Contractual Pricing with Incentive Constraints

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Abstract

Extending the notion of contractual pricing introduced by Makowski and Ostroy (2003) to economies with moral hazard, this paper develops a model of team formation and organized competition with three main contributions. Firstly, a team's organization is defined as an allocation of (public or private) commodities together with incentive compatible actions and information to its members. Secondly, a version of price taking equilibrium is proposed to formalize the idea that individuals compete to play games, and that both the games being played as well as the (incentive compatible) outcome of those games are determined competitively. This yields a general equilibrium foundation for efficient correlated equilibrium. Finally, our approach is contrasted with the recent theory of clubs, with emphasis on differences in the nature of prices for team versus club membership that lead to incentive efficiency. Specifically, anonymous pricing of jobs within clubs fails to decentralize incentive efficient allocations in the presence of binding incentive constraints.

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1 Introduction

Households, small businesses, and community centers are just a few examples of some of the most fundamental institutions in society, referred to broadly as *teams*. This paper develops a formal economic model of team formation and organization. Using the framework of Makowski and Ostroy (2003) as a point of departure, the tools of general equilibrium and game theory are employed in a unified framework to identify just how economic forces lead individuals to participate in organized teams.

The organization of a team is interpreted as a *contract* consisting of three broadly-construed components: individual behavior (assumed incentive compatible), trade of physical commodities (which may include contractual payments to individuals), and the allocation of information to team members (which may differ across individuals). The model developed below addresses all three components explicitly by describing the interaction amongst team members as a (normal-form) game with communication, whose outcome is the result of market forces. In other words, a team's organization is an economic outcome, where individuals, bargaining competitively, agree to join a team with an organizational design that maximizes its net worth.

Competition is modeled formally as price taking equilibrium, where individuals take as given *contractual prices*, i.e., personalized prices for membership in organized teams. Contractual prices fulfill two roles: to lead individuals towards joining the right team, and to induce team members to agree on the right contract. The personalized nature of contractual prices is generally unavoidable if such prices are to decentralize efficient organizations. Since individual preferences determine the set of incentive compatible contracts, prices must distinguish between team members with different preferences in order for the right organization to emerge. Otherwise, individuals might be led (by prices) towards suboptimal organizations.

The main technical challenge of this paper involves formulating a team's organizational problem and developing a notion of price taking equilibrium where individuals trade organized team membership in such a way that game-theoretic considerations are accommodated in general equilibrium. To this end, the organizational role of mediation in a team is emphasized throughout the paper. Formally, mediation builds a natural bridge between general equilibrium and game theory.

This section continues with an introductory overview of the paper, followed by a comparison with the relevant literature.

1.1 Overview and Discussion

This paper begins by exploring the view of a team as a family of normal-form games indexed by commodity trades. Such games are augmented to include an additional player above and beyond the original team members, who is indifferent between every possible outcome of the game and whose available strategies consist of the team's net trades. A team's organization is understood as any *correlated equilibrium* of this augmented game. Since this additional player is indifferent, he may be interpreted as an "invisible" *organizer*, or mediating principal: a disinterested party whose responsibility within the team is twofold: to recommend behavior to other team members (tell people what to do), and to implement the team's trading strategy (decide the team's net trade, possibly contingent on his recommendations).

Correlated equilibrium underlines the potential role of secrets in organizations towards improving a team's welfare. For instance, it may be in the team's interest that some individual members know about the team's trades and/or other team members' actions and for others not to know. This informational allocation is in principle an integral part of a team's organization when individual behavior is subject to incentive compatibility. (Without incentive compatibility, there is no need for secrets.) Possibly asymmetric information is allocated to team members by the organizer: on the one hand, he provides information by recommending individual behavior; on the other hand, he withholds it by doing so privately.

With this definition of organization, a model of competition for participation in such organizations is introduced in Section 3. We begin by formalizing the intuition that competition ought to drive individuals towards joining teams and organizing them in such a way that maximizes social welfare. Three structural assumptions drive our formulation of the planning problem. The first is that the goings-on in a team do not affect in any direct way the utility possibilities of members of any other team. The second assumption is that individuals find membership in different teams and/or organizations perfectly substitutable. Finally, we focus on economies with transferable utility.[1]

The planner's problem turns out to be a linear program. Using duality, we define price taking equilibrium so that the first welfare theorem automatically applies.

[1]Arguably, the main role of these assumptions is to simplify exposition, their relaxation is considered beyond the scope of this paper and the subject of further research.

Our notion of price taking equilibrium has three salient features. Firstly, contracts are explicitly priced, and are subject to specific linearity requirements: the price of any convex combination (or lottery) of contracts must equal the same convex combination of prices for the original contracts (otherwise there will be arbitrage opportunities). Secondly, organizers are viewed as operating a reproducible and non-proprietary technology, as (non-scarce) entrepreneurs who sell to individuals the right to participate in a specifically organized team and make the associated commodity transactions subject to any contracts sold being incentive compatible. In equilibrium, organizers' net revenue from all these transactions equals zero. Finally, contractual prices are personalized in that they must generally distinguish between individuals with different preferences to decentralize efficient allocations.

We then consider team membership markets from a different angle. We assume that individuals compete for "occupations" and that all they care about is their occupation as well as that of their co-workers together with what the team does, rather than on any individual's particular type or preferences. Without incentive constraints, this translates into an anonymous "job" market where occupations in teams are priced as private goods would. Individuals of different types purchase their occupation as well as their employer (team), in accordance with Ellickson et al. (1999,2003). With incentive constraints, this anonymous job market fails to be incentive efficient, since an individual's tastes for occupations affects a team's utility possibilities by virtue of determining whether or not actions are incentive compatible.

1.2 Location in the Literature

With hindsight, one of the first formal approaches for addressing social interactions in general equilibrium might be attributed to Lindahl (1919). Much of this work is founded to a large extent on the insight that the outcome of a game is a public good and that as such it may be priced in the spirit originally proposed by Lindahl.

The most relevant literature on general equilibrium may be categorized into the theory of *clubs* and literature based on the *assignment model*.

An economic theory of clubs began with Buchanan (1965), who focused on the issue of crowding. It was rekindled by Ellickson (1973) and his "constituencies," which evolved back into clubs in Cole and Prescott (1997) and Ellickson et al. (1999,2003). Such club theories involve neither incentive constraints nor information problems.

Individuals compete for a club membership that includes their occupation, which lends an interpretation that individuals compete for "jobs" in anonymous markets. In Section 4, we consider this alternative to conclude that, with incentive constraints, anonymous job markets generally fail to decentralize efficient allocations.

Prescott and Townsend (2000) introduced a rudimentary version of incentive constraints into the framework of clubs. Their paper deals with the problem of hiring monitors to relax incentive constraints for workers in general equilibrium. However, by assuming that all agents have the same preferences over consumption, effort, and job, they obviate away the problems alluded to in the previous paragraph. Recently, Zame (2005) developed a formal, general theory of clubs with incentive constraints that is compared to the present approach in Section 4.

As for literature stemming from the assignment model of Shapley and Shubik (1972), the work of Gretsky et al. (1999,2003) may be thought of as studying teams to trace the boundaries of perfect competition. Makowski and Ostroy (2003) are the first to include games in general equilibrium, with their study of *transparent teams*. They focus their analysis on environments where individual members' actions are freely (i.e., at no cost) observable and contractible by all members of any team. Equivalently, teams have access to binding contracts for individual actions, thereby assuming away behavioral incentive constraints within a team. Makowski and Ostroy (2003) develop a model where individuals—led by market forces—form teams that trade in goods markets and bargain competitively over team actions as local public goods. This paper's incremental contribution is the analysis of incentive constraints within a team in general equilibrium by relaxing the assumption of binding contracts.

As regards organization specifically, communication between team members is viewed as a crucial facet, and as such we rely on the correlated equilibrium of Aumann (1987), and on Myerson (1997), who describes it as the solution of a linear program. For specific organizational questions such as the allocation of monitoring, residual claims, and control rights, see Rahman (2005), on which this paper is based.

Finally, there is a vast literature on general equilibrium with incentive constraints. Arguably, the most prevalent papers in this literature are Prescott and Townsend (1984a,1984b), who rely on duality to define Walrasian equilibrium with "built-in" incentive compatibility constraints. Also relevant is the work of Jerez (2003), who emphasizes a linear programming approach to economies with moral hazard and adverse selection.

2 Economic Organization

In this section we define an organization as well as a team's problem of finding an efficient one. We begin by describing a *team production technology* as in Makowski and Ostroy (2003), where team members are thought of playing a given normal-form game. This game involves a virtual team member, referred to as the team's *organizer*, in charge of the team's trading strategy. Trades may include local public goods, such as a household's appliances, or may consist of private goods, such as incentive payments to a firm's employees.

The organizer fulfills two tasks: to make recommendations to players (in private) and to strike (possibly recommendation-contingent) trades for the team. Specifically, he picks a team action/trade pair according to some probability distribution and then privately recommends players to play their part of the drawn team action. Team members, taking the organizer's recommendation into account, recognize the likely recommendations to other team members as well as the team's trades when deciding whether or not to obey the organizer. As such, individuals obtain information from the organizer's recommendation. Just how much information is allocated to individual members thus becomes part of a team's organizational problem.

A *contract* is a probability distribution over team actions and net trades with the property that team members are willing to obey the organizer's recommendations, in other words, a *correlated equilibrium.* A team's organizational problem is to maximize welfare by choosing a correlated equilibrium from amongst all correlated equilibria that on average make net trades consistent with the team's available physical resources.

According to our assumptions on a team's technology, an economic organization (a solution to the team's problem) always exists. Furthermore, the value function that arises by solving the team's problems with different resource constraints is a well-defined, concave function. It is interpreted as the utility of a "representative team" whose members play no game at all and whose utility function is defined on the space of net trades.

We conclude this section deriving an alternative, dual version of a team's organizational problem, which will be useful in Section 3 to motivate the interpretation that *teams maximize profit.*

2.1 A Team Production Technology

This subsection follows closely Makowski and Ostroy (2003). A *team* is a finite collection of individuals. A *type of team* is a finite collection t of types of individual. Teams engage in team production. Given a team of any type t, each of its members $i \in t$ has a finite set[2] of available actions collected in A_t^i, with typical element a_t^i.

Team actions, i.e., profiles of individual actions indexed by team members, are denoted by a_t and belong to the product space

$$A_t = \prod_{i \in t} A_t^i.$$

Team actions have two consequences for the team. First of all, they have repercussions on utilities. Every individual member $i \in t$ is assumed to have an intrinsic utility function over team actions and net trades. Thus, if the team action is $a_t \in A_t$ and the net trade is $z_t \in \mathbb{R}^\ell$, let $v_t^i(a_t, z_t)$ denote i's associated utility.

Our first formal assumption will be on the topological properties of $v_t^i(a_t)$.

Assumption 2.1. For every $i \in t$ and $a_t \in A_t$, the function $v_t^i(a_t)$ is Lipschitz on its effective domain, dom $v_t^i(a_t) = \{z_t \in \mathbb{R}^\ell : v_t^i(a_t, z_t) > -\infty\}$, which is a compact, convex set containing the zero net trade vector.

Team actions also have a direct effect on the team's *trading possibilities.* Every team of any type is assumed to take as given a function

$$v_t^0 : A_t \times \mathbb{R}^\ell \to \{0, -\infty\},$$

called the team's *trading possibilities indicator*, where $v_t^0(a_t, z_t) = 0$ means that it is technologically possible, feasible, for the team to trade z_t when their team action is a_t; the value $-\infty$ means that it is impossible. This leads us to our second assumption, that constrains the set of feasible trading possibilities.

Assumption 2.2. For every $a_t \in A_t$, dom $v_t^0(a_t) = v_t^0(a_t)^{-1}(0)$ is a compact, convex set that contains the zero trade vector.

[2] Although, in the interest of clarity, the main body of the paper assumes finite games, the proofs in Appendix A extend all results to continuous games. See Appendix B for a definition of such games as well as correlated equilibrium therein.

Given a team action a_t and a trade z_t, I will denote the *team's utility* by $v_t(a_t, z_t)$ and define it by the following summation:

$$v_t(a_t, z_t) := \sum_{i \in t} v_t^i(a_t, z_t) + v_t^0(a_t, z_t).$$

The value $v_t(a_t, z_t)$ may be interpreted as the team's welfare (with equal welfare weights by assuming transferable utility, see Section 3) when the team action is a_t and the team's net trade is z_t, for any z_t that is feasible with respect to a_t. If z_t is not feasible with respect to a_t, then z_t lies outside the effective domain of $v_t(a_t)$.

A *correlated team strategy* is any (regular, Borel) probability measure $\sigma_t \in \Delta(A_t \times \mathbb{R}^\ell)$ on the space of team actions and net trades. The quantity $\sigma_t(a_t, z_t)$ stands for the probability (density) that the team takes action a_t and trades z_t.

2.2 The Organization of Team Production

We begin by stating the team's problem of finding an incentive efficient *contract*.[3]

$$\begin{aligned} V_t(z_t) := \sup_{\sigma_t \geq 0} \sum_{(a_t, \hat{z}_t)} \sigma_t(a_t, \hat{z}_t) v_t(a_t, \hat{z}_t) \qquad & \text{s.t.} \\ \sum_{(a_t, \hat{z}_t)} \sigma_t(a_t, \hat{z}_t) &= 1 \\ \sum_{(a_t^{-i}, \hat{z}_t)} \sigma_t(a_t, \hat{z}_t)[v_t^i(b_t^i, a_t^{-i}, \hat{z}_t) - v_t^i(a_t, \hat{z}_t)] &\leq 0 \\ \sum_{(a_t, \hat{z}_t)} \sigma_t(a_t, \hat{z}_t) \hat{z}_t &= z_t. \end{aligned}$$

The team chooses a lottery over team actions and trades subject to a family of incentive constraints indexed by team members $i \in t$ and individual action pairs $a_t^i, b_t^i \in A_t^i$, as well as a resource constraint. The resource constraint requires a team's expected trade to equal some given trade z_t that indexes a family of organizational problems and defines a value function for the team which subsumes its normal-form games. The incentive constraints lead to correlated equilibrium as a game-theoretic solution concept by imagining a virtual "zeroth" player—called the team's *organizer*—whose available actions are the team's trades and whose utility function equals the team's trading possibilities indicator.

[3] The supremum is defined on the set of regular, Borel probability measures on $A_t \times \mathbb{R}^\ell$; for any $f : A_t \times \mathbb{R}^\ell \to \mathbb{R}$, the sum $\sum_{(a_t, \hat{z}_t)} f(a_t, \hat{z}_t)\sigma_t(a_t, \hat{z}_t)$ is shorthand for $\int_{A_t \times \mathbb{R}^\ell} f(a_t, \hat{z}_t) d\sigma_t(a_t, \hat{z}_t)$.

