capital to be delivered at state (s,η). A standard arbitrage argument implies that $q_{(s,1)} + q_{(s,2)} = v_s$. The sequential markets budget constraint for debt constrained markets differs from the liquidity constrained budget constraint in two ways. First, as in the deterministic case, we have $\Theta > 0$ rather than $\Theta = 0$. Second, and significantly, in the liquidity constrained case consumers are restricted to trades in which $\theta_{(s,1)}^i = \theta_{(s,2)}^i$.

In the stochastic case, we define a *symmetric stochastic steady state* by consumption x^g when productivity is good and x^b when productivity is bad, and the rule

$$x_s^i = \begin{cases} x^g & w_s^i = \omega^g \\ x^b & w_s^i = \omega^b. \end{cases}$$

In the debt constrained economy, stochastic steady states are much like deterministic steady states: we decrease x^g from ω^g until we either achieve the symmetric first best at $x^g = \omega/2$ or until the individual rationality constraint begins to bind. As in the deterministic case, we define a function proportional to the difference between the utility from the steady state consumption plan and consumption in autarky. A recursive calculation shows that this function is

$$f^D(x^g) = \big(1 - \delta(1-\pi)\big)\big(u(x^g) - u(\omega^g)\big) + \delta\pi\big(u(\omega - x^g) - u(\omega^b)\big).$$

By exactly the same argument as that leading to Proposition 3, we obtain the following result.

Proposition 5: *A symmetric stochastic steady state x^g of the debt constrained economy is characterized by*

$f^D(\omega/2) \ge 0$ and $x^g = \omega/2$ or
$\omega^g > \omega/2$, $f^D(x^g) = 0$ and $x^g \in [\omega/2, \omega^g]$.

When $\pi = 1$ the function f^D is concave and satisfies $f^D(\omega^g) > 0$, and we have concluded that a symmetric steady state exists and is unique. Since when $0 < \pi < 1$ it is still true that f^D is concave and satisfies $f^D(\omega^g) > 0$, we reach exactly the same conclusions.

Proposition 6: *A symmetric stochastic steady state exists in the debt constrained economy. There is only one symmetric stochastic steady state.*

An interesting question is how the steady state level of consumption depends on the parameter $1-\pi$ measuring the persistence of the shock. From the implicit function theorem, in the case where the debt constraint binds, we can compute

$$\frac{dx^g}{d(1-\pi)} = \frac{\partial f^D / \partial \pi}{\partial f^D / \partial x^g}.$$

We already observed that at an interior steady state f^D must intersect the axis from below, so $\partial f^D / \partial x^g$ is positive. We can also rewrite f^D as

$$f^D(x^g) = (1-\delta)\big(u(x^g) - u(\omega^g)\big) + \delta\pi\big(u(\omega - x^g) - u(\omega^b) + u(x^g) - u(\omega^g)\big).$$

When $f^D(x^g) = 0$, since the first term is negative, the second term is positive, and since $\partial f^D / \partial \pi$ is proportional to the second term, it is also positive. We conclude that

$$\frac{dx^g}{d(1-\pi)} > 0,$$

meaning that a more persistent shock results in greater consumption by the good productivity consumer, or equivalently less trade between the two consumers.

This result is reinforced by reexamination of the numerical example. Recall that $u(x) = \log x$, while the endowments and discount factors are $\omega^g = 24$, $\omega^b = 9$, $r = 1$, $\delta = 1/2$. Recall that in the deterministic case, $\pi = 1$, we had $x^g = 18$, $x^b = 16$. By way of contrast, if $\pi = 1/2$, we can compute

$$f^D(x^g) = \frac{3}{4}(\log x^g - \log 24) + \frac{1}{4}\big(\log(34 - x^g) - \log 9\big) = 0,$$

from which it follows that $x^g = 21.52$, $x^b = 12.48$, a considerable reduction in the amount of consumption smoothing.

