1. Sketch the cone generated by the columns of the following matrix:

\[
A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}
\]

What is the cone generated by the first and third columns of the matrix? If 
\( b = (1, 0) \), decide whether or not the system \( Ax = b \) has a solution \( x \geq 0 \).

2. Sketch the cone generated by the columns of the following matrix:

\[
A = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 3 & -2 \end{bmatrix}
\]

3. Sketch the hyperplane generated by the following equation in \( \mathbb{R}^2 \):

\[
x_1 - 2x_2 = 3.
\]

Identify the set of vectors in \( \mathbb{R}^2 \) such that \( x_1 - 2x_2 < 3 \), as well as the set of points such that \( x_1 - 2x_2 > 3 \). Now repeat this exercise with the equation \( x_1 + 2x_2 = 3 \) instead.

4. Prove the following statements given a matrix \( A \in \mathbb{R}^{m \times n} \) and a vector \( b \in \mathbb{R}^m \):

(a) Either \( Ax \geq b \) has a non-negative solution \( x \) or there is a non-negative solution \( y \) such that \( yA \leq 0 \) and \( yb > 0 \), but not both.

(b) Either \( Ax = 0, \sum_i x_i = 1 \) has a non-negative solution or there exists \( y \in \mathbb{R}^m \) such that \( yA \gg 0 \), but not both.

(c) Either \( Ax = 0 \), has a nonzero, non-negative solution or there exists \( y \in \mathbb{R}^m \) such that \( yA \gg 0 \), but not both.

(d) Either \( Ax \leq b \) has a solution or there exists \( y \in \mathbb{R}^m_+ \) such that \( yA \geq 0 \) and \( yb < 0 \), but not both.

(e) Either there exists \( x \gg 0 \) such that \( Ax = 0 \) or there exists \( y \in \mathbb{R}^m \) such that \( yA > 0 \), but not both.

(f) The system \( Ax \ll b \) has a solution if and only if \( y = 0 \) is the only solution to \( \{ yA = 0, yb \leq 0, y \geq 0 \} \).

(g) Let \( F = \{ x \in \mathbb{R}^n : Ax \leq 0 \} \), \( c \in \mathbb{R}^n \) and \( G = \{ x \in \mathbb{R}^n : cx \leq 0 \} \). Prove that \( F \subset G \) if and only if there exists \( y \in \mathbb{R}^m_+ \) such that \( c = yA \).
5. A put option gives the holder the right to sell a stock at a prearranged price, $K$. If tomorrow’s stock price $S$ satisfies $S < K$ then the option holder’s optimal strategy is to exercise her put option, i.e., sell the stock at price $K$ to obtain a benefit of $K - S$. Otherwise, if $S \geq K$ then the put option becomes worthless.

(a) Sketch the graph of the terminal value of a put option with strike price $K$ as a function of the terminal value of the stock $S$.

(b) If the stock price increases by a factor of $u$ with probability $p$ and decreases by a factor of $d$ with probability $1 - p$ and interest rates on bonds are equal to $r$, determine the price of a put option.

6. Consider the following multi-period version of the previous model. There are three periods, indexed by $t = 0, 1, 2$, with $t = 0$ signifying today. There are two assets: a bond that pays an interest rate equal to $r$ every period, and a stock. In every period, the price of the stock can either increase by a factor of $u$ with probability $p$ or decrease by a factor of $d$ with probability $1 - p$.

(a) A European call with maturity date $T$ and strike price $K$ gives its holder the right to buy a stock at price $K$ on date $T$. Suppose that $T = 2$. Calculate the price of the European call . . .

   i. . . . in period $t = 1$ assuming that the stock price went up.
   ii. . . . in period $t = 1$ assuming that the stock price went down.
   iii. . . . in period $t = 0$.

(b) An American call with maturity date $T$ and strike price $K$ gives its holder the right to buy a stock at price $K$ on any date up to and including $T$. Suppose that $T = 2$. Calculate the price of the American call . . .

   i. . . . in period $t = 1$ assuming that the stock price went up.
   ii. . . . in period $t = 1$ assuming that the stock price went down.
   iii. . . . in period $t = 0$.

(c) Is it ever optimal to exercise the American call early, i.e., at any period $t$ less than $T$?

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