Indeed, consider the following *augmented* normal-form game. The set of players is $t \cup \{0\}$. Let $A_t^0 = \mathbb{R}^\ell$ and

$$A_{t\cup\{0\}} \ := \ A_t \times A_t^0,$$

where A_t^0 is the organizer's action space. The organizer's utility function over $A_{t\cup\{0\}}$ is given by $v_t^0(a_t, z_t)$; all other players $i \in t$ have the same utility function as before, namely $v_t^i(a_t, z_t)$.

By definition, a correlated strategy of the augmented game $\sigma_t \in \Delta(A_{t\cup\{0\}})$ is a *correlated equilibrium* of the augmented game or a *contract* if for every team member $i \in t$ and every pair of individual actions $a_t^i, b_t^i \in A_t^i$,

$$\sum_{(a_t^{-i}, z_t)} \sigma_t(a_t, z_t)[v_t^i(b_t^i, a_t^{-i}, z_t) - v_t^i(a_t, z_t)] \ \leq \ 0.$$

Being indifferent to every (feasible) correlated strategy of the augmented game, the organizer's incentive constraints never bind, so we might as well ignore them. This does not mean, however, that the organizer has no strategic role, as demonstrated by Example 2.3 below.

Example 2.3. Suppose that $\ell = 1$ and that there is only one active player $i \in t$ with the set of actions available to him given by $A_t = \{a_t, b_t, c_t\}$. Let's say that dom $v_t^0(a_t) =$ dom $v_t^0(b_t) =$ dom $v_t^0(c_t) = [0, 1]$. Utility functions are

$$\begin{aligned}
v_t^i(a_t, z_t) &= \min\{2z_t, \tfrac{1}{2}z_t + \tfrac{1}{2}\} \ = \ v_t(a_t, z_t) \\
v_t^i(b_t, z_t) &= 0.6, \qquad v_t(b_t, z_t) \ = \ 1.1 \\
v_t^i(c_t, z_t) &= \min\{2(1 - z_t), \tfrac{1}{2}(1 - z_t) + \tfrac{1}{2}\} \ = \ v_t(c_t, z_t).
\end{aligned}$$

Note that b_t *is never incentive compatible when the organizer plays a pure strategy.* Indeed, for $\frac{1}{3} \leq z_t \leq 1$, player i finds it a best response to play a_t, whereas for $0 \leq z_t \leq \frac{2}{3}$, player i finds it a best response to play c_t instead.

On the other hand, *there are mixed strategies by the organizer that make* b_t *a best response.* For instance, consider the mixed strategy $\mu = \frac{1}{2}[0] + \frac{1}{2}[1]$.[4] Indeed, $v_t^i(a_t, \mu) = v_t^i(c_t, \mu) = \frac{1}{2}$, which is clearly less than $0.6 = v_t^i(b_t, \mu)$. It now follows that b_t is a best response for player i to μ.

Therefore, $V_t(\frac{1}{2}) = 1.1$, whereas the most the team could achieve with i knowing the team's trade would be 1, since b_t is never a best response if the organizer is restricted to pure trading strategies.

[4] Here $[z_t]$ stands for Dirac measure: for any $A \subset \mathbb{R}^\ell$, $[z_t](A) = 1$ if $z_t \in A$ and zero otherwise.

An immediate lesson from this example is that even when all utility functions are concave, there may still be gains to randomization of net trades. Furthermore, gains from such randomization with concave utilities must arise from the relaxation of incentive constraints. More generally, the example suggests that *output uncertainty might be the solution to a team's incentive problem rather than the problem itself.*

Calculating the entire function V_t in Example 2.3, it is noteworthy to remark that it is concave. Concavity turns out to be a general phenomenon, as claimed in the next result. (Its proof, as well as all other proofs, may be found in Appendix A.)

Proposition 2.4. *The function V_t is concave. The supremum in the team's problem is always attained by some σ_t.*

Proposition 2.4 asserts that V_t subsumes a team's goings-on into some concave utility function over possible trades. Concavity points towards possible interpretations of correlated equilibrium. Specifically, the team's problem may be thought to involve two kinds of "lottery." The first kind provides mixed trading strategies in order to create incentives, and the second specializes teams to playing different correlated equilibria. For instance, both such types of lottery are used to calculate the entire function V_t in Example 2.3.

The first kind of lottery allocates *uncertainty* across team members in order to relax incentive constraints (this is related to the incentive-constraint-relaxing lotteries of Arnott and Stiglitz (1988) and Cole (1989)), whereas the second kind is comparable to "public randomization" or the so-called "convexifying effect of large numbers." For example, the distinction between ex ante and ex post lotteries by Bennardo and Chiappori (2003) is similar to ours, except that in this model, what is ex ante and what is ex post will generally depend endogenously on the particular team member at a given correlated equilibrium.

To illustrate, consider the next example, where it turns out to be optimal for one active player to remain ignorant of the team's trading strategy (ex ante lotteries) and another active player to be perfectly informed (ex post lotteries). Such "secrets" involve the organizer's mediation strategy as a crucial part of the team's optimal contract. The motivation for secrets is simply that one player ought to be ignorant of the team's trades for a welfare-improving action to be incentive compatible without making the other active player ignorant, who would benefit from knowing the team's trade.

Example 2.5. Once again, $\ell = 1$ and the relevant trading space is the unit interval. There are two active players, 1 and 2, with action space

$$A_t = A_t^1 \times A_t^2 = \{u_t, d_t\} \times \{l_t, m_t, r_t\}.$$

Let the payoffs to player 1 be given by

$$\begin{aligned} v_t^1(u_t, l_t, z_t) &= v_t^1(u_t, m_t, z_t) = v_t^1(u_t, r_t, z_t) = 1 - z_t^2 \\ v_t^1(d_t, l_t, z_t) &= v_t^1(d_t, m_t, z_t) = v_t^1(d_t, r_t, z_t) = 1 - (1 - z_t)^2, \end{aligned}$$

and let the utility functions for player 2 be

$$\begin{aligned} v_t^2(u_t, l_t, z_t) &= \min\{2z_t, \tfrac{1}{2}z_t + \tfrac{1}{2}\} = v_t^2(d_t, l_t, z_t) \\ v_t^2(u_t, m_t, z_t) &= 0.6 = v_t^2(d_t, m_t, z_t) \\ v_t^2(u_t, r_t, z_t) &= \min\{2(1 - z_t), \tfrac{1}{2}(1 - z_t) + \tfrac{1}{2}\} = v_t^2(d_t, r_t, z_t). \end{aligned}$$

The team's overall utility is given by $v_t(a_t, z_t) = v_t^1(a_t, z_t) + v_t^2(a_t, z_t)$.

Notice that active players don't really care about their opponent's play. However, they care about the strategy of the organizer. For $z_t = \frac{1}{2}$, the team's best correlated equilibrium of its augmented game is given by

$$\sigma_t((u_t, m_t), 0) = \sigma_t((d_t, m_t), 1) = \tfrac{1}{2}.$$

Looking at players' incentive constraints, it follows that when recommended to play m, player 2 attaches equal probability to the trades $z_t = 0$ and $z_t = 1$. On the other hand, when recommended to play u_t, player 1 *knows* that the team's trade will be $z_t = 0$ and when recommended to play d_t he knows that the trade will be $z_t = 1$. Being risk averse, player 1 prefers to know. Similarly, the organizer always knows the recommended moves of active players. Finally, it follows that $V_t(\frac{1}{2}) = 1.6$.

The organizer, indifferent amongst every feasible outcome, may be thought of as a correlating, mediating, or randomizing device, a machine or disinterested party who leads the team by *coordinating team members, striking trades on the team's behalf, and accepting the burden of trading feasibility.* We might imagine a market for organizers where the technology for organization services is perfectly reproducible and non-proprietary and whose cost is negligible. With an unlimited supply of organizers, the price to a team of acquiring organization services should also be negligible. This is explored in Section 3, where competition drives organizational efficiency.

2.3 A Team's Dual Problem

An alternative form of the team's problem is now considered. Notice that the team's problem above is a linear program, and as such has an equivalent dual problem.

Proposition 2.6. *The team's problem's dual is the following linear program.*

$$\begin{aligned} W_t(z_t) &:= \inf_{\lambda\geq 0, v, \overline{p}} \overline{p}\cdot z_t + v \qquad \text{s.t.} \\ v_t(a_t,\hat{z}_t) - \overline{p}\cdot\hat{z}_t - \sum_{(i,b_t^i)} \lambda_t^i(a_t^i,b_t^i)[v_t^i(b_t^i,a_t^{-i},\hat{z}_t) - v_t^i(a_t,\hat{z}_t)] &\leq v \end{aligned}$$

for every $a_t \in A_t$ and $\hat{z}_t \in \mathbb{R}^\ell$. Moreover, $V_t(z_t) = W_t(z_t)$ for every z_t.

The dual above chooses "shadow" prices for primal constraints. The λ's are prices for incentive constraints, v for the probability constraint, and $\overline{p}$ for the team's resources. The dual objective is to minimize a team's utility *gross* of commodity purchases, $\overline{p}\cdot z_t$.

Lemma 2.7. *If $(\lambda, v, \overline{p})$ solves the dual at z_t then v equals the team's* indirect utility*,*

$$V_t^*(\overline{p},\lambda) := \sup_{\sigma_t\in\Delta} \sum_{(a_t,\hat{z}_t)} [v_t(a_t,\hat{z}_t) - \overline{p}\cdot\hat{z}_t - \sum_{(i,b_t^i)} \lambda_t^i(a_t^i,b_t^i)[v_t^i(b_t^i,a_t^{-i},\hat{z}_t) - v_t^i(a_t,\hat{z}_t)]]\sigma_t(a_t,\hat{z}_t).$$

If σ_t solves the team's problem at the same z_t then it also solves the indirect utility.

This indirect utility maximizes a team's welfare by picking correlated strategies without regard to incentive compatibility. At a dual solution, a welfare-maximizing choice exists that is also a correlated equilibrium. Another way of describing the team's problem, called the "*sesquial*," restricts the team to choose correlated equilibria.

$$\begin{aligned} V_t^*(\overline{p}) &:= \sup_{\sigma_t\in\Delta} \sum_{(a_t,\hat{z}_t)} (v_t(a_t,\hat{z}_t) - \overline{p}\cdot\hat{z}_t)\sigma_t(a_t,\hat{z}_t) \quad \text{s.t.} \\ \sum_{(a_t^{-i},\hat{z}_t)} \sigma_t(a_t,\hat{z}_t)[v_t^i(b_t^i,a_t^{-i},\hat{z}_t) - v_t^i(a_t,\hat{z}_t)] &\leq 0. \end{aligned}$$

Proposition 2.8. *Suppose that $(\lambda, v, \overline{p})$ solves a team's dual at some z_t. If σ_t solves the team's problem at the same z_t then it also solves the sesquial, and*

$$\sum_{(a_t,\hat{z}_t)} \sum_{(i,b_t^i)} \lambda_t^i(a_t^i,b_t^i)[v_t^i(b_t^i,a_t^{-i},\hat{z}_t) - v_t^i(a_t,\hat{z}_t)]\sigma_t(a_t,\hat{z}_t) = 0,$$

therefore $V_t^*(\overline{p},\lambda) = V_t^*(\overline{p}) = \sup_{\hat{z}_t}\{V_t(\hat{z}_t) - \overline{p}\cdot\hat{z}_t\}$.

Therefore, a team's organization may be subsumed beneath its commodity purchasing problem, where the team's game is entirely implicit.

The key difference between a team's indirect utility and its sesquial is that in the former the team may choose any correlated strategy and at dual-solving prices *chooses* to purchase a correlated equilibrium, whereas in the latter the team may only choose correlated strategies amongst the set of correlated equilibria. By Proposition 2.8, both views are compatible with the current model of a team's organizational problem. Such compatibility also applies to an economy where individuals may choose to join different teams, as will be seen in Section 3.

But first, we present an example to illustrate the duality derived above.

Example 2.9. Consider a two-member team playing the following normal-form game without commodity trades.

	L	R
U	4, 4	1, 5
D	5, 1	0, 0

The team's problem of maximizing the sum of utilities simplifies to

$$\begin{aligned} \sup_{\sigma \geq 0} \quad 8\sigma(U,L) + 6\sigma(U,R) + 6\sigma(D,L) \quad \text{s.t.} & \\ \sigma(U,L) + \sigma(U,R) + \sigma(D,L) &\leq 1 \\ 4\sigma(U,L) + \sigma(U,R) &\geq 5\sigma(U,L) \\ 4\sigma(U,L) + \sigma(D,L) &\geq 5\sigma(U,L) \end{aligned}$$

and is solved by $\sigma(U,L) = \sigma(U,R) = \sigma(D,L) = 1/3$. The team's welfare equals 20/3.

The dual problem simplifies to

$$\begin{aligned} \inf_{\lambda \geq 0, v} \quad v \quad \text{s.t.} & \\ 8 - \lambda_1(U,D) - \lambda_2(L,R) &\leq v \\ 6 + \lambda_1(U,D) &\leq v \\ 6 + \lambda_2(L,R) &\leq v \end{aligned}$$

and is solved by $\lambda_1(U,D) = \lambda_2(L,R) = 2/3$. The value of the dual clearly equals the value of the primal, 20/3.

3 Organized Competition

This section introduces a version of price taking equilibrium, called *contractual pricing equilibrium*, that incorporates contracts as per Section 2. It extends the version of Makowski and Ostroy (2003) to incorporate incentive compatibility constraints in economies with moral hazard.

We begin by defining the planner's problem of assigning individuals to teams and organizing teams (incentive) efficiently. Doing so requires a *commodity representation* of economic activity. Such a representation defines an activity as a team, a net trade, and a team action. The planner "activates" activities subject to availability of individuals, availability of physical commodities, and contractual incentive compatibility.

We continue by using duality to study a relevant price system for the economy that will decentralize the planner's allocations. Taking personalized activity prices as given, individuals purchase participation in organizations to maximize utility without regard to whether or not they are feasible, incentive compatible. It is argued that with the "right" price system, individuals—led by prices—choose to participate not only in feasibly, incentive compatibly organized teams, but also that their contracts are incentive efficient.