In the debt constrained economy, when the economy becomes stochastic, consumption smoothing is reduced, and consumption of x^g and x^b fluctuates randomly. Conceptually, however, the equilibria are very similar in the deterministic and stochastic cases.

The case of liquidity constraints is strikingly different. As in the deterministic case, the good productivity consumer trades goods to the bad productivity consumer in exchange for physical capital. Since the good productivity consumer holds physical capital at the end of the period, his first order condition

$$\frac{v_s Du(x^g)}{(v_{s+1}+r)(1-\pi)Du(x^g)+(v'_{s+1}+r)\pi Du(\omega-x^g)}=\delta$$

continues to determine prices, where v_{s+1} is the price of capital when the state at $t+1$ is the same as that in the previous period, and v'_{s+1} is the price when a reversal of the state takes place. In addition, we show in the appendix (see also Levine and Zame [1996], for example) that the capital prices v_s must be uniformly bounded, say by $\bar{v}$, or else no one would be willing to hold capital.

This boundedness of physical capital prices poses a dilemma, however. The consumer with bad productivity must purchase $x^b-\omega^b$ units of consumption each period, and so must expend at least $(1/\bar{v})(x^b-\omega^b)$ units of physical capital each period. Since there is only one unit of capital in the economy, a consumer can have bad productivity no more than $\bar{v}/(x^b-\omega^b)$ periods before he will have expended all of his physical capital. If $0<\pi<1$, however, then there is a positive probability that a consumer will have a run of bad luck with his productivity that exceeds this length of time. We conclude that $x^b-\omega^b=0$, that is, the only possible stochastic steady state is autarky. In autarky, however, the consumer with bad productivity is free to borrow, so prices must be determined also by his first order condition

$$\frac{Du(\omega^b)}{(1-\pi)Du(\omega^b)+\pi Du(\omega^g)}=\delta\frac{v+r}{v}=\frac{Du(\omega^g)}{(1-\pi)Du(\omega^g)+\pi Du(\omega^b)}.$$

This is possible only if $\omega^g=\omega^b$, violating the assumption that $\omega^g>\omega^b$. We can summarize this discussion with a proposition.

Proposition 7: *If* $0<\pi<1$ *there is no symmetric stochastic steady state with liquidity constraints.*

There is a simple intuition for this result: Suppose that a consumer has bad productivity for the first time. Then he should sell some of his physical capital to smooth his consumption. If the consumer is unlucky enough to have bad productivity in the subsequent period, he has less physical capital and so is in a different situation than when he had bad productivity for the first time. Consequently, with liquidity constraints, consumption must depend not only on the current state, but also on the distribution of physical capital between the two types. Notice that this proposition is not sensitive to permitting borrowing of physical capital: any fixed debt constraint will eventually be exceeded by a very long run of bad luck.

7. Dynamic Analysis

The debt constrained economy is sufficiently simple that we can give a complete analysis even without the steady state assumption. Given an arbitrary initial condition $\theta_0^1,\ \theta_0^2$ there is a unique equilibrium. This equilibrium can have two distinct phases: an initial phase and a final phase. In the deterministic case the initial phase is just the first period; more generally, the initial phase lasts until the two consumer types exchange roles. The equilibrium is one of two types. If the parameters are such that the symmetric first best satisfies the debt constraints, then in the final phase, each consumer's consumption over time is constant. If the symmetric first best does not satisfy the debt constraint, then the final phase is the symmetric steady state. The striking fact in this case is that, even if the initial condition and initial phase are quite asymmetric, once roles have reversed, from that point on consumption depends only a consumer's endowment, and not on his type.

This characterization of dynamic equilibrium formally presented in the appendix. Recall that the first welfare theorem holds for these economies. There are two separate cases, the *non-binding case* in which the debt constraint does not bind at the symmetric first best and the *binding case* in which it does. In the non-binding case, efficiency requires that consumption remain constant until the first point in time at which the debt constraint does begin to bind. If the debt constraint never binds, then the equilibrium is a steady state, although not necessarily a symmetric one. If the debt constraint does bind at some point, then from that point on, the stationarity of the model forces the economy to a steady state with constant consumption in which the debt constraints binds on just one consumer type.

