1. Consider a production function

\[ y = A \prod_{j=1}^{n} x_j^{\alpha_j} \]

where \( y \) is output, the \( x_j \)'s are inputs, and \( A \) and the \( \alpha_j \)'s are positive constants. Let \( w = (w_1, \ldots, w_n) \) be a vector of input prices. Show that the minimum cost of producing a given output level \( y \) is

\[ C(w, y) = \beta (y/A)^{1/\beta} \prod_{j=1}^{n} (w_j/\alpha_j)^{\alpha_j/\beta}, \]

where \( \beta = \sum_j \alpha_j \). Assuming that \( \beta < 1 \), calculate the corresponding maximum profit function \( \pi(p, w) \), where \( p \) is the output price. What goes wrong if \( \beta \geq 1 \)?

2. Suppose that the direct utility function is \( U(x, y) = (\alpha x^\rho + \beta y^\rho)^{1/\rho} \), with \( \alpha, \beta > 0 \) and \( \rho < 1 \).

(a) Show that the expenditure function is of the form

\[ E(p, q, u) = \rho u (ap^r + bq^r)^{1/r}, \]

where \( p \) and \( q \) are the prices of each of the two goods, \( u \) is the utility level, and \( a, b \) and \( r \) are constants that can be expressed in terms of \( \alpha, \beta \) and \( \rho \).

(b) Find the compensated demand functions and show that the ratio of the expenditure-minimizing quantities equals

\[ x/y = (a/b)(q/p)^{1-r}. \]

(c) The elasticity of \( (x/y) \) with respect to \( (q/p) \) is given by

\[ \frac{d \ln(x/y)}{d \ln(q/p)} \]

and is called the elasticity of substitution. Show that in this example it is constant and equal to \( 1 - r \). What condition must be imposed on \( \rho \) to ensure a non-negative elasticity of substitution, i.e., \( r < 1 \)?
3. Consider an economy with two agents, \( A \) and \( B \). Agent \( A \) is endowed with a left glove and agent \( B \) with a right glove. Let \( v_i > 0 \) be the utility to agent \( i \) if he receives one left glove and one right glove. If an agent receives only one glove or no gloves then he gets a utility of 0.

(a) Find the core of this economy.

(b) Calculate the set of equilibrium prices in this economy. (You may assume that even if an agent receives more than one pair of gloves his utility does not increase beyond \( v_i \).)

(c) Now suppose that there are \( n \) identical copies of agent \( A \) and \( n+1 \) copies of agent \( B \), for any natural number \( n \). Find the core of this economy as well as the set of equilibrium prices. (Hint: You may assume equal treatment of equals.)

*Acknowledgment:* Questions 1 and 2 above were taken from Dixit’s (1995) textbook.