Therefore, economic organizations *emerge* as a result of market competition. Activity prices lead individuals to agree not only upon the team's behavior, but also on the division of the team's surplus. It is also be established that *solving a team's problem is a competitive outcome.* Organizers in the spirit of Section 2 are assumed to be on the long side of the market, competing to deliver organizational services with a technology exhibiting constant returns to scale, and as such earn zero profit in equilibrium. This may also be motivated by interpreting organizers as entrepreneurs, and assuming that "entrepreneurship" is not scarce.

Technically, the version of price taking equilibrium proposed is an expression of complementary slackness conditions derived from the dual to the planner's problem. This approaches relies on the assumption that the economy avails *transferable utility.*

Finally, this section concludes with an illustrative example where individuals may or may not join organized teams as well as trade in commodity markets.

3.1 The Planner's Problem

Consider an economy with finitely many types of *individual* in the set $I = \{1, \dots, n\}$ (which may include type repetitions). The set of possible types of *team*, with typical type of team t, is given by the collection

$$T = \{t \subset I \ : \ t \neq \emptyset\}.$$

There is a continuum of each type of individual. A *population* is a vector $q \in \mathbb{R}^I_{++}$, where $q_i > 0$ stands for the mass of individuals of type i in the economy.

Definition 3.1. An *activity* is a triple (t, a_t, z_t) such that $t \in T$ is a type of team, $a_t \in A_t$ is a team action, and $z_t \in \mathbb{R}^\ell$ is a net trade. Let $\mathcal{A}$ denote the set of all activities, and $\mathcal{A}_i = \{(t, a_t, z_t) \in \mathcal{A} : i \in t\}$. An *allocation* is a positive (regular, Borel) measure on $\mathcal{A}$. Let X be the set of allocations, with typical element x.

An individual's utility over an activity (t, a_t, z_t) is given by $v_i(t, a_t, z_t) = v^i_t(a_t, z_t)$, where v^i_t is given. As a normalization, we assume that $v_i(t, a_t, z_t) = -\infty$ if $i \notin t$. We also write $v_0(t, a_t, z_t) = v^0_t(a_t, z_t)$ for the trading possibilities indicator.

Definition 3.2. *The planner's problem* of allocating (human and physical) resources to maximize welfare incentive compatibly is given by the following linear program.

$$\begin{aligned}
V(q) \ := \ \sup_{x_i \geq 0} \ \sum_{i=0}^{n} \sum_{(t,a_t,z_t)} v_i(t, a_t, z_t) x_i(t, a_t, z_t) \qquad & \text{s.t.} \\
\sum_{(t,a_t,z_t)} x_i(t, a_t, z_t) \quad &= \quad q_i \\
\sum_{(a_t^{-i},z_t)} x_0(t, a_t, z_t)[v_i(t, b^i_t, a^{-i}_t, z_t) - v_i(t, a_t, z_t)] \quad &\leq \quad 0 \\
\sum_{(t,a_t,z_t)} z_t x_0(t, a_t, z_t) \quad &= \quad 0 \\
(x_i - x_0)|_{\mathcal{A}_i} \quad &= \quad 0.
\end{aligned}$$

The planner's objective is to maximize the sum of individual utilities by choosing the type of team t to which every individual belongs, the team's action profile a_t, and its commodity trade z_t. The mass of individuals of type i allocated to activity (t, a_t, z_t) equals $x_i(t, a_t, z_t)$. Every individual of type i allocated to activity (t, a_t, z_t) obtains a utility $v_i(t, a_t, z_t)$; the equal-weighted total utility amongst all individuals of type i consuming activity (t, a_t, z_t) is $v_i(t, a_t, z_t) x_i(t, a_t, z_t)$.

The first family of constraints, indexed by $i \in I$, stipulates that the planner's allocation of individuals of any type i to activities must equal the available mass of those individuals. (In other words, the net supply of individuals equals zero.)

The second family of constraints is indexed by types of team t, individual team members $i \in t$, and individual action pairs $a_t^i, b_t^i \in A_t^i$. To interpret these constraints game-theoretically, suppose that the planner allocates individuals of each type to activities randomly, and even though they may be informed of the planner's allocation, they remain ignorant of the precise activity to which they are associated, until the activity is realized. For instance, take any two activities $\alpha = (t, a_t, z_t)$ and $\beta = (t, b_t, y_t)$ and suppose that the planner intends to activate a positive mass of each. Take any individual of a given type $i \in t$ which by assumption is completely unaware of the activity to which he is contributing until he receives a recommendation from the planner to play a_t^i or b_t^i. Taking a "frequentist" approach, this individual interprets the mass of activities to which he *might* belong (as per the planner's allocation) and constructs therefrom a probability function with which to evaluate his incentive to obey the planner's recommendation. With only two activities, if $a_t^i \neq b_t^i$ then the individual learns the precise activity as soon he receives his recommendation. If $a_t^i = b_t^i$ then the individual attaches a probability of belonging to α and β according to population proportions: $x_0(\alpha)/(x_0(\alpha)+x_0(\beta))$ and $x_0(\beta)/(x_0(\alpha)+x_0(\beta))$. Since the right-hand side equals zero, without loss we may divide both sides by $x_0(\alpha) + x_0(\beta) > 0$ to read the incentive constraints in terms of population masses and probability interchangeably.

The third constraint in the planner's problem is the economy's restriction on physical resources, imposing a zero net supply of trades.

Finally, the fourth family of constraints, indexed by individual types i, requires that allocations be consistent across types of individual. Formally, this is an equality of measures captured by stating that the difference between x_i and x_0 must be zero on the subspace $\mathcal{A}_i := \{(t, a_t, z_t) \in \mathcal{A} : i \in t\}$.[5] Strictly speaking, this constraint is redundant in that the planner's problem could have been equivalently written without it by replacing x_i with x_0 and restricting v_i to $t \ni i$. Nevertheless, including it facilitates an interpretation of activities as consumption goods and allocations as consumption bundles. Acknowledging that *the outcome of a game is a public good* (albeit local), this constraint makes explicit the public nature of activity consumption: every team member must consume exactly the same activity adopted by a team.

[5]Formally, given a measure x_0 and any measurable set B, we define $x_0|_{\mathcal{A}_i}(B) = x_0(B \cap \mathcal{A}_i)$.

3.2 Price Taking Equilibrium

Next, we define a version of price taking equilibrium applicable to this economy and argue its efficiency properties. To this end, we begin by deriving the dual linear program associated with the planner's problem. This will lead us both towards a definition of equilibrium and towards proving the First Welfare Theorem.

Definition 3.3. A *price system* is a family $p = \{\overline{p}, p_i : i \in I\}$ where $\overline{p} \in \mathbb{R}^\ell$ is a vector of *commodity prices* and $p_i : \mathcal{A} \to \mathbb{R}$ is a continuous function of personalized *activity prices* for every $i \in I$.

Theorem 3.4. *The planner's problem's* dual program *is given by:*

$$\begin{aligned} W(q) \ := \ \inf_{\lambda \geq 0, \pi, p} \sum_{i \in I} \pi_i q_i & & \text{s.t.} \\ v_i(t, a_t, z_t) - p_i(t, a_t, z_t) & \leq & \pi_i \\ v_0(t, a_t, z_t) - p_0(t, a_t, z_t) & \leq & 0 \\ \sum_{i \in t \cup \{0\}} p_i(t, a_t, z_t) - \sum_{(i, b_t^i)} \lambda_t^i(a_t^i, b_t^i)[v_i(t, b_t^i, a_t^{-i}, z_t) - v_i(t, a_t, z_t)] & = & \overline{p} \cdot z_t, \end{aligned}$$

where p is a price system. Moreover, $V(q) = W(q)$.

The choice variables in the dual program above correspond to "shadow prices" of relaxing each constraint in the planner's problem. The π_i are associated with the planner's constraint on human resources (the marginal social value of increasing q_i), the vector $\overline{p}$ reflects the marginal social value of relaxing the zero net supply constraint on commodity trades, the functions p_i are associated with the public good constraints, and finally λ is associated with the incentive constraints.

The objective in the dual is interpretable as the sum of "expenditures" on individuals, calculated by adding across individual types the utility of each individual of a given type i, π_i, multiplied by the mass of individuals of that type, q_i.

The first family of constraints is indexed by individual types i and activities (t, a_t, z_t). Interpreting $p_i(t, a_t, z_t)$ as the "money" price (i.e., in units of transferrable utility) to individuals of type i for participating in activity (t, a_t, z_t), the constraints may be read as requiring that the utility $v_i(t, a_t, z_t)$ to an individual of type i from (t, a_t, z_t) *net of its price* $p_i(t, a_t, z_t)$ must be bounded above by π_i for every activity (t, a_t, z_t). Hence, the supremum of all such net utilities must be bounded above by π_i, too. In fact, at a solution of the dual program, the bound on this supremum is tight.

Proposition 3.5. *If* (λ, π, p) *solves the dual then*

$$\sup\{v_i(t, a_t, z_t) - p_i(t, a_t, z_t)\} \;=\; \pi_i$$

for every $i \in I \cup \{0\}$*, with* $\pi_0 = 0$*. Moreover, the* indirect utility

$$v_i^*(p_i) \;:=\; \sup_{\mu \in \Delta(\mathcal{A})} \sum_{(t,a_t,z_t)} [v_i(t, a_t, z_t) - p_i(t, a_t, z_t)]\mu(t, a_t, z_t)$$

also satisfies $v_i^*(p_i) = \pi_i$ *for every* $i \in I \cup \{0\}$*. Every supremum above is attained.*

Proposition 3.5 shows how activity prices p_i determine *contractual prices*: for any type of team t, every $i \in t \cup \{0\}$, and organization $\sigma_t \in \Delta(A_{t\cup\{0\}})$,

$$p_i(\sigma_t) \;:=\; \sum_{(a_t,z_t)} p_i(t, a_t, z_t)\sigma_t(a_t, z_t)$$

may be interpreted as i's price to join t and play σ_t.

An individual's indirect utility reflects his decision problem. Given personalized prices and an endowment of one unit of "self," every i purchases participation in organized teams to maximize utility.[6] By construction, the organizer becomes responsible for the feasibility of a team's commodity trades (since $v_0 = -\infty$ affects his "bottom line" and no other team member's). Neither the organizer nor individual team members care about feasibility or incentive compatibility of organizations they trade. As will be argued subsequently, prices are such that everyone is willing to purchase feasible, incentive compatible organizations.

Next, consider the relationship between contractual prices and commodity prices. For any type of team t and any correlated strategy $\sigma_t \in (A_{t\cup\{0\}})$, let

$$m_t(\sigma_t) \;:=\; \sum_{(a_t,z_t)} \sum_{i \in t\cup\{0\}} (p_i(t, a_t, z_t) - \overline{p} \cdot z_t)\sigma_t(a_t, z_t)$$

be the net money expenditure associated with σ_t.

Proposition 3.6. *If* (λ, π, p) *solves the dual problem then for any type of team* t *and any* $\sigma_t \in \Delta(A_{t\cup\{0\}})$*,* $m_t(\sigma_t) > 0$ *only if* σ_t *is not a correlated equilibrium, and* $m_t(\sigma_t) < 0$ *if* σ_t *is feasible but suboptimal.*

[6]Specifically, we assume the existence of a "money" good m_i that enters utility linearly and may be consumed in arbitrarily positive or negative amounts (and whose price equals one). Therefore, $\sup\{v_i(\mu) + m_i : m_i + p_i(\mu) \leq 0\} = v_i^*(p_i)$, giving the economy *transferable utility*.

Corollary 3.7. *If* (λ, π, p) *solves the dual and* x *solves the primal then for all* t*,*

$$\sum_{(a_t,z_t)} \sum_{i \in t} p_i(t, a_t, z_t) x(t, a_t, z_t) \;=\; \sum_{(a_t,z_t)} \overline{p} \cdot z_t x(t, a_t, z_t)$$

and the organizer's expected expenditure equals zero:

$$\sum_{(a_t,z_t)} p_0(t, a_t, z_t) x(t, a_t, z_t) \;=\; 0.$$

The last two results describe the role of λ in the dual problem. The λ's are shadow prices of a team's incentive constraints, reflecting the opportunity cost of incentive compatibility. The amount $\sum_{(i,b_t^i)} \lambda_t^i(a_t^i, b_t^i)[v_i(t, a_t, z_t) - v_i(t, b_t^i, a_t^{-i}, z_t)]$ may be interpreted as a "tax" on incentive incompatibility, imposed on team members in order for incentive compatible team behavior to be demanded by them. By Corollary 3.7, such taxes add up to zero on average: the sum of personalized contractual prices equals the team's expected net revenue from commodity trades. Corollary 3.7 also implies that $p_0(t, a_t, z_t) = 0$ *for every activity* $(t, a_t, z_t) \in \operatorname{supp} x$. In other words, organizer's don't just earn zero profit on average, but on every such activity.

The previous results describe individual and team behavior when facing dual-solving contractual prices and point out some connections between individual price-taking behavior and the planner's problem. This leads to the following notion of price taking equilibrium, which reconciles the primal and the dual.

Definition 3.8. An *contractual pricing equilibrium* (CPE) is a pair (x, p) such that x is an allocation and p is a price system with which markets clear, contracts clear, and individuals optimize:

$$\begin{aligned}
\sum_{(t,a_t,z_t)} z_t x(t, a_t, z_t) &= 0 \\
\sum_{(t,a_t,z_t) \in \mathcal{A}_i} x(t, a_t, z_t) &= q_i \\
\sum_{(a_t,z_t)} \sum_{i \in t \cup \{0\}} p_i(t, a_t, z_t) x(t, a_t, z_t) &= \sum_{(a_t,z_t)} \overline{p} \cdot z_t x(t, a_t, z_t) \\
\sum_{(a_t,z_t)} p_0(t, a_t, z_t) x(t, a_t, z_t) &= 0 \\
\sum_{(a_t^{-i},z_t)} x(t, a_t, z_t)[v_i(t, b_t^i, a_t^{-i}, z_t) - v_i(t, a_t, z_t)] &\leq 0 \\
(v_i(t, a_t, z_t) - p_i(t, a_t, z_t)) x(t, a_t, z_t) &= v_i^*(p_i) x(t, a_t, z_t).
\end{aligned}$$

The objects that define CPE are a feasible solution to the primal, x, and a price system from which a feasible solution to the dual may be derived. In CPE, the pair (x, p) is associated with two mutually dual linear programs that are linked by Theorem 3.4. The conditions for CPE simply list the requirements for x to be a feasible solution to the primal, for p to yield a feasible solution to the dual, and that there be no duality gap. We interpret each condition below.