The binding case, where the debt constraint is binding at the symmetric first best, is more interesting. Here, it is easy to show that in equilibrium the debt constraint eventually binds on both types of consumers. When it binds for the first time, the equilibrium jumps immediately to the symmetric steady state. The intuition is clear: Efficiency requires that the equilibrium allocation solve the problem of maximizing the expected discounted utility of the unconstrained type from that date onward, subject to the individual rationality constraints for the other consumer type. If the unconstrained type is always the type with the smaller endowment, then this problem is symmetric between the two consumers and the equilibrium after the debt constraint binds for the first time must be both symmetric and stationary.

The dynamic path of consumption and capital can be illustrated by our numerical example. For simplicity we discuss the deterministic case. We first consider an example in the non-binding case. In any such example, we should first check to see if the constant allocation that satisfies the budget constraints also satisfies the individual rationality constraints: if it does then this is the unique equilibrium. To do this check, we observe that in any steady state with non-binding individual rationality constraints the price of capital is $v = \delta r/(1-\delta)$. Letting the first consumer type have the high endowment first, the sequential markets budget constraints are

$$x^1 + v\theta_1^1 = \omega^g + (v+r)\theta_0^1, x^1 + v\theta_2^1 = \omega^b + (v+r)\theta_1^1.$$

We can easily solve these two equations for θ_1^1 and x^1 to find

$$x^1 = \omega^g + \delta(\omega^b - \omega^g)/(1-\delta) + r\theta_0^1$$

and similarly we can find

$$x^2 = \omega^b + \delta(\omega^g - \omega^b)/(1-\delta) + r\theta_0^2 .$$

We then simply check that $u(x^i) + \delta u(x^i) \geq u(\omega^g) + \delta u(\omega^b)$, for $i = 1,2$. To make things interesting, let us suppose that $u(x) = \log x$, $\omega^g = 24$, $\omega^b = 9$, $r = 1$, and $\delta = 3/4$. If $\theta_0^2 < 0.33$, then $x^2 < 15.76$, which can easily be checked to violate the individual rationality constraint. Take then the case in which $\theta_0^2 = 0$. Beginning at $t = 1$, when the individual rationality constraint starts to bind, we nevertheless have $x_t^1 = 18.24$, $x_t^2 = 15.76$. What about $t = 0$? The first order condition for type 1, who is unconstrained in borrowing is

$$\frac{Du(x_0^1)}{\delta Du(x^1)} = \frac{v+r}{v_0}$$

and he faces budget constraint

$$x_0^1 + v_0\theta_1^1 = \omega^b + (v+r)\theta_0^1 .$$

These must be solved for x_0^1 and v_0. Since after $t=0$ we will be at the steady state, both consumers must hold the same capital shares going into period 1 that they will hold going into any odd period. Using this fact, we calculate $x_0^1 = 19.27, \quad x_0^2 = 14.73, \quad v_0 = 3.17$.

Next we examine the binding case. Suppose we lower the discount factor from $\delta = 3/4$ to $\delta = 1/2$. Then we can easily check that at the symmetric first best, the individual rationality constraints bind. Computing the symmetric steady state, we find that $v = 1.29, \quad \theta^g = -1.32, \quad \theta^b = 2.32, \; x^g = 18, \quad x^b = 16$. In period $t=0$ we use the budget constraint and the first order condition,

$$\frac{Du(x_0^1)}{\delta Du(x^b)} = \frac{v+r}{v_0}$$

to solve for $x_0^1,\ v_0$; in the example we have

$$v_0 = \frac{24+\theta_0^1}{16.32-\theta_0^1}$$
$$x_0^1 = \frac{336+14\theta_0^1}{16.32-\theta_0^1}.$$

