1. Prove the following statements given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$:

(a) Either $Ax \geq b$ has a non-negative solution $x$ or there is a non-negative solution $y$ such that $yA \leq 0$ and $yb > 0$, but not both.

(b) Either $Ax = 0$, $\sum_i x_i = 1$ has a non-negative solution or there exists $y \in \mathbb{R}^m$ such that $yA \gg 0$, but not both. ($x \gg 0$ means $x_i > 0$ for all $i$.)

(c) Either $Ax = 0$, has a nonzero, non-negative solution or there exists $y \in \mathbb{R}^m$ such that $yA \gg 0$, but not both.

(d) Either $Ax \leq b$ has a solution or there exists $y \in \mathbb{R}^m_+$ such that $yA \geq 0$ and $yb < 0$, but not both.

(e) Either there exists $x \gg 0$ such that $Ax = 0$ or there exists $y \in \mathbb{R}^m$ such that $yA > 0$, but not both.

(f) The system $Ax \ll b$ has a solution if and only if $y = 0$ is the only solution to $\{yA = 0, yb \leq 0, y \geq 0\}$.

(g) Let $F = \{x \in \mathbb{R}^n : Ax \leq 0\}$, $c \in \mathbb{R}^n$ and $G = \{x \in \mathbb{R}^n : cx \leq 0\}$. $F \subset G$ if and only if there exists $y \in \mathbb{R}^m_+$ such that $c = yA$.

2. Let $D \in \mathbb{R}^{N \times S}$ be a dividend matrix and $p \in \mathbb{R}^N$ a vector of asset prices. An arbitrage is a portfolio $\theta \in \mathbb{R}^N$ such that either $\theta D \geq 0$ and $p\theta < 0$ or $\theta D > 0$ and $p\theta \leq 0$. ($x > 0$ means $x \geq 0$ and $x \neq 0$.) A state price vector is any vector $\psi \in \mathbb{R}^S_+$ such that $D\psi = p$.

(a) Prove that there is no arbitrage if and only if there is a state-price vector.

(b) Suppose that the first asset is a bond, i.e., $d_{1s} = 1$ for all $s \in S$. Prove that there is no arbitrage if and only if there exist beliefs $\pi \in \Delta(S)$ that attach positive probability to every state and a discount factor such that asset pricing prices may be computed as the discounted expected future dividends, i.e., asset prices “obey” risk neutral valuation.

3. Write down the duals of the following LPs: (a) $\max \{cx : Ax = b, x \geq 0\}$, (b) $\min \{cx : Ax = b, x \geq 0\}$, (c) $\max \{cx : Ax \leq b\}$ (d) $\min \{cx : Ax \geq b\}$.

4. Find the optimal primal and dual solutions to the following LP:

$$\max_{x \geq 0} \quad x_1 + x_2 - 3x_3 \quad \text{s.t.} \quad x_1 + 2x_2 - 3x_3 = 4, \quad 4x_1 + 5x_2 - 9x_3 = 13.$$
5. Convert the following optimization into a linear program.

\[ \begin{align*}
\min_{x,y,z} & \quad |x| + |y| + |z| \\
\text{s.t.} & \quad x + y \leq 1, \quad 2x + z = 3.
\end{align*} \]

6. Let \( V = \max\{\sum_{j=1}^{n} c_{j} x_{j} : \sum_{j=1}^{n} a_{j} x_{j} \leq b, \ x \geq 0\} \). Assume that all \( c_{j} \) and \( a_{j} \) are positive. Show that \( V = b \max_{j} c_{j} / a_{j} \).

7. Use strong duality to prove the Theorem of the Alternative.

8. Calculate the functions \( V(b) \) and \( V'(b) \) (the derivative of \( V \)), where

\[ V(b) = \max_{x \geq 0} x_{1} + 2x_{2} \quad \text{s.t.} \quad x_{1} + \frac{8}{3} x_{2} \leq 4, \ x_{1} + x_{2} \leq b, \ 2x_{1} \leq 3. \]

How do these compare to the set of optimal dual solutions \( y^{*}_{2}(b) \)?

9. Consider the following three-player normal-form game.

\[
\begin{array}{ccc|ccc|ccc}
L & R & & L & R & & L & R & \\
U & 0, 0, 3 & 0, 0, 0 & U & 2, 2, 2 & 0, 0, 0 & U & 0, 0, 0 & 0, 0, 0 \\
D & 1, 0, 0 & 0, 0, 0 & D & 0, 0, 0 & 2, 2, 2 & D & 0, 1, 0 & 0, 0, 3 \\
A & & & B & & & C & & \\
\end{array}
\]

Find: (a) all pure-strategy Nash equilibria, (b) a correlated equilibrium where player 3 plays \( B \) with probability 1, and (c) an information structure such that the correlated equilibrium distribution in (b) is a Bayesian-Nash equilibrium of the game augmented by the information structure.

10. Consider the following three-player game. Player 1 has two actions, \( A_{1} = \{U, D\} \). Player 2 also has two actions, \( A_{2} = \{L, R\} \). Player 3 has \( \text{infinitely many} \) actions, \( A_{3} = \mathbb{Z} \), with typical element \( z \in \mathbb{Z} \). Payoffs are depicted below.

\[
\begin{array}{ccc|ccc|ccc}
L & R & & L & R & & L & R & \\
U & 2, 2, 3 + 1/z & 0, 0, 8 & U & 2, 2, 2 & 0, 0, 0 & U & 2, 2, 0 & 0, 0, 0 \\
D & 0, 0, 0 & 2, 2, 0 & D & 0, 0, 0 & 2, 2, 2 & D & 0, 0, 8 & 2, 2, 3 - 1/z \\
\end{array}
\]

\( z < 0 \) \hspace{2cm} \( z = 0 \) \hspace{2cm} \( z > 0 \)

Intuitively, player 1 chooses a row, player 2 chooses a column, and player 3 chooses a matrix. Thus, if players 1 and 2 choose \( (U, L) \) and player 3 chooses \( z = -4 \) then players 1 and 2 each get a payoff of 2 and player 3 gets 2.75.

(a) Characterize the set of pure-strategy Nash equilibria. (ii) Characterize the set of mixed-strategy Nash equilibria.

(b) (i) Find a correlated equilibrium. (ii) Characterize the set of correlated equilibria.