The first condition is an equality of trade vectors, requiring that overall trades of physical commodities add up to zero, so commodity markets must clear. The second condition, indexed by I, requires consistency in the assignment of individuals to teams with respect to individual availability, so people markets must also clear. Organizers are not included in these constraints because by assumption they are not scarce. The third condition, indexed by T, requires that individual money payments made in equilibrium by team members coincide with payments associated with a team's trade. By the first constraint, organizers' payments when trading commodities add up to zero, so the money market clears. Therefore all markets clear in CPE.

The fourth condition requires organizers to trade an average of zero units of money in equilibrium. Organizers only trade feasible organizations (i.e., $\sigma_t \in \Delta(A_{t\cup\{0\}})$ such that supp $\sigma_t \subset$ dom v_t^0), therefore $v_0^*(p_0) = 0$, in other words, organizers make no profit. This requirement could have been incorporated in other equivalent ways. For instance, we could have explicitly introduced a mass of organizers that far exceeded the possible mass of teams and assumed that organizer's reservation utility equals zero. Equivalently, we could have assumed that organizational technology exhibits constant returns to scale, and that organizers' indirect utility involves them choosing positive measures instead of probability measures, thereby requiring zero profit for organizers in order for organizational markets to clear. To avoid complications, we may as well require that $p_0(\alpha) = 0$ for every activity.

The fifth condition requires equilibrium allocations to be incentive compatible. This is the intended meaning of "contractual clearance" in the definition of CPE. Without this condition, contractual demand would not necessarily equal supply.

The last condition, indexed by $I \cup \{0\}$, requires that allocations maximize individual utility, including that of organizers. Such utility maximization, net of monetary transfers, does not take into account feasibility or incentive compatibility of organizations. Individuals are led towards feasible, incentive compatible organizations purely by prices. We now study the extent to which they are also led towards efficient ones.

Definition 3.9. An allocation $x \in X$ is called *incentive efficient* if it solves the planner's problem.

This definition agrees with incentive-constrained Pareto efficiency in economies with transferable utility, as follows. Redefine an allocation by (x, m) to include money transfers, where $m : I \to \mathbb{R}$ (assume organizers get no money), and call it incentive-constrained Pareto efficient if it is feasible (i.e., x is feasible in the planner's problem and $\sum_i m_i q_i = 0$) and there exists no other (x', m') such that

$$v_i(x'|_{\mathcal{A}_i}) + m_i' q_i \geq v_i(x|_{\mathcal{A}_i}) + m_i q_i$$

for every i and $v_i(x'|_{\mathcal{A}_i}) + m_i' q_i > v_i(x|_{\mathcal{A}_i}) + m_i q_i$ for some i.

Proposition 3.10. *Let $m : I \to \mathbb{R}$ be any feasible allocation of money. An allocation x is incentive efficient if and only if (x, m) is incentive-constrained Pareto efficient.*

We may now state the main result of this paper.

Theorem 3.11. *Contractual pricing equilibrium exists and is incentive efficient.*

Led by equilibrium prices, individuals choose to join teams and enter into contractual arrangements in such a way that resources are allocated incentive efficiently. Formally, Theorem 3.11 bridges the planner's problem and its dual with the help of organizers. Viewing organizers as entrepreneurs, the fact that in equilibrium they make zero profit illustrates the implicit assumption that entrepreneurship is not scarce. Entrepreneurs may be viewed as team organizers, facilitating contractual efficiency and incentive compatibility. Otherwise there would be gains from organizational trade that such entrepreneurs would be sure to exploit. In equilibrium, no such gains are available beyond equilibrium allocations.

Before presenting an illustrative example, we conclude this subsection by remarking that, in equilibrium, *teams maximize profit.*

Proposition 3.12. *If (x, p) is a contractual pricing equilibrium then*

$$V_t^*(\overline{p}) \;=\; \sum_{i \in t \cup \{0\}} v_i^*(p_i)$$

for any type of team t such that $x_t = x(t \times A_{t \cup \{0\}}) > 0$, and x/x_t solves every individual member's indirect utility as well as the team's.

3.3 Example

An example that immediately comes to mind is the famous assignment model, which is clearly a special case of this economy. We now present an example that incorporates individual matching, incentive constraints, and trade of physical commodities.

Example 3.13. Consider an economy with $\ell = 1$ and $I = \{h, w\}$, where h stands for 'husband,' and w for 'wife.' The population of individuals is given by $q = (1, 1)$. Individuals belonging to singleton teams have the following utility functions:

$$v_h(z_h) \;=\; z_h - \tfrac{1}{2}z_h^2, \qquad v_w(z_w) \;=\; -\tfrac{1}{2}z_w^2,$$

where dom $v_h = [0, 1]$ and dom $v_w = (-\infty, 0]$. Intuitively, husbands are buyers and wives are sellers. Husbands and wives may get 'married' to form a team of type $m = \{h, w\}$ and play the following game. Let $A_m = \{c_m^h, d_m^h\} \times \{c_m^w, d_m^w\}$, and suppose that utility functions are given by dom $v_m^0(a_m) = [0, 1]$ and

	c_m^w	d_m^w
c_m^h	$\frac{9}{16}z_m,\ \frac{9}{16}z_m$	$\frac{3}{16}z_m,\ \frac{12}{16}z_m$
d_m^h	$\frac{12}{16}z_m,\ \frac{3}{16}z_m$	$0,\ 0$

The team's problem is solved by $\sigma_m(c_m^h, c_m^w) = \sigma_m(c_m^h, d_m^w) = \sigma_m(d_m^h, c_m^w) = 1/3$ for any $z_m > 0$. (Random trades won't add value in this case.) The organized team's utility is therefore given by $V_m(z_m) = \min\{z_m, 1\}$.

After a little symbolic arithmetic, it follows that the planner's problem is solved by

$$z_m = 1, \qquad z_w = -\tfrac{1}{\sqrt{2}}, \qquad z_h = 1 - \tfrac{1}{\sqrt{2}},$$

and letting the mass of married couples equal $x_m = 1 - \frac{1}{\sqrt{2}}$. The market price for the commodity is given by $\overline{p} = \frac{1}{\sqrt{2}}$, since wives' partial equilibrium problem equates their quantity supplied with the price.

As regards activity prices, for singleton teams let $p_i(i, z_i) = \overline{p} \cdot z_i$. Equilibrium prices for married couples must leave individuals indifferent between purchasing (m, c_m^h, c_m^w, z_m), (m, c_m^h, d_m^w, z_m), and (m, d_m^h, c_m^w, z_m), and clear the money market:

$$\begin{aligned} v_i(m, c_m^h, c_m^w, z_m) - p_i(m, c_m^h, c_m^w, z_m) \;&=\; v_i(m, c_m^h, d_m^w, z_m) - p_i(m, c_m^h, d_m^w, z_m) \\ &=\; v_i(m, d_m^h, c_m^w, z_m) - p_i(m, d_m^h, c_m^w, z_m), \\ \sum_{i \in \{h,w\}} p_i(m, c_m^h, c_m^w, z_m) + p_i(m, c_m^h, d_m^w, z_m) + p_i(m, d_m^h, c_m^w, z_m) \;&=\; \overline{p} \cdot z_m. \end{aligned}$$

Furthermore, since there are both matched and unmatched husbands and wives, they must be indifferent between marrying and not marrying, which implies that $v_h(m, \sigma_m, z_m) - p_h(m, \sigma_m, z_m) = \frac{3}{4} - \frac{1}{\sqrt{2}}$ and $v_w(m, \sigma_m, z_m) - p_w(m, \sigma_m, z_m) = \frac{1}{4}$. (As for other activities, e.g., ones that are not "activated," many prices would support them not being purchased.) Since $v_h(m, \sigma_m, z_m) = v_w(m, \sigma_m, z_m) = \frac{1}{2}$, it follows that $p_h(m, \sigma_m, z_m) = \frac{1}{\sqrt{2}} - \frac{1}{4}$ and $p_w(m, \sigma_m, z_m) = \frac{1}{4}$. Finally,

$$\begin{aligned} p_h(m, c^h_m, c^w_m, z_m) &= \tfrac{9}{16} - (\tfrac{3}{4} - \tfrac{1}{\sqrt{2}}), \quad p_h(m, c^h_m, d^w_m, z_m) = \tfrac{3}{16} - (\tfrac{3}{4} - \tfrac{1}{\sqrt{2}}), \\ p_h(m, d^h_m, c^w_m, z_m) &= \tfrac{12}{16} - (\tfrac{3}{4} - \tfrac{1}{\sqrt{2}}), \quad p_w(m, c^h_m, c^w_m, z_m) = \tfrac{9}{16} - \tfrac{1}{4}, \\ p_w(m, c^h_m, d^w_m, z_m) &= \tfrac{12}{16} - \tfrac{1}{4}, \quad p_w(m, d^h_m, c^w_m, z_m) = \tfrac{3}{16} - \tfrac{1}{4}. \end{aligned}$$

By the remarks after Corollary 3.7, without loss we may assume that organizer's prices are identically zero. This completes the description of contractual pricing equilibrium in this economy. Incentive efficiency is readily verified.

4 Occupational Equilibrium

So far, we have relied exclusively on the view that individuals of a certain type face personalized prices for team membership in the role dictated by their type. Let us now consider a different view of teams and the market for team membership. Specifically, let us relax this assumption by allowing individuals the possibility of having different occupations within the team. Of course, we might interpret the interaction of team members as including the choice of occupation within a team, rendering this view a special case of the previous one. The main difference here is that individuals are allowed preferences over different occupations. As such, we consider team membership prices indexed by occupations rather than types of individual.

An "occupational equilibrium" that distinguishes the market for individuals from the market for occupations in a team is discussed as an alternative to Section 3, in line with the theory of clubs (see Section 1.2). Individuals compete for jobs in teams, where tastes are defined over what team members do without reference to their particular identity. In other words, we assume that only what "coworkers" do, rather than who they are, matters to prospective team members. The "job market," or the market for occupations, is anonymous in that individuals may purchase participation in a team in any occupation at prices that are not personalized, and competing individuals purchase those jobs that best suit their tastes at prevailing market prices.

Without incentive constraints, this pricing approach suffices to decentralize efficient allocations. However, in the presence of incentive constraints, and as soon as individuals have differing tastes over actions within the same occupation, this approach fails to decentralize (incentive) efficient allocations. This is because individual tastes determine the set of incentive compatible allocations, so without taste-indexed prices there are not enough "degrees of freedom" for efficient decentralization. We develop this argument in more detail and provide an illustrative example below.

4.1 Occupational Market

As usual, let $I = \{1, \ldots, n\}$ denote the set of types of individual, and $q \in \mathbb{R}^I_{++}$ the population of such individual types. Define the set of *occupations* as $\Omega = \{1, \ldots, m\}$, which may include repetitions, with typical element ω. A *type of team* is now a non-empty subset $\tau \subset \Omega$ together with a map $\varphi : \tau \to I$, called a "fill," specifying the type of individual $\varphi(\omega) \in I$ that fills occupation ω in τ.[7] We redefine

$$T \ := \ \{t = (\tau, \varphi) \ : \ \tau \subset \Omega, \tau \neq \emptyset, \varphi : \tau \to I\}$$

to be the set of all types of team. A team of type $t = (\tau, \varphi)$ plays a normal-form game just as before. We make the following normalization.

Assumption 4.1. For every $t = (\tau, \varphi)$ and $t' = (\tau, \psi)$, $A_t = A_{t'}$.

In other words, an occupation involves the same choice of actions for every type of individual. This assumption is, of course, without loss of generality, since we could always augment individual action spaces and suitably restrict utility levels to ensure its satisfaction. Having made this assumption, we may denote the space of team actions available to a team of type $t = (\tau, \varphi)$ by A_τ for any φ.

Next, we define individual preferences. In principle, every individual of any type $i \in I$ has preferences over occupations, types of team, team actions, and net trades. A type of team no longer determines an individual's occupation, since φ need not be injective. Denote by $v_i(\omega, t, a_t, z_t)$ the utility representing such preferences, with $v_i(\omega, t, a_t, z_t) = -\infty$ if $t = (\tau, \varphi)$ and $i \neq \varphi(\omega)$. Let $\mathcal{A}$ be the set of all (t, a_t, z_t) and $\mathcal{A}' = \Omega \times \mathcal{A}$. Finally, let $\mathcal{A}_i = \{(\tau, \varphi, a_\tau, z_\tau) \in \mathcal{A} : (\exists \omega \in \tau)(\varphi(\omega) = i)\}$ and $\mathcal{A}_i^\omega = \{(\tau, \varphi, a_\tau, z_\tau) \in \mathcal{A} : \varphi(\omega) = i\}$.

[7] Thus, it is possible for individuals of the same type to fill more than one occupation in some τ by letting $\varphi(\omega) = i$ for more than one $\omega \in \tau$.

Next, we make an assumption that is often emphasized in the clubs literature, namely that a team's assignment of other individuals to occupations, φ, has no effect on any individual's utility, so individual utility does not vary with coworkers' identities.

Assumption 4.2. For every individual type $i \in I$ as well as every pair $t = (\tau, \varphi)$ and $t' = (\tau, \psi)$, $v_i(\omega, t, a_\tau, z_\tau) = v_i(\omega, t', a_\tau, z_\tau)$. Furthermore, $v_0(t, a_\tau, z_\tau) = v_0(t', a_\tau, z_\tau)$.

Therefore, individuals do not intrinsically care about other team members' types. Furthermore, the team's trading possibilities are also unaffected by the individual types of its members, they are only affected by team actions.

4.2 Equilibrium without Incentive Constraints

From the economy just defined, we now derive a version of the planner's problem assuming that individuals are able to make binding agreements on team actions. Therefore, individual behavior in a team is not subject to any incentive constraints. The planner assigns individuals to occupations in organized teams that play team actions and strike commodity trades.

Consider the following version of the planner's problem, discussed below.

$$V(q) \;:=\; \sup_{x_i \geq 0} \sum_{i=1}^{n} \sum_{(\omega,t,a_t,z_t)} v_i(\omega,t,a_t,z_t)x_i(\omega,t,a_t,z_t) + v_0(t,a_t,z_t)x_0(t,a_t,z_t) \qquad \text{s.t.}$$

$$\begin{aligned} \sum_{(t,a_t,z_t)} x_i(\omega,t,a_t,z_t) &= q_i \\ \sum_{(t,a_t,z_t)} z_t x_0(t,a_t,z_t) &= 0 \\ x_i(\omega) - x_0|_{\mathcal{A}_i^\omega} &= 0. \end{aligned}$$

Intuitively, the planner maximizes welfare by allocating individuals of each type to economic activities. The planner maximizes the sum of utilities of all team members by choosing the mass of individuals assigned to a particular set of activities and occupations, involving types of team, team actions and net trades. The planner chooses positive (regular, Borel) measures x_0 on $\mathcal{A}$ and $x_i(\omega)$ on $\mathcal{A}_i^\omega$ for every (i, ω).