8. Conclusion

With liquidity constraints the structure of equilibrium in the stochastic case is complicated: it cannot be a stochastic steady state. In a sense the picture is worse than this. Equilibria have been computed in a few special cases, as in Scheinkman and Weiss [1986] and Kehoe, Levine, and Woodford [1992]. There is a general theorem about existence of Markov equilibrium due to Duffie, Geanakoplos, Mas-Colell, and McLennan [1994], and a method of computing approximate equilibria due to Levine [1993]. The equilibria are Markov on a very large state space, however, and as far as we know no model that combines both idiosyncratic and aggregate risk has been successfully used for calibration or estimation. The debt constrained economy is much simpler. Stochastic steady states do exist, and are easy to compute. This is because in a stochastic steady state short run shocks have no long run effects.

Appendix

We treat the general case of random productivity. The results also apply to the special case of the deterministic model of the first half of the paper when we set the reversal probability $\pi = 1$.

Lemma: *Equilibrium capital prices v_s are bounded.*

Proof: Let $\vec{x}^i \in \ell_\infty^{++}$ denote the equilibrium consumption plan of type *i*, and let $\vec{\theta}^i$ be the corresponding plan for holding capital. The strategy of proof is to construct, for any state history *s,* an alternative consumption plan $\vec{x}$ for one of types that satisfies the budget and liquidity constraints. The fact that the utility from the equilibrium plan is at least as good as that from the alternative plan gives rise to an inequality. We can then manipulate this inequality, using the fact that equilibrium consumption must be socially feasible, to derive an upper bound on the capital price v_s. (Capital prices are bounded below because they are nonnegative.)

Fix $s \in S$. One type, say *i*, must hold at least half the physical capital stock in equilibrium at *s*. Consider the alternative plan for type *i,* $\vec{x}$, that consumes $x_s = v_s / 2$ at *s*, $x_\sigma = w_\sigma^i$ for state histories $\sigma > s$ that follow *s*, and $x_\sigma = x_\sigma^i$ (the same as the equilibrium plan) for all other state histories. Since type *i* holds at least half of the physical capital at *s* this plan satisfies the budget and liquidity constraints if we choose capital holding $\theta_\sigma = 0$ for $\sigma \ge s$ and $\theta_\sigma = \theta_\sigma^i$ otherwise. For $0 \le \lambda \le 1$, define the consumption plan $\vec{x}^\lambda$

$$x_\sigma^\lambda = (1-\lambda)x_\sigma^i + \lambda x_\sigma .$$

Since $\vec{x}^\lambda$ is a convex combination of $\vec{x}^i$ and $\vec{x}$, it also satisfies the budget and liquidity constraints if we choose capital holdings $\theta_\sigma^\lambda = (1-\lambda)\theta_\sigma^i + \lambda\theta_\sigma$.

Since at equilibrium prices $\vec{x}^\lambda$ is feasible for *i* and $\vec{x}^i$ is optimal, we must have

$$(1-\delta)\sum_{\sigma\in S}\delta^{t(\sigma)}\pi_\sigma u(x_\sigma^i) \ge (1-\delta)\sum_{\sigma\in S}\delta^{t(\sigma)}\pi_\sigma u(x_\sigma^\lambda).$$

Since $\vec{x}^i$ and $\vec{x}$ differ only along the branch of the tree of state histories that begins at *s* this inequality holds also where the sum on both sides is only over states the equal or follow *s*. Dividing the resulting inequality by $\delta^{t(s)}\pi_s$, we can write

$$(1-\delta)\sum_{\sigma>s}\delta^{t(\sigma)}\pi_\sigma\left[u(x_\sigma^i) - u(x_\sigma^\lambda)\right] \ge -(1-\delta)\delta^{t(s)}\pi_s\left[u(x_\sigma^i) - u(x_\sigma^\lambda)\right].$$

The concavity of u implies that $Du(x_s)\left|x_\sigma^i - x_\sigma^\lambda\right| \geq u(x_\sigma^i) - u(x_\sigma^\lambda)$. This inequality is immediate if $x_\sigma^i \leq x_\sigma^\lambda$, since then $u(x_\sigma^i) - u(x_\sigma^\lambda)$ is nonpositive. If $x_\sigma^i \geq x_\sigma^\lambda$, then $x_\sigma^i \geq x_\sigma$ so for all x'_σ, $x_\sigma^i \geq x'_\sigma \geq x_\sigma^\lambda, Du(x_\sigma) \geq Du(x'_\sigma)$ and the inequality follows from Taylor's theorem. Since in addition, $\left|x_\sigma^i - x_\sigma^\lambda\right| \leq \lambda\omega$, and for $\sigma > s$ $x_\sigma \geq \omega^b$, we conclude that

$$Du(\omega^b)\lambda\omega \geq u(x_\sigma^i) - u(x_\sigma^\lambda).$$

Substituting back into the previous utility inequality, this gives

$$\delta^{t(s)}\pi_s Du(\omega^b)\lambda\omega \geq -(1-\delta)\delta^{t(s)-1}\pi_s\left[u(x_\sigma^i) - u(x_\sigma^\lambda)\right].$$

Dividing both sides by $\delta^{t(s)-1}\pi_s\lambda$ and taking the limit as $\lambda \to 0$ we find

$$\delta Du(\omega^b)\omega \geq (1-\delta)Du(x_\sigma^i)\left[x_s - x_s^i\right].$$

Since $x_s^i \leq \omega$ and $x_s = v_s / 2$, we conclude that if $v_s / 2 \geq \omega$

$$\delta Du(\omega^b)\omega \geq (1-\delta)Du(\omega)\left[v_s / 2 - \omega\right].$$

Consequently

$$v_s \leq \max\left\{2\omega, 2\left[\frac{\delta Du(\omega^b)\omega}{(1-\delta)Du(\omega)} + \omega\right]\right\}$$

❑

Proposition A: *If p_s, x_s^1, x_s^2 is an equilibrium of the debt constrained economy with Arrow-Debreu budget constraints, then there exist prices q_s and v_s and asset holdings θ_s^1 and θ_s^2 such that q_s, v_s, x_s^1, x_s^2, θ_s^1, θ_s^2 is an equilibrium of the economy with sequential markets constraints. Conversely, if q_s, v_s, x_s^1, x_s^2, θ_s^1, θ_s^2 is an equilibrium of the economy with sequential markets constraints, then there exist prices p_s such that p_s, x_s^1, x_s^2 is an equilibrium of the economy with Arrow-Debreu budget constraints*

Proof: Consider first an equilibrium of the economy with Arrow-Debreu budget constraints. The budget constraint is

$$\sum_{s\in S} p_s x_s^i \leq \sum_{s\in S} p_s(w_s^i + \theta_0^i r)$$

If wealth $\sum_{s\in S} p_s(w_s^i + \theta_0^i r)$ is unbounded or if $p_s = 0$ for some s, the consumer's problem has no solution, so in equilibrium neither of these can be the case. From finite wealth and the fact that $w_s^i + \theta_0^i r$ is uniformly bounded away from zero, we conclude that the infinite price vector $\vec{p}$ is an element of ℓ^1; that is, the sequence $\vec{p}$ is summable.

Since $p_s > 0$ and $\vec{p} \in \ell^1$, we can define

$$q_{(s,\eta)} = \frac{r \sum_{\sigma \geq (s,\eta)} p_\sigma}{p_s}$$

$$\theta^i_{(s,\eta)} = \frac{\sum_{\sigma \geq (s,\eta)} p_\sigma (x_\sigma^i - w_\sigma^i)}{p_s q_{(s,\eta)}}.$$

If we now plug these definitions into $x_s^i + q_{(s,1)}\theta^i_{(s,1)} + q_{(s,2)}\theta^i_{(s,2)}$, we find that

$$x_s^i + q_{(s,1)}\theta^i_{(s,1)} + q_{(s,2)}\theta^i_{(s,2)} = w_s^i + (v_s + r)\theta_s^i.$$

Since in equilibrium $x_\sigma^i - w_\sigma^i \geq -\omega$, we also find from the definitions that

$$\theta^i_{(s,\eta)} \geq -\omega / r.$$

Consequently, if $\Theta \geq \omega / r$, any budget feasible plan with respect to the Arrow-Debreu budget constraint is budget feasible with respect to the sequential markets budget constraint. Moreover, the consumption alternative defined in the lemma is budget feasible with respect to the present value budget constraint, so it follows also that the v_s are uniformly bounded.