The first family of constraints, indexed by $i \in I$, requires the planner's assignment of individuals to be compatible with their availability. The second constraint requires that net trades add up to zero.

The third family of constraints, indexed by $(i, \omega) \in I \times \Omega$, imposes an equality in the space of measures on $\mathcal{A}_i^\omega$. It requires that the mass of individuals participating in any activity coincide with the mass of other team members participating in the same activity. Thus, if a unit mass of teams of type $t = (\tau, \varphi)$ for φ is activated by the planner, then the same mass of individuals of type $\varphi(\omega)$ must participate in t's activity with occupation ω for every $\omega \in \tau$.

With the construction above, and relying on the material developed in Section 3, one might appeal to duality in order to obtain a notion of "occupational equilibrium." However, by Assumptions 4.1 and 4.2, individuals do not care about a team's fill, φ. This suggests an alternative formulation of the planner's problem, discussed below.

Proposition 4.3. *$U(q) = V(q)$ for every q, where*

$$\begin{aligned} U(q) \ := \ \sup_{x_i \geq 0} \sum_{(i,\omega,\tau,a_\tau,z_\tau)} v_i(\omega, \tau, a_\tau, z_\tau) x_i(\omega, \tau, a_\tau, z_\tau) + v_0(\tau, a_\tau, z_\tau) x_0(\tau, a_\tau, z_\tau) \quad \text{s.t.} \\ \sum_{(\omega,\tau,a_\tau,z_\tau)} x_i(\omega, \tau, a_\tau, z_\tau) \ &= \ q_i \\ \sum_{(\tau,a_\tau,z_\tau)} z_\tau x_0(\tau, a_\tau, z_\tau) \ &= \ 0 \\ \sum_{i \in I} x_i(\omega) - x_0|_{\mathcal{A}'_\omega} \ &= \ 0, \end{aligned}$$

and $\mathcal{A}'_\omega = \{(\tau, a_\tau, z_\tau) : \omega \in \tau\}$.

Relying on Assumption 4.2, this version of the planner's problem derives individual preferences without reference to a team's fill, i.e., the identity of other team members. The planner chooses positive (regular, Borel) measures x_0 on $\mathcal{A}'$ and $x_i(\omega)$ on $\mathcal{A}'_\omega$ for every ω. The first and second families of constraints carry the same intuition as before, namely that the assignment of individuals coincide with their availability, and that the economy's resource constraint is satisfied. The third family of constraints, indexed by $\omega \in \Omega$, ensures the supply of occupations (given by the mass of teams with such occupations) equals the demand for such occupations, $\sum_i x_i(\omega)$.

In U, occupation-contingent team membership $(\omega, \tau, a_\tau, z_\tau)$ is viewed as a private good, whereas in V, $(\omega, \tau, \varphi, a_\tau, z_\tau)$ is viewed as part of a local public good. The condition in U that replaces the Lindahl constraint in V is that the sum of individual demands for activities adds up to aggregate supply. The planner's problem is "folded" to reflect the provision of private rather than public goods.

The equality between U and V is interpreted as a "commodification" of the market for occupations, leading to an anonymous job market. This is confirmed by the anonymous price system that emerges from the dual below.

Proposition 4.4. *The dual of U is given by the following linear program:*

$$\begin{aligned} U(q) \;&=\; \inf_{\pi,p} \;\sum_{i\in I} \pi_i q_i \quad \text{s.t.} \\ v_i(\omega,\tau,a_\tau,z_\tau) - p_\omega(\tau,a_\tau,z_\tau) \;&\le\; \pi_i \\ v_0(\tau,a_\tau,z_\tau) + \overline{p}\cdot z_\tau - \sum_{\omega\in\tau} p_\omega(\tau,a_\tau,z_\tau) \;&\le\; 0 \end{aligned}$$

and there is no duality gap.

The dual minimizes "expenditure" on individuals subject to the usual constraint that the expense on each individual corresponds to his indirect utility (see Proposition 3.5) and that monetary transfers amongst feasible activities add up to zero. (The interpretation of an "organizer" also applies here.) The planner selects a vector π indexed by individual types, a vector $\overline{p}$ of commodity prices, and a family $\{p_\omega(\tau,a_\tau,z_\tau)\}$ of activity prices, indexed by occupational choices, which turn out to be *anonymous*: an individual's price for participating in (τ,a_τ,z_τ) as $\omega\in\tau$ does not depend on his identity. We interpret this as an anonymous, or "commodified," job market.

Definition 4.5. An *occupational equilibrium* is a pair (x,p) such that markets clear and individuals optimize:

$$\begin{aligned} \sum_{(\tau,a_\tau,z_\tau)} z_\tau x_0(\tau,a_\tau,z_\tau) \;&=\; 0, \\ \sum_{(\omega,\tau,a_\tau,z_\tau)} x_i(\omega,\tau,a_\tau,z_\tau) \;&=\; q_i, \\ (v_0(\tau,a_\tau,z_\tau) + \sum_{\omega\in\tau} p_\omega(\tau,a_\tau,z_\tau) - \overline{p}\cdot z_\tau)x_0(\tau,a_\tau,z_\tau) \;&=\; 0, \\ (v_i(\omega,\tau,a_\tau,z_\tau) - p_\omega(\tau,a_\tau,z_\tau))x_i(\omega,\tau,a_\tau,z_\tau) \;&=\; v_i^*(p_i)x_0(\tau,a_\tau,z_\tau). \end{aligned}$$

Individuals compete anonymously for occupations. By duality, equilibrium is efficient.

Theorem 4.6. *An occupational equilibrium exists and satisfies Pareto efficiency.*

Next, we introduce incentive constraints, to argue that incentive efficient allocations (solutions to the planner's incentive-constrained problem) cannot be decentralized by an anonymous job market, unlike the present case.

4.3 Failure of Efficiency with Incentive Constraints

Let us now introduce incentive constraints. Consider the planner's problem below.

$$\begin{aligned}
V(q) \;:=\; \sup_{x_i \geq 0} \sum_{(i,\omega,t,a_t,z_t)} v_i(\omega,t,a_t,z_t)x_i(\omega,t,a_t,z_t) + v_0(t,a_t,z_t)x_0(t,a_t,z_t) \quad \text{s.t.}& \\
\sum_{(\omega,t,a_t,z_t)} x_i(\omega,t,a_t,z_t) &= q_i \\
\sum_{(t,a_t,z_t)} z_t x_0(t,a_t,z_t) &= 0 \\
\sum_{(a_t^{-\omega},z_t)} x_0(t,a_t,z_t)[v_{\varphi(\omega)}(\omega,t,b_t^\omega,a_t^{-\omega},z_t) - v_{\varphi(\omega)}(\omega,t,a_t,z_t)] &\leq 0 \\
x_i(\omega) - x_0|_{\mathcal{A}_i^\omega} &= 0.
\end{aligned}$$

The planner's objective is the same as that for V in the previous subsection, to maximize welfare with respect to x, which must be a positive measure on the same space of activities as before. In fact, the only difference that has been introduced here is the third family of constraints, indexed by $t \in T$, $\omega \in \tau$, and $a_t^\omega, b_t^\omega \in A_\tau^\omega$. The constraints stipulate that the individual of type $\varphi(\omega)$ playing the role of ω must find it incentive compatible to play a_t^ω when recommended to do so by an organizer, as in Section 2.

Notice that we could have omitted φ altogether if it weren't for the third constraint, which imposes incentive compatibility. Regardless of Assumption 4.2, which requires that individual preferences depend only on actions and trades, not others' individual types, an individual's tastes over his own actions determine whether or not some action is incentive compatible given what others play. This has an effect on the dual of the planner's problem below, where efficiently decentralizing prices will generally depend on individual types.

$$\begin{aligned}
V(q) \;=\; \inf_{\lambda \geq 0, p} \sum_{i \in I} \pi_i q_i \quad \text{s.t.}& \\
v_i(\omega,t,a_t,z_t) - p_i(\omega,t,a_t,z_t) &\leq \pi_i, \\
v_0(t,a_t,z_t) - p_0(t,a_t,z_t) &\leq 0 \\
p_0(t,a_t,z_t) + \sum_{\omega \in \Omega} p_{\varphi(\omega)}(\omega,t,a_t,z_t)& \\
- \sum_{(\omega,b_t^\omega)} \lambda_{\varphi(\omega)}(\omega,t,a_t^\omega,b_t^\omega)[v_{\varphi(\omega)}(\omega,t,b_t^\omega,a_t^{-\omega},z_t) - v_{\varphi(\omega)}(\omega,t,a_t,z_t)] &= \overline{p} \cdot z_t.
\end{aligned}$$

In the dual, individuals generally *must* face personalized prices for team membership in order for the planner's allocation to be decentralized, where this now includes an assignment of individuals to occupations. Furthermore, even though by assumption individuals do not care about the identity of those taking up various occupations, it will generally matter to team membership prices, since the incidence of incentive constraints might vary with individual types, thereby affecting a team's contractual possibilities. To illustrate, consider the following example.

Example 4.7. Let $I = \{m, f_1, f_2\}$ and $\Omega = \{h, w\}$, with $q = (1, 1, 1)$. For simplicity, suppose that there is no trade of physical commodities. On their own, individuals of type f_1 obtain a utility of 1, everyone else gets zero. In pairs, they play different games depending on their occupation. Assume that individuals of type m ("males") obtain a sufficiently negative utility from taking the occupation w ("wife"), as well as individuals of type f_1 or f_2 ("females") taking the occupation h ("husband") that only males will be husbands and only females will be wives. Also, therefore, there will be no teams consisting of two females, since one of them would then have to become a husband. If a male marries a female (i.e., forms a doubleton team where the male is the husband and the female is the wife) then the couple plays the following normal form game depending on the female's type (the left bi-matrix corresponds to marrying a female of type f_1 and the right bi-matrix corresponds to f_2):

	L	R
U	4, 4	1, 5
D	5, 1	0, 0

	L	R
U	4, 4	1, 5
D	5, 1	0, 2

In each game, males are row players and females choose columns. The only difference between the two games is that in the left game, f_1 gets a utility of 0 if (D, R) is played, whereas f_2 gets a utility of 2. Husbands' payoffs do not depend on their wives' identity (i.e., whether their type is f_1 or f_2), in line with Assumption 4.2.[8]

By Example 2.9, there is a unique correlated equilibrium for a marriage t_1 between a male and a female of type f_1 that maximizes the sum of utilities, namely $\sigma_{t_1}(U, L) = \sigma_{t_1}(D, L) = \sigma_{t_1}(U, R) = 1/3$. However, this correlated strategy is not incentive compatible in a game played by a marriage t_2 consisting of a male and a female of type 2, since then L is strictly dominated by R. The unique contract that maximizes the sum of utilities is given by $\sigma_{t_2}(U, R) = 1$, a pure-strategy Nash equilibrium.

[8]I thank Bill Zame for help with this example.

Without personalized prices, i.e., prices for correlated strategies that depend on the type of a female, it is impossible to decentralize this economy's efficient allocation, which is to match all the males with all the females of type f_1 and for them to play the correlated strategy σ_{t_1} defined in the previous paragraph.

Indeed, any commodified price system must reward females of type f_1 with at least one unit of net utility, implying that females of type f_2 will want to participate in marriage, since their utility from any team action in a marriage is greater than or equal to that of a female of type f_1. Since females of type f_2 get zero if unmatched and at least 1 if matched, it is necessary for equilibrium that all females of type f_2 be married. (No *personalized* rationing is allowed.) It follows that anonymous prices cannot decentralize the economy's incentive efficient allocation.

5 Conclusions

This paper has identified and explored the concept of organized competition. In doing so we have resorted to general equilibrium and game theory as our main theoretical tools, and found a framework in which to unify them. We have developed a model that captures important economic phenomena: individual substitution possibilities across teams, the allocation of incentive compatible individual as well as team actions and commodities, the allocation of information in organizations, and the interpretation that teams maximize profit.

As regards organizations, our reliance on correlated equilibrium as a game-theoretic solution yielded on the one hand a succinct description of communication amongst team members as part of a team's organization. In particular, correlated strategies gave a complete description of the potential uses of uncertainty for a team.

On the other hand it provided a useful technical tool for placing a team's partial equilibrium problem of finding a welfare-maximizing contract in general equilibrium. By facilitating a simple construction of a team's "indirect utility" from those of its individual members, the profit-maximizing interpretation of team behavior becomes a natural one that is riveted to economic behavior.

As regards organized competition, an overall conclusion might be that individuals compete to play games, and the outcome of those games is determined competitively.

For instance, principals and agents might compete for each other to play some contractual game which may be improved upon by the participation of a monitor. This suggests that the incidence of incentive constraints may also be thought of as a competitive outcome with benefits (better-aligned incentives) and (monitoring) costs.

Formally, a notion of price taking equilibrium was introduced in an economy with transferable utility, whose equilibrium allocations were incentive efficient. Individuals were viewed as owning a single unit of "self" which they sold to purchase team membership. Contractual pricing equilibrium describes how the outcome of a game may be priced, and how individuals, in purchasing team membership, might agree to enter a contractual organization in which they remain less than fully informed of other team members' behavior. In equilibrium, personalized team-membership prices will tend to reflect individual substitution possibilities across teams.

Finally, we commented on the recent theory of clubs, and one important way in which the present approach differs from it. We argued that introducing incentive constraints in a general equilibrium model of teams makes it generally impossible for incentive efficiency to be attained with "anonymous prices" for occupations. Unless all individuals have the same preferences over actions under each occupation (or by a strange coincidence), incentive efficiency is bound to fail. Our objection to such an assumption on preferences further motivates the "teams" approach when those teams are subject to incentive constraints (or equivalently, teams fail to have access to binding contracts) on the grounds that some notion of (incentive) efficiency is desirable for any positive model of competition.

As a technical comment, the proofs of results in the text that appear in Appendix A are written in order to automatically accommodate continuous games, as well as correlated equilibrium therein. See Appendix B for definitions and preliminaries.

Many further questions arise naturally as a result of this paper. Some such questions include more detailed results about a team's economic organization, the effects of private information before and after teams have formed, an explicit introduction of time (and space) into the model, and, of course, relaxing some of the key assumptions above, such as transferable utility and that teams are perfectly substitutable. Answers to those questions are the subject of subsequent papers, and beyond the scope of this particular one. The broad objective in this paper was to lay down a theoretical foundation with which to be able to ask questions about team formation and organization.