Now consider an equilibrium of the economy with sequential markets budget constraints. We want show that, if the prices v_s are uniformly bounded, a consumption plan that is feasible with respect to the sequential markets budget constraints is feasible with respect to the corresponding Arrow-Debreu budget constraint.

Let $s = (\eta_1, \ldots, \eta_t)$, and let $\sigma_1 = (\eta_1), \sigma_2 = (\eta_1, \eta_2), \ldots, \sigma_t = s$. We define Arrow-Debreu prices by

$$p_s = \prod_{\tau=1}^{t} \frac{q_{\sigma_\tau}}{v_{\sigma_{\tau-1}} + r}.$$

Since $q_{(s,1)} + q_{(s,2)} = v_s$, and v_s is uniformly bounded, it follows that $\vec{p} \in \ell^1$.

We now recursively work the budget constraint forward, solving the budget constraint for asset holding

$$\theta^i_{(s,\eta)} = \frac{x^i_{(s,\eta)} + q_{(s,\eta,1)}\theta^i_{(s,\eta,1)} + q_{(s,\eta,2)}\theta^i_{(s,\eta,2)} - w^i_{(s,\eta)}}{v_{(s,\eta)} + r}$$

and substituting back into the previous period budget constraint

$$x_s^i + q_{(s,1)}\theta_{(s,1)}^i + q_{(s,2)}\theta_{(s,2)}^i = w_s^i + (v_s + r)\theta_s^i.$$

Using the definition of present value prices, this yields a sequence of budget constraints of the form

$$\sum_{s\in S,t(s)<T} p_s(x_s^i - w_s^i - \theta_0^i r) + \sum_{s\in S,t(s)=T} p_s\theta_s^i \le 0\,.$$

Since $\vec{p} \in \ell^1$ and in equilibrium $1+\Theta \ge \theta_s^i \ge -\Theta$, the final sum vanishes as $T \to \infty$, and the Arrow-Debreu budget constraint is satisfied.

❑

Proposition B: *There is a unique equilibrium of the debt constrained model. During the initial phase, equilibrium consumption is constant. In the binding case, following the initial phase, consumption follows the symmetric steady state. In the non-binding case, following the initial phase, consumption is constant, although possibly different than during the initial phase.*

Proof: We first consider the binding case. Since the first welfare theorem holds, it suffices to show that all efficient allocations have the required property. Uniqueness of equilibrium follows directly since the values of individual consumers' allocations at the supporting prices are monotone in the welfare weights.

To study efficient allocation, we formulate the Pareto problem recursively as the problem of maximizing the utility of a consumer initially in the good state subject to social feasibility, individual rationality and a utility constraint for the other consumer. Denote by $\hat{U}^g, \hat{U}^b$ the average present value utilities received in the good and bad state respectively at the symmetric steady state. We denote by $\overline{U}^g, \overline{U}^b$ the average present value utilities from the endowment in the good and bad state respectively. Note that $\hat{U}^g = \overline{U}^g$. It is convenient also to define the function $\overline{u}(u)$ as the utility a consumer receives when the other consumer receives utility u within a period. Notice that this function is smooth, strictly concave, and that it is its own inverse.