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A Proofs

The support of a regular Borel measure μ is denoted by supp μ and defined as the complement of the union of all open sets with measure zero according to μ.

Lemma A.1. *For any $\sigma_t \in \Delta(A_t \times \mathbb{R}^\ell)$, $v_t(\sigma_t) > -\infty$ if and only if* $\operatorname{supp} \sigma_t \subset \operatorname{dom} v_t$*, where* $\operatorname{dom} v_t = \{(a_t, z_t) : v_t(a_t, z_t) > -\infty\}$.

Proof. For necessity, since dom v_t is compact and v_t is continuous on it, by Weierstrass' Theorem (Folland, 1999, page 129, Proposition 4.26) the range $v_t(\operatorname{dom} v_t)$ is compact, hence bounded below by some $v > -\infty$. By monotonicity of Lebesgue's integral, and since σ_t is a probability measure, it follows that $v_t(\sigma_t) = \int v_t d\sigma_t \geq v\sigma_t(\operatorname{dom} v_t) = v$.

For sufficiency, suppose that there is an $(a_t, z_t) \in \operatorname{supp} \sigma_t$ outside dom v_t. Since dom v_t is closed (by virtue of being compact by Assumptions 2.1 and 2.2), the complement of dom v_t, call it U_t, is open. By additivity,

$$v_t(\sigma_t) \;=\; \int v_t d\sigma_t \;=\; \int_{U_t^c} v_t d\sigma_t + \int_{U_t} v_t d\sigma_t.$$

If $\sigma_t(U_t) = 0$ then $\operatorname{supp} \sigma_t \subset U_t^c = \operatorname{dom} v_t$, and in particular $(a_t, z_t) \notin \operatorname{supp} \sigma_t$, a contradiction. Therefore, $\sigma_t(U_t) > 0$, so $\int_{U_t} v_t d\sigma_t = -\infty$, since $v_t = -\infty$ on U_t. Since the range $v_t(\operatorname{dom} v_t)$ is compact, it is bounded above by some $V < +\infty$. Therefore, $\int_{U_t^c} v_t d\sigma_t \leq V\sigma_t(U_t^c) \leq V$, so $v_t(\sigma_t) = -\infty$. □

Proposition 2.4. Take any z_t for which there exists some σ_t with $\operatorname{supp} \sigma_t \subset \operatorname{dom} v_t$ and $\sum \sigma_t(a_t, \hat{z}_t)\hat{z}_t = z_t$. (Otherwise $v_t(\sigma_t) = -\infty$ by Lemma A.1 for every feasible σ_t, so $V_t(z_t) = -\infty$, too.) Viewed as a linear functional on $\Delta(\operatorname{dom} v_t)$, v_t is clearly bounded (being Lipschitz and dom v_t being compact by Assumptions 2.1 and 2.2), hence also continuous (Folland, 1999, page 153, Proposition 5.2). By Lemma A.1, without loss we restrict attention to $\sigma_t \in \Delta(\operatorname{dom} v_t)$. This set of probability measures is the closed unit ball of the dual of the space of continuous functions on dom v_t with the supremum norm. Therefore, by Alaoglu's Theorem (Folland, 1999, page 169), $\Delta(\operatorname{dom} v_t)$ is weak$*$ compact. The set of feasible σ_t (i.e., those satisfying the resource constraint as well as incentive compatibility) is a weak$*$ closed subset of $\Delta(\operatorname{dom} v_t)$, being defined by finitely many linear inequalities, therefore it is weak$*$ compact. By Weierstrass' Theorem, the supremum in the team's problem is attained by some feasible σ_t whenever the feasible set is nonempty.

For concavity of V_t, take any two trades $y_t, z_t \in \mathbb{R}^\ell$, and let μ_t, σ_t solve their associated team's problems. For any $\lambda \in (0, 1)$, the mixture $\lambda\mu_t + (1 - \lambda)\sigma_t$ satisfies incentive compatibility (since μ_t and σ_t satisfy it individually), so it is a feasible solution for the team's problem given the trade $\lambda y_t + (1-\lambda)z_t$. Finally, by definition of supremum, it follows that $V_t(\lambda y_t + (1-\lambda)z_t) \geq \lambda v_t(\mu_t) + (1-\lambda)v_t(\sigma_t) = \lambda V_t(y_t) + (1-\lambda)V_t(z_t)$. $\square$

Proposition 2.6. Let $X = \Delta(A_{t\cup\{0\}})$ and $Y = \mathbb{R} \times \mathbb{R}^{K_t} \times \mathbb{R}^{\ell}$, where $K_t = \prod_{i\in t} \left|A_t^i\right| \times \left|A_t^i\right|$. With the notation of Appendix C, the team's (primal) organizational problem is given by (F, g, h^*), where $F : X \to Y$ determines the primal left-hand side constraints, g determines the right-hand side constraints, and $h^* = v_t$. By inspection, the dual parameters $(\widehat{F}, \widehat{g}, \widehat{h}^*)$ satisfy $\widehat{F} : Y^* \to X^*$, where $X^* = C(A_{t\cup\{0\}})$, since we endow X with the weak$*$ topology (see Lemma B.3), $\widehat{g} = h^*$, and $\widehat{h}^* = g$, so it remains only to show that $\widehat{F} = F^*$, the adjoint of F. Given $(v, \lambda, \overline{p}) \in Y^*$, and $\sigma_t \in X$, by definition $\sigma_t(F^*(v, \lambda, \overline{p})) = (v, \lambda, \overline{p})(F(\sigma_t))$, which in turn equals

$$\int \overline{p} \cdot \hat{z}_t - v + \sum_{(i,b_t^i)} \lambda_t^i(a_t^i, b_t^i)[v_t^i(b_t^i, a_t^{-i}, \hat{z}_t) - v_t^i(a_t, \hat{z}_t)]d\sigma_t.$$

But this is the same as $\sigma_t(\widehat{F}(v, \lambda, \overline{p}))$, yielding duality. To show that there is no duality gap (i.e., $V_t(z_t) = W_t(z_t)$), by Theorem C.1 it suffices to argue that V_t is subdifferentiable with respect to its right-hand side constraints. To do so we apply Theorem C.1 and Theorem C.2 and show that V_t has bounded steepness. First we show that V_t is subdifferentiable with respect to z_t, and then we include the remaining right-hand sides. For bounded steepness, it suffices to show that the directional derivative $V_t'(z_t; y) = \lim_{\lambda\downarrow 0}(V_t(z_t + \lambda y) - V_t(z_t))/\lambda$ is bounded above by a constant times $\|y\|$. For a contradiction, suppose the contrary, that is, $V_t'(z_t; y) = +\infty$. Therefore there is a sequence $\{\lambda_n > 0\}$ such that $\lambda_n \downarrow 0$ and for every $m > 0$ there is an $N_m \in \mathbb{N}$ with $(V_t(z_t + \lambda_n y) - V_t(z_t))/\lambda_n > m$ for all $n > N_m$. (Pick a subsequence such that $N_m = m$.) By Theorem 2.4, for every n there is a σ_t^n such that $\int z d\sigma_t^n(a, z) = z_t + \lambda_n y$ and σ_t^n is a correlated equilibrium with $\int v_t d\sigma_t^n = V_t(z_t + \lambda_n y)$. By the Maximum Theorem (see, e.g., Stokey and Lucas, 1989, Theorem 3.6, page 62), the efficient correlated equilibrium correspondence is upper hemicontinuous (the feasible correlated equilibrium correspondence is clearly continuous since σ_t is feasible for z_t and σ_t' for z_t' only if $\gamma\sigma_t + (1-\gamma)\sigma_t'$ is feasible for $\gamma z_t + (1-\gamma)z_t'$ given $0 < \gamma < 1$), so we may pick σ_t^n converging to some σ_t. Since σ_t^n is a probability measure, there exists $(a_t^n, \hat{z}_t^n) \in \operatorname{supp} \sigma_t^n$ such that $v_t(a_t^n, \hat{z}_t^n) \geq V_t(z_t + \lambda_n y)$ and $v_t(a_t^n, \hat{z}_t^n)/\lambda_n > n$ for every $n \in \mathbb{N}$. But because $A_t \times \operatorname{dom} V_t$ is compact, there is a subsequence $\{(a_t^{n_k}, \hat{z}_t^{n_k})\}$ that converges to some $(a_t, \hat{z}_t)$. Hence, for any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that $|v_t(a_t, \hat{z}_t^{n_k}) - v_t(a_t^{n_k}, \hat{z}_t^{n_k})| < \varepsilon$ for every $k > K$. Finally, $(v_t(a_t, \hat{z}_t^{n_k}) - v_t(a_t, z_t))/\lambda_n \geq (v_t(a_t^{n_k}, \hat{z}_t^{n_k}) - \varepsilon - v_t(a_t, z_t))/\lambda_n = (v_t(a_t^{n_k}, \hat{z}_t^{n_k}) - v_t(a_t, z_t))/\lambda_n - \varepsilon/\lambda_n$. As n tends to infinity, $(v_t(a_t, \hat{z}_t^{n_k}) - v_t(a_t, z_t))/\lambda_n$ tends to $v_t'(a_t, z_t; a_t, \hat{z}_t)$, whereas ε/λ_n tends to zero. However, $(v_t(a_t^{n_k}, \hat{z}_t^{n_k}) - v_t(a_t, z_t))/\lambda_n$ tends to infinity, contradicting that $v_t(a_t)$ is Lipschitz (Assumptions 2.1 and 2.2). Therefore, the directional derivative $V_t'(z_t; y)$ is bounded. Since it is also positively homogeneous, its bound may be written as $V_t'(z_t; y) \leq C\,\|y\|$.

It remains to show that V_t has bounded steepness with respect to all its right-hand side constraints perturbed together. A small perturbation in the incentive constraints' right-hand sides still leaves the set of feasible correlated strategies a compact, convex set. As long as it remains nonempty (which is the only case of interest), the arguments above follow through just the same. Finally, including the probability constraint is subsumed by the z_t constraint since going from a measure whose total mass is one to a mass m is equivalent to going from z_t to z_t/m and multiplying the objective by m. $\square$

Remark A.2. The proof of Proposition 2.6 applies also to continuous games *word for word*. See Appendix B for a description of such games.

Lemma 2.7. By complementary slackness (see the second condition in Proposition C.4), σ_t maximizes

$$\int_{A_{t\cup\{0\}}} v_t(a_t, \hat{z}_t) - \overline{p}\cdot\hat{z}_t - v - \sum_{(i,b_t^i)} \lambda_t^i(a_t^i, b_t^i)[v_t^i(b_t^i, a_t^{-i}, \hat{z}_t) - v_t^i(a_t, \hat{z}_t)]d\sigma_t(a_t, \hat{z}_t)$$

amongst all positive measures, and equals zero since there is no duality gap by Proposition 2.6. Therefore, σ_t maximizes

$$\int_{A_{t\cup\{0\}}} v_t(a_t, \hat{z}_t) - \overline{p}\cdot\hat{z}_t - \sum_{(i,b_t^i)} \lambda_t^i(a_t^i, b_t^i)[v_t^i(b_t^i, a_t^{-i}, \hat{z}_t) - v_t^i(a_t, \hat{z}_t)]d\sigma_t(a_t, \hat{z}_t) - v$$

amongst all probability measures and equals zero, too. The result now follows. $\square$

Proposition 2.8. By definition, if σ_t' is a correlated equilibrium then for any $\lambda \geq 0$,

$$\int \sum_{(i,b_t^i)} \lambda_t^i(a_t^i, b_t^i)[v_t^i(b_t^i, a_t^{-i}, \hat{z}_t) - v_t^i(a_t, \hat{z}_t)]d\sigma_t'(a_t, \hat{z}_t) \ \leq \ 0.$$

By complementary slackness (Proposition C.4), this holds with equality if $\sigma_t' = \sigma_t$ solves the team's primal problem.

Looking at the indirect utility, by Lemma 2.7, for any correlated equilibrium σ_t',

$$\int v_t(a_t, \hat{z}_t) - \overline{p}\cdot\hat{z}_t d\sigma_t(a_t, \hat{z}_t) \ \geq$$
$$\int v_t(a_t, \hat{z}_t) - \overline{p}\cdot\hat{z}_t - \sum_{(i,b_t^i)} \lambda_t^i(a_t^i, b_t^i)[v_t^i(b_t^i, a_t^{-i}, \hat{z}_t) - v_t^i(a_t, \hat{z}_t)]d\sigma_t' \ \geq$$
$$\int v_t(a_t, \hat{z}_t) - \overline{p}\cdot\hat{z}_t d\sigma_t'(a_t, \hat{z}_t).$$

Therefore, σ_t solves the sesquial and $V_t^*(\overline{p}, \lambda) = V_t^*(\overline{p})$. It remains to show that $V_t^*(\overline{p}) = \sup_{\hat{z}_t}\{V_t(\hat{z}_t) - \overline{p}\cdot\hat{z}_t\}$. But this is immediate because replacing $\hat{z}_t$ with $\int \hat{z}_t d\sigma_t(a_t, \hat{z}_t)$ everywhere in $\sup_{\hat{z}_t}\{V_t(\hat{z}_t) - \overline{p}\cdot\hat{z}_t\}$ is exactly the same optimization problem as the sesquial, so they must have the same solution and value. □

Theorem 3.4. In the notation of Appendix C, the primal is defined by the triple (F, g, h^*) where $F : X \to Y$, $g \in Y$, and $h^* \in X^*$, together with

$$X \;=\; \prod_{i=0}^{n} M(\mathcal{A}), \qquad Y \;=\; \mathbb{R}^I \times \mathbb{R}^K \times \mathbb{R}^\ell \times \prod_{i=1}^{n} M(\mathcal{A}),$$

where

$$K = \prod_{t\in T}\prod_{i\in t} |A_t^i| \times |A_t^i|$$

is finite. It can be shown (Conway, 1990, Theorem V.1.3, page 125) that if X is any locally convex space then $(X^*, wk*)^* = X$. Therefore, $(M(\mathcal{A}), wk*)^* = C(\mathcal{A})$.

Endowing $M(\mathcal{A})$ with the weak$*$ topology implies that

$$X^* \;=\; \prod_{i=0}^{n} C(\mathcal{A}), \qquad Y^* \;=\; \mathbb{R}^I \times \mathbb{R}^K \times \mathbb{R}^\ell \times \prod_{i=1}^{n} C(\mathcal{A}).$$

It is clear that the dual constraints map Y^* into X^*, since their left-hand sides involve finite sums of continuous functions, hence are also continuous. It is also clear that g and h^* from the primal correspond to h^* and g in the dual. Therefore, all that remains to prove is that the dual constraints operator $\widehat{F} : Y^* \to X^*$ equals the adjoint F^*. By definition of the adjoint, $x(F^*(y^*)) = y(F(x))$. For any $(\pi, \lambda, p) \in Y^*$ and any $(x_0, x_i) \in X$, this equals

$$\sum_{i=1}^{n}\int \pi_i dx_i - \sum_{(t,i,a_t^i,b_t^i)} \int_{A_t^{-i}\times\mathbb{R}^\ell} \lambda_t^i(a_t^i, b_t^i)(v_i(t, b_t^i, a_t^{-i}, z_t) - v_i(t, a_t, z_t))dx_0(t, a_t, z_t)$$
$$- \int \overline{p}\cdot z_t dx_0(t, a_t, z_t) - \sum_{i=1}^{n}\int_{\mathcal{A}_i} p_i d(x_i - x_0),$$

which is precisely what is obtained from $\widehat{F}$ when the v_i's are interpreted as right-hand side constraints in the purported dual. This completes the derivation of the dual problem.

Next we show that $V(q) = W(q)$. By Theorem C.1, it suffices to show that V is subdifferentiable at all its right-hand side constraints. By Theorem C.2, it suffices to show that V has bounded steepness. Fixing the right-hand side Lindahl constraints at zero, this follows from Proposition 2.6. To incorporate the Lindahl constraints, let $\eta_i \in M(\mathcal{A}_i)$ be any regular, Borel measure on the space of activities for type i individuals. Denote by $V_t(\eta)$, where

$\eta : i \mapsto \eta_i$, the primal value function of the right-hand side Lindahl constraints. The directional derivative $V_t(0;\eta) = \lim_{\lambda \downarrow 0}(V_t(\lambda\eta) - V_t(0))/\lambda$ will be shown to be bounded by a similar technique as in the proof of Proposition 2.6. Suppose $\lambda_n \downarrow 0$, and let $x^n = \{x_0^n, x_i^n : i \in I\}$ solve the primal with $\lambda_n \eta$ for right-hand side constraints. By construction, $x_i^n = x_0^n + \lambda_n \eta_i$. Therefore, $V_t(\lambda_n\eta) = \sum \int v_i dx_i^n = \sum \int v_i d(x_0^n + \lambda_n \eta_i) \leq \sum \int v_i dx_0^n + \int \overline{v}_i \lambda_n d\,|\eta_i|$, where $\overline{v}_i < \infty$ is an upper bound on v_i. Let the primal solution x^∞ when $\lambda = 0$ be a limit point of x^n (this is possible because the primal solution correspondence is upper hemicontinuous), so that $V_t'(0;\eta) \leq \sum \int \overline{v}_i d\,|\eta_i| = \sum \overline{v}_i \,\|\eta_i\|$, implying that V_t has bounded steepness with respect to the Lindahl constraints. The argument for bounded steepness when all right-hand side constraints are perturbed jointly is a repetition of the argument above and that of Proposition 2.6 at the same time. Finally, there is no duality gap. □

Proposition 3.5. Firstly, $v_i(t, a_t, z_t) - p_i(t, a_t, z_t) \leq \pi_i$ for every (t, a_t, z_t), therefore $\pi_i' = \sup\{v_i(t, a_t, z_t) - p_i(t, a_t, z_t)\} \leq \pi_i$. If $\pi_i' < \pi_i$ and $\pi_j' \leq \pi_j$ for $j \neq i$ then $\sum \pi_k' q_k < \sum \pi_k q_k$, contradicting optimality of π and proving the first claim for $i \neq 0$. For the organizer's bound, if $\sup\{v_0 - p_0\} = -\varepsilon < 0$ then $v_0(t, a_t, z_t) - p_0(t, a_t, z_t) < 0$ for every activity, so for any $i \in I$ we may increase p_i to $p_i' = p_i + \varepsilon$ such that $\sup\{v_0 - p_0'\} = 0$. But this implies that $v_i(t, a_t, z_t) - p_i'(t, a_t, z_t) < v_i(t, a_t, z_t) - p_i(t, a_t, z_t)$ and furthermore that $\sup\{v_i - p_i'\} < \sup\{v_i - p_i\} = \pi_i$. Therefore, there is a $\pi' \leq \pi$ that is feasible and leads to a lower objective than π, contradicting optimality. This establishes the first claim.

By Lemma A.1, without loss supp μ is compact, and by Alaoglu's Theorem the choice space for $v_i^*(p_i)$ is compact. Viewed as a linear functional on $\Delta(\mathcal{A})$, $v_i - p_i$ is continuous, so a maximum μ^* exists. By monotonicity of Lebesgue's integral, $\int (v_i - p_i) d\mu^* \leq \sup\{v_i - p_i\}$. Conversely, by definition of supremum there is a sequence $\{\alpha_n\}$ of activities such that $(v_i - p_i)([\alpha_n]) \to \sup\{v_i - p_i\}$, and since Dirac measure[9] is regular, Borel (so in $\Delta(\mathcal{A})$), for any $\varepsilon > 0$ there is an N such that $\sup\{v_i - p_i\} - \varepsilon < (v_i - p_i)([\alpha_n]) \leq \int (v_i - p_i) d\mu^*$ for all $n > N$. Since $\varepsilon > 0$ was arbitrary, $\sup\{v_i - p_i\} \leq \int (v_i - p_i) d\mu^*$, so $\sup\{v_i - p_i\} = \int (v_i - p_i) d\mu^*$ and the result follows. Finally, supremal attainment follows by Weierstrass' Theorem. □

Proposition 3.6. We develop a very similar argument to the proof of Proposition 2.8. If σ_t is a correlated equilibrium then

$$m_t(\sigma_t) \;=\; \int \sum_{(i, b_t^i)} \lambda_t^i(a_t^i, b_t^i)[v_t^i(b_t^i, a_t^{-i}, \hat{z}_t) - v_t^i(a_t, \hat{z}_t)] d\sigma_t(a_t, \hat{z}_t) \;\leq\; 0,$$

where the equality follows from the dual formulation (see Theorem 3.4) and the inequality follows by definition of correlated equilibrium. This establishes the first part of the claim.

[9] The measure $[\alpha]$ is defined as follows: for any $B \subset \mathcal{A}$, $[\alpha](B) = 1$ if $\alpha \in B$ and zero otherwise.

For the second claim, notice that if x solves the primal then $\sigma_t = x/x_t \in \Delta(A_{t\cup\{0\}})$ (where $x_t = x(t \times A_t \times \mathbb{R}^\ell) > 0$) is a correlated equilibrium such that $m_t(\sigma_t) = 0$ by complementary slackness (Proposition C.4). $\square$

Corollary 3.7. Since there is no duality gap by Theorem 3.4, complementary slackness with respect to the primal incentive constraints implies that

$$\int_{A_{t\cup\{0\}}} \sum_{(i,b_t^i)} \lambda_t^i(a_t^i, b_t^i)[v_t^i(b_t^i, a_t^{-i}, \hat{z}_t) - v_t^i(a_t, \hat{z}_t)]dx(t, a_t, \hat{z}_t) \;=\; 0$$

for every t. This establishes the first equality. The second equality is a consequence of Proposition 3.5. According to it, the organizer's indirect utility equals zero. Therefore, supp $x \subset$ dom v_0. But since $v_0(\alpha) = 0$ for all $\alpha \in$ dom v_0, the second claim follows. $\square$

Proposition 3.10. If (x, m) fails to be incentive-constrained Pareto efficient then there exists some other feasible (x', m') such that $v_i(x'|_{\mathcal{A}_i}) + m_i' \geq v_i(x|_{\mathcal{A}_i}) + m_i$ for every i and $v_i(x'|_{\mathcal{A}_i}) + m_i' > v_i(x|_{\mathcal{A}_i}) + m_i$ for some i. Adding across i and by feasibility of monetary transfers, $\sum_i v_i(x'|_{\mathcal{A}_i}) > \sum_i v_i(x|_{\mathcal{A}_i})$, therefore x is not incentive efficient. Conversely, if x is any non-incentive efficient, feasible allocation (if it isn't feasible then immediately it isn't Pareto efficient), then there is another feasible x' that attains a greater value in the planner's problem: $\sum_i v_i(x'|_{\mathcal{A}_i}) - v_i(x|_{\mathcal{A}_i}) = \overline{\beta} > 0$. If $\beta_i q_i = v_i(x|_{\mathcal{A}_i}) - v_i(x'|_{\mathcal{A}_i}) + m_i q_i + \overline{\beta}/n$ then $\sum_i \beta_i q_i = 0$ and $v_i(x'|_{\mathcal{A}_i}) + \beta_i q_i = v_i(x|_{\mathcal{A}_i}) + m_i q_i + \overline{\beta}/n > v_i(x|_{\mathcal{A}_i}) + m_i q_i$ for every i, therefore (x, m) is not Pareto efficient. This completes the proof. $\square$

Theorem 3.11. By Theorem 3.4 there is no duality gap between the planner's problem and its dual, and by Assumptions 2.1 and 2.2 together with Theorem B.11 the feasible set for the planner's problem is not empty, implying that there is a solution to both the primal and the dual by Theorem 3.4 and Theorem C.1 as well as that there is no duality gap. By definition of equilibrium, both x is feasible for the primal and without loss p is feasible for the dual (the third condition for CPE is satisfied on all correlated equilibria by complementary slackness, and p_0 may be redefined if necessary off correlated equilibria, if necessary). By market clearance, individual money payments add up to zero, therefore the value of the primal equals the value of the dual, from which it follows that x solves the primal and that it is incentive efficient. $\square$

Proposition 3.12. Since $x_t > 0$, it follows, by the last condition for CPE, that

$$\sum_{(a_t, z_t)} (v_i(t, a_t, z_t) - p_i(t, a_t, z_t))x(t, a_t, z_t)/x_t \;=\; v_i^*(p_i).$$

Summing across $t \cup \{0\}$ and using the third condition for CPE yields

$$\sum_{(a_t,z_t)} \sum_{i \in t \cup \{0\}} (v_i(t, a_t, z_t) - \overline{p} \cdot z_t) x(t, a_t, z_t)/x_t \;=\; \sum_{i \in t \cup \{0\}} v_i^*(p_i).$$

But by the fourth condition for CPE, it follows that x/x_t must maximize the team's welfare amongst all incentive compatible contracts with the same expected net trade, otherwise the planner's problem would not be solved, therefore, by Proposition 2.8,

$$V_t^*(\overline{p}) \;=\; \sum_{(a_t,z_t)} \sum_{i \in t \cup \{0\}} (v_i(t, a_t, z_t) - \overline{p} \cdot z_t) x(t, a_t, z_t)/x_t,$$

and the result finally follows. □

Proposition 4.3. Both U and V have optimal solutions by Weierstrass' Theorem. (The details of this argument mimic the proof of Proposition 2.4.) Let $\mathcal{A} = \{(\tau, \varphi, a_\tau, z_\tau)\}$ and $\mathcal{A}' = \{(\tau, a_\tau, z_\tau)\}$. First we prove that $U(q) \geq V(q)$. For any $x = \{x_i : i \in I \cup \{0\}\}$ that is feasible for V and any Borel subset B' of $\mathcal{A}'$, let $\widehat{B} = \{(\tau, \varphi, a_\tau, z_\tau) : (\tau, a_\tau, z_\tau) \in B'\}$ and define $x_0'(B') := x_0(\widehat{B})$. Then x_0' satisfies the resource constraint associated with U, since by construction

$$\int z_\tau dx_0'(\tau, a_\tau, z_\tau) \;=\; \int z_t dx_0(\tau, \varphi, a_\tau, z_\tau) \;=\; 0.$$

For any Borel subset B' of $\mathcal{A}'_\omega$ and any (i, ω), define the subset

$$B_i^\omega = \{(\tau, \varphi, a_\tau, z_\tau) \in \mathcal{A} : (\omega, \tau, a_\tau, z_\tau) \in B', \varphi(\omega) = i\},$$

and let $x_i'(\omega, B') := x_i(\omega, B_i^\omega)$. The population constraint on individuals is also satisfied by x_i', since by definition $x_i'(\omega, \mathcal{A}') = x_i(\omega, \mathcal{A})$, and adding across ω yields the required equality.

Next, the private-goods constraint associated with U is derived from the Lindahl constraint associated with V. By definition, $x_i'(\omega, B') = x_i(\omega, B_i^\omega)$, which in turn equals $x_0(B_i^\omega \cap \mathcal{A}_i^\omega)$ according to the Lindahl constraint in V. Adding across i, it follows that

$$\sum_{i \in I} x_i'(\omega, B') \;=\; \sum_{i \in I} x_i(\omega, B_i^\omega) \;=\; \sum_{i \in I} x_0(B_i^\omega) \;=\; x_0(\widehat{B}) \;=\; x_0'(B'),$$

since the B_i^ω are disjoint and their union equals $\widehat{B}$. Therefore, x' derives a feasible solution for U from any feasible solution for V. By Assumption 4.2, $v_i(\omega, \tau, \varphi, a_\tau, z_\tau) = v_i(\omega, \tau, a_\tau, z_\tau)$ for any φ such that $i = \varphi(\omega)$, therefore

$$\int v_i(\omega, \tau, \varphi, a_\tau, z_\tau) dx_i(\omega, \tau, \varphi, a_\tau, z_\tau) \;=\; \int v_i(\omega, \tau, a_\tau, z_\tau) dx_i(\omega, \tau, \varphi, a_\tau, z_\tau),$$

which in turn equals $\int v_i(\omega, \tau, a_\tau, z_\tau) dx_i'(\omega, \tau, a_\tau, z_\tau)$. Hence, the value of the objective in V at x equals that in U at x'. It follows that $U(q) \geq V(q)$, since for any feasible solution x

to V there is a feasible solution x' to U with the same value (so an optimal solution to V is feasible for U).