Notice first that the average present value utility a consumer initially in the bad state receives must be at least $\overline{U}^b$, and, since the constraint binds on the other consumer at the symmetric steady state, no more than $\hat{U}^b$. Let $V^g(U^b)$ for $U^b \in [\overline{U}^b, \hat{U}^b]$ be the solution to the problem of maximizing the utility of a consumer initially in the good state subject to social feasibility, individual rationality and a utility constraint for the other consumer. The inverse of this function is denoted by $V^b(U^g)$. Exploiting the symmetry between the two consumers, let u^b be the initial utility of the consumer in the bad state, let $\tilde{U}^b$ be his second period average present value if he remains in the bad state, and let $\tilde{\tilde{U}}^b$ be his second period average present value if he switches to the good state. The Bellman equation is

$$V^g(U^b) = \max_{u^b, \tilde{U}^b, \tilde{\tilde{U}}^b} (1-\delta)\overline{u}(u^b) + \delta(1-\pi)V^g(\tilde{U}^b) + \delta\pi V^b(\tilde{\tilde{U}}^b)$$

subject to

$$\begin{aligned}
&(1-\delta)u^b + \delta(1-\pi)\tilde{U}^b + \delta\pi\tilde{\tilde{U}}^b \geq U^b \\
&\tilde{U}^b \geq \overline{U}^b \\
&\tilde{\tilde{U}}^b \geq \overline{U}^g \\
&V^g(\tilde{U}^b) \geq \overline{U}^g \\
&V^b(\tilde{\tilde{U}}^b) \geq \overline{U}^b.
\end{aligned}$$

The objective function is strictly concave, so this problem has a unique solution. Consequently, it suffices to verify that our proposed plan of time constant consumption in the initial phase, and the symmetric steady state thereafter solves this problem.

Under this proposal $\tilde{U}^b = \hat{U}^b, \tilde{\tilde{U}}^b = \hat{U}^g$, and the utility constraint should bind, so that first constraint holds with equality and is used to determine u^b

$$u^b = U^b + \frac{\delta\pi}{1-\delta}\left(U^b - \hat{U}^g\right).$$

Plugging these guesses into the Bellman equation, and observing that $V^b(\hat{U}^g) = \hat{U}^b$, we can solve to find our proposed value function

$$V^g(U^b) = \frac{(1-\delta)\overline{u}\left(U^b + \frac{\delta\pi}{1-\delta}\left(U^b - \hat{U}^g\right)\right) + \delta\pi\hat{U}^b}{1-\delta(1-\pi)},$$

and we may also solve for the inverse function

$$V^b(U^g) = \frac{(1-\delta)\overline{u}\left(U^g + \frac{\delta\pi}{1-\delta}\left(U^g - \hat{U}^b\right)\right) + \delta\pi\hat{U}^g}{1-\delta(1-\pi)}.$$

We need only show that the first order conditions and constraints are satisfied by our proposed solution.

We begin by verifying the constraints hold at the proposed solution. The first constraint holds with equality by construction. The second constraint holds because the U^b must be in the range $[\overline{U}^b, \hat{U}^b]$. The third constraint holds because the symmetric steady state satisfies the individual rationality restrictions; indeed it holds with equality.

Turning to the fourth constraint, since $\overline{u}$ is strictly decreasing, since $U^b \le \hat{U}^b$, it suffices to show that

$$\overline{u}\left(\hat{U}^b + \frac{\delta\pi}{1-\delta}\left(\hat{U}^b - \hat{U}^g\right)\right) \ge \overline{U}^g + \frac{\delta\pi}{1-\delta}\left(\overline{U}^g - \hat{U}^b\right).$$

To show this, observe that at the symmetric steady state

$$(1-\delta)\hat{u}^\eta + \delta(1-\pi)\hat{U}^\eta + \delta\pi\hat{U}^{-\eta} = \hat{U}^\eta,$$

so that

$$\hat{u}^\eta = \hat{U}^\eta + \frac{\delta\pi}{1-\delta}(\hat{U}^\eta - \hat{U}^{-\eta}).$$

Since $\hat{U}^g = \overline{U}^g$ the inequality in question reads $\overline{u}(\hat{u}^b) \ge \hat{u}^g$. Since the symmetric steady state is socially feasible, in fact $\overline{u}(\hat{u}^b) = \hat{u}^g$. So the fourth inequality is verified.

Similarly for the fifth inequality, we may write it as

$$\overline{u}\left(\hat{U}^g + \frac{\delta\pi}{1-\delta}\left(\hat{U}^g - \hat{U}^b\right)\right) \ge \overline{U}^b + \frac{\delta\pi}{1-\delta}\left(\overline{U}^b - \hat{U}^g\right).$$

Making use of the equations for $\hat{u}^\eta$ above and $\overline{u}(\hat{u}^g) = \hat{u}^b$, this becomes

$$\hat{U}^b + \frac{\delta\pi}{1-\delta}(\hat{U}^b - \hat{U}^g) \ge \overline{U}^b + \frac{\delta\pi}{1-\delta}\left(\overline{U}^b - \hat{U}^g\right),$$

which follows directly from the fact that at the symmetric steady state $\hat{U}^b \ge \overline{U}^b$.

To verify the first order conditions, we guess that all the Lagrange multipliers are zero except for those corresponding to the first and third constraints. So we write the Lagrangean

$$\begin{aligned}&(1-\delta)\overline{u}(u^b) + \delta(1-\pi)V^g(\tilde{U}^b) + \delta\pi V^b(\tilde{\tilde{U}}^b)\\&+\lambda\left((1-\delta)u^b + \delta(1-\pi)\tilde{U}^b + \delta\pi\tilde{\tilde{U}}^b\right) + \mu\tilde{\tilde{U}}^b.\end{aligned}$$

The corresponding first order conditions are

$$\lambda = -D\overline{u}(u^b),\ \lambda = -DV^g(\widetilde{U}^b),\ \mu = -\delta\pi\left(DV^b(\widetilde{\widetilde{U}}^b) + \lambda\right).$$

It suffices to show, therefore, that $D\overline{u}(u^b) = DV^g(U^b)$, and $D\overline{u}(u^b) \geq DV^b(\hat{U}^g)$. From the definition of V^g, V^b above, we have

$$DV^g(U^b) = D\overline{u}\left(U^b + \frac{\delta\pi}{1-\delta}\left(U^b - \hat{U}^g\right)\right)$$

$$DV^b(U^g) = D\overline{u}\left(U^g + \frac{\delta\pi}{1-\delta}\left(U^g - \hat{U}^b\right)\right).$$

By construction,

$$u^b = U^b + \frac{\delta\pi}{1-\delta}\left(U^b - \hat{U}^g\right).$$

This gives $D\overline{u}(u^b) = DV^g(U^b)$ immediately. Substituting into the final inequality, we must show

$$D\overline{u}(U^b + \frac{\delta\pi}{1-\delta}\left(U^b - \hat{U}^g\right)) \geq D\overline{u}\left(\hat{U}^g + \frac{\delta\pi}{1-\delta}\left(\hat{U}^g - \hat{U}^b\right)\right).$$

Since $U^b \leq \hat{U}^b \leq \hat{U}^g$, this follows from the fact that $D\overline{u}$ is strictly decreasing.

Turning to the non-binding case, observe that when the utility constraints do not bind, the unique efficient allocation is for each consumer to have a constant consumption stream in all periods. If we increase the utility of the consumer initially in the bad state, eventually the constraint binds on the consumer in the good state. Since this is an efficient allocation, the consumer initially in the bad state can receives no higher utility in any feasible allocation satisfying the utility constraints.

In the opposite case, where we reduce the utility of the consumer initially in the bad state, eventually the constraint binds on that consumer in the first period following the initial phase. To reduce his utility further, we simply reduce his consumption in the initial phase only, leaving the constraint after the initial phase just binding. It is easy to show that efficiency demands a constant consumption stream during the initial phase, and this allocation can easily be verified to be efficient using exactly the same type of dynamic programming argument used above.

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