To show that $U(q) \leq V(q)$, consider any $\widehat{x} = \{\widehat{x}_i : i \in I \cup \{0\}\}$ feasible for U. By the team constraint in U, $\sum_i \widehat{x}_i(\omega) = \widehat{x}_0$, and in particular $\widehat{x}_i(\omega)$ is absolutely continuous with respect to $\widehat{x}_0$. By the Radon-Nikodym Theorem (see Folland, 1999, page 90), $\widehat{x}_i(\omega)$ has a Radon-Nikodym derivative $\widehat{g}_i^\omega \in L^1_+(\widehat{x}_0)$ such that $x_i(\omega, B) = \int_B \widehat{g}_i^\omega dx_0$. Fix any $\alpha = (\tau, a_\tau, z_\tau)$ with $\omega \in \tau$ and let $I(\omega, \alpha) = \{i \in I : \widehat{g}_i^\omega(\alpha) > 0\}$. If $I(\omega, \tau, a_\tau, z_\tau)$ is a singleton for every $\omega \in \tau$ then let $\widehat{\varphi}(\omega) = I(\omega, \tau, a_\tau, z_\tau)$ and

$$g_i^\omega(\tau, \varphi, a_\tau, z_\tau) \;=\; \begin{cases} \widehat{g}_i^\omega(\tau, a_\tau, z_\tau) & \text{if } \varphi = \widehat{\varphi} \\ 0 & \text{otherwise.} \end{cases}$$

Next, suppose $I(\omega, \tau, a_\tau, z_\tau)$ is not a singleton. Pick any $i(\omega) \in I(\omega, \tau, a_\tau, z_\tau)$ for every $\omega \in \tau$. This defines a fill $\widehat{\varphi}$, given by $\widehat{\varphi}(\omega) = i(\omega)$. Let $\widehat{g}(\alpha) = \min\{g^\omega_{i(\omega)}(\omega, \alpha) : \omega \in \tau\}$, which is greater than zero since there are finitely many $\omega \in \tau$ and $i(\omega) \in I(\omega, \alpha)$, and define

$$g^\omega_{i(\omega)}(\omega, \tau, \varphi, a_\tau, z_\tau) \;=\; \widehat{g}(\alpha).$$

Subtract $\widehat{g}(\alpha)$ from $\widehat{g}^\omega_{i(\omega)}(\alpha)$ for every $\omega \in \tau$ and repeat the previous algorithm. At every stage of this algorithm, at least one element from at least one $I(\omega, \alpha)$ is removed, and therefore (since each $I(\omega, \alpha)$ is finite and there are finitely many $\omega \in \tau$) all the elements in $I(\omega, \alpha)$ will be removed after finitely many repetitions of the algorithm. Applying this finitely iterated algorithm to each α eventually leads to a family of functions $\{g_i^\omega\}$. We construct $\{x_i(\omega)\}$ as follows. From $\widehat{x}_i(\omega)$, let For any Borel subset $B \subset \mathcal{A}_i^\omega$, let $x_i(\omega, B) = \int_{B'} \sum_\varphi g_i^\omega(\tau, \varphi, a_\tau, z_\tau) dx_0(\tau, a_\tau, z_\tau)$, where $B' = \{(\tau, a_\tau, z_\tau) : (\exists \varphi)((\tau, \varphi, a_\tau, z_\tau) \in B)\}$. Define x_0 according to the Lindahl constraint for V. This leads to an x that is feasible for V, since it is feasible at every step of the algorithm. It now follows that any $\widehat{x}$ feasible for U leads to an x that is feasible for V with the same value (since individual preferences as well as trading possibilities are independent of φ), so an optimal solution for U leads to a feasible solution for V with the same value. Therefore, $U(q) \leq V(q)$, and finally $U(q) = V(q)$. $\square$

Proposition 4.4. We will use the notation developed in the proof of Proposition 4.3. The proof is similar to the previous results resting on duality. Let

$$X \;=\; M(\mathcal{A}') \times \prod_{i=1}^n \prod_{\omega=1}^m M(\mathcal{A}'_\omega), \qquad Y \;=\; \mathbb{R}^I \times \mathbb{R}^\ell \times \prod_{\omega=1}^m M(\mathcal{A}'_\omega),$$

with dual spaces

$$X^* \;=\; C(\mathcal{A}') \times \prod_{i=1}^n \prod_{\omega=1}^m C(\mathcal{A}'_\omega), \qquad Y^* \;=\; \mathbb{R}^I \times \mathbb{R}^\ell \times \prod_{\omega=1}^m C(\mathcal{A}'_\omega).$$

That these spaces coincide with the domain and range spaces of the primal and the dual, respectively, is verified by inspection. The confirmation that indeed the primal and dual have mutually adjoint constraint operators is verified by the approach of Theorem 3.4 as well as that U is Lipschitz. This completes the proof. □

Theorem 4.6. By Proposition 3.10, Pareto efficiency is equivalent to solving the planner's problem. By Proposition 4.4, there is no duality gap, so by Theorem C.1, the result follows, since there exists a solution to the primal (by Weierstrass' Theorem). □

B Correlated Equilibrium in Continuous Games

Let I be a finite set of individuals, let (A_i, d_i) be a compact metric space of individual actions for every $i \in I$, and let $\mathcal{B}_i = \mathcal{B}(A_i)$ denote its Borel σ-algebra, the smallest σ-algebra that includes every open set defined according to the metric d_i. Let

$$A = \prod_{i=1}^{n} A_i$$

be the product space of action profiles, endowed with the product σ-algebra. Let $\Delta(A)$ stand for the set of regular, Borel probability measures on A, with typical element μ, called a *correlated strategy.* Every individual i has a given utility function $v_i : A \to \mathbb{R}$ that is assumed continuous with respect to $a \in A$.

Definition B.1. $\mu \in \Delta(A)$ is a *correlated equilibrium* if given $i \in I$, $b_i \in A_i$, and $B_i \in \mathcal{B}_i$,

$$\int_{B_i \times A_{-i}} v_i(b_i, a_{-i}) d\mu(a) \ \leq \ \int_{B_i \times A_{-i}} v_i(a) d\mu(a).$$

Clearly, this definition coincides with that for finitely many actions when A is a finite set.

Lemma B.2. *The set of correlated equilibria is a compact, convex set.*

Proof. Let μ and ν be correlated equilibria and consider $\lambda = p\mu + (1-p)\nu$, for any $p \in [0,1]$. By the linearity of integration with respect to measures, it follows that $\int f d\lambda = p \int f d\mu + (1-p) \int f d\nu$, for any bounded, Borel measurable f, and convexity follows. Compactness follows because the set of correlated equilibria is weak$*$ closed (if $\mu_n \to_{wk*} \mu$ then $\int f d\mu_n \to \int f d\mu$ for any bounded, continuous $f : A \to \mathbb{R}$, and $v_i \chi_{B_i}$ is approximable by bounded, continuous functions by Egoroff's Theorem, where χ is a characteristic function), the set of correlated strategies is weak$*$ compact by Alaoglu's Theorem, and a closed subset of any compact set is compact. □

Next, we show existence of correlated equilibrium by extending Myerson's (1997) proof, taken from Nau and McCardle (1990), to continuous games. We begin by defining the continuous version of the strategic incentive problem, followed by its dual. Then we appeal to Theorem 1 in Gretsky et al.(2002), which requires a Lipschitz calculation to finally prove existence. Their relevant results and some preliminaries are collected in Appendix C.

B.1 Primal

Consider the following linear program.

$$\inf_{\alpha \geq 0, \beta} \beta \qquad \text{s.t.}$$

$$(\forall i \in I, a_i \in A_i) \qquad \int_{A_i} d\alpha_i(b_i|a_i) \;=\; 1$$

$$(\forall a \in A) \quad \beta + \sum_{i=1}^{n} \int_{A_i} [v_i(b_i, a_{-i}) - v_i(a)] d\alpha_i(b_i|a_i) \;\geq\; 0.$$

The spaces X of choice variables and Y of right-hand side constraints are given by

$$X \;=\; \mathbb{R} \times \prod_{i=1}^{n} C(A_i, M(A_i)), \qquad Y \;=\; C(A) \times \prod_{i=1}^{n} C(A_i).$$

The space X of choice variables consists of the real number β and $\alpha_i \in C(A_i, M(A_i))$ for every i, a continuous map from A_i to $M(A_i)$, the space of regular, Borel measures on A_i endowed with the weak$*$ topology. It follows that for any $a_i, \widehat{a}_i \in A_i$ sufficiently close to one another, the measures $\alpha_i(\cdot|a_i), \alpha_i(\cdot|\widehat{a}_i)$ will be arbitrarily close.

The primal problem is to minimize β subject to $\alpha_i(\cdot|a_i)$ being a probability measure for every $a_i \in A_i$ and for every $a \in A$, the sum of deviation gains across players exceeds $-\beta$. The measures α_i may be interpreted as players' reactions to recommended behavior. Multiplying the second constraint by -1, $-\beta$ may be interpreted as the minimum sum of utility surpluses to players from obeying recommendations versus deviating according to α.

Now we turn to defining the dual problem associated with this primal. To do so, we will define the relevant dual spaces of choice variables and right-hand side constraints. Once both the primal and dual are defined we will proceed to prove existence of correlated equilibrium in continuous games.

B.2 Dual

To find the dual problem, we must calculate the spaces X^* and Y^*. First we cite a useful result (Conway, 1990, Theorem V.1.3, page 125).

Lemma B.3. *If X is a locally convex space then $(X^*, wk*)^* = X$.*

It follows that $(M(A_i), wk*)^* = C(A_i)$. Therefore:

Proposition B.4. *The dual spaces X^* and Y^* of X and Y are given by*

$$\begin{aligned} X^* &= \mathbb{R} \times \prod_{i=1}^{n} M(A_i, C(A_i)) \\ Y^* &= M(A) \times \prod_{i=1}^{n} M(A_i). \end{aligned}$$

Proof. That Y^* is the dual of Y follows directly from the Riesz Represenation Theorem, whereas the duality of X^* follows from Singer's Representation Theorem (which extends the Riesz Representation Theorem to vector-valued measures). □

Theorem B.5. *The dual problem is given by*

$$\begin{aligned} \sup_{\mu \geq 0, \lambda} \sum_{i=1}^{n} \int_{A_i} d\lambda_i & \qquad \text{s.t.} \\ \int_A d\mu &= 1 \\ (\forall i \in I, b_i \in A_i, B_i \in \mathcal{B}(A_i)) \quad \lambda_i(B_i) + \int_{B_i \times A_{-i}} [v_i(b_i, a_{-i}) - v_i(a)] d\mu(a) &\leq 0 \end{aligned}$$

Before proving the theorem, we state and prove the following useful results.

Lemma B.6. *For any $(\mu, \lambda) \in X$, the map $\Phi : A_i \to M(A_i)$ defined by*

$$\Phi(b_i)(B_i) = \lambda_i(B_i) + \int_{B_i \times A_{-i}} [v_i(b_i, a_{-i}) - v_i(a)] d\mu(a)$$

is continuous.

Proof. Since v_i is continuous, if $\{b_i^n\}$ is a sequence in A_i such that $b_i^n \to b_i \in A_i$ then $v_i(b_i^n, a_{-i}) \to v_i(b_i, a_{-i})$. Since A is compact, v_i is bounded above by some $\overline{v_i} < +\infty$, for which $\int_{B_i \times A_{-i}} \overline{v_i} d\mu = \overline{v_i} \mu(B_i \times A_{-i}) \leq \overline{v_i}$.

By the Bounded Convergence Theorem,

$$\int_{B_i \times A_{-i}} v_i(b_i^n, a_{-i}) d\mu(a) \to \int_{B_i \times A_{-i}} v_i(b_i, a_{-i}) d\mu(a),$$

therefore $\Phi(b_i^n)(B_i) \to \Phi(b_i)(B_i)$ for every Borel set $B_i \in \mathcal{B}(A_i)$, as required. □

Corollary B.7. *For any $(\mu, \lambda) \in X$, the map $\Psi : \mathcal{B}(A_i) \to C(A_i)$ defined by*

$$\Psi(B_i)(b_i) \;=\; \lambda_i(B_i) + \int_{B_i \times A_{-i}} [v_i(b_i, a_{-i}) - v_i(a)] d\mu(a)$$

is a vector measure.

Proof. For any countable family $\{B_i^n\}$ of disjoint Borel sets, we must show that $\Psi(\bigcup_n B_i^n) = \sum_n \Psi(B_i^n)$. For any $b_i \in A_i$, $\Psi(\bigcup_n B_i^n)(b_i) = \sum_n \Psi(B_i^n)(b_i)$ follows by σ-additivity of the integral with respect to μ. That $\Psi(\bigcup_n B_i^n) = \sum_n \Psi(B_i^n)$ is a continuous function of b_i follows by the proof of Lemma B.6. $\square$

Proof of Theorem B.5. We must show that the constraint set is generated by the adjoint $\mathcal{A}^*$ of $\mathcal{A}$, where $\mathcal{A} : X \to Y$ is defined by

$$\mathcal{A}(\alpha, \beta) \;=\; \left(\left[\int d\alpha_i(b_i|a_i) \right]_{i \in I, a_i \in A_i}, \left[\beta + \sum_{i=1}^{n} \int [v_i(b_i, a_{-i}) - v_i(a)] d\alpha_i(b_i|a_i) \right]_{a \in A} \right).$$

By construction, $\mathcal{A}(x) \in Y$, so $y^*(\mathcal{A}(x)) \in \mathbb{R}$ for any $y^* \in Y^*$. The adjoint of $\mathcal{A}$ is defined as the map $\mathcal{A}^*$ that solves $y^*(\mathcal{A}(x)) = x(\mathcal{A}^*(y^*))$.

For any $y^* = (\mu, \lambda) \in Y^*$, the dual problem identifies $\mathcal{A}^*$ as

$$\mathcal{A}^*(\mu, \lambda) \;=\; \left(\int d\mu, \left[\lambda_i(B_i) + \int_{B_i \times A_{-i}} [v_i(b_i, a_{-i}) - v_i(a)] d\mu(a) \right]_{i \in I, b_i \in A_i, B_i \in \mathcal{B}(A_i)} \right).$$

By Corollary B.7, $\mathcal{A}^*$ indeed maps Y^* to X^*. Notice that $(\mu, \lambda)(\mathcal{A}(\alpha, \beta))$ equals

$$\sum_{i=1}^{n} \int_{A_i} \int_{A_i} d\alpha_i(b_i|a_i) d\lambda_i(a_i) + \beta \int_A d\mu + \sum_{i=1}^{n} \int_A \int_{A_i} [v_i(b_i, a_{-i}) - v_i(a)] d\alpha_i(b_i|a_i) d\mu(a),$$

whereas $(\alpha, \beta)(\mathcal{A}^*(\mu, \lambda))$ equals

$$\beta \int d\mu + \sum_{i=1}^{n} \int_{A_i} \int_{A_i} d\alpha_i(b_i|a_i) d\lambda_i(a_i) + \int_A \int_{A_i} [v_i(b_i, a_{-i}) - v_i(a)] d\alpha_i(b_i|a_i) d\mu(a),$$

establishing the required duality between the primal and the dual problems. $\square$

B.3 Existence of Correlated Equilibrium

To prove existence of correlated equilibrium, we show that the value function of the primal problem is subdifferentiable at the primal constraint, after which Theorem 1 of Gretsky et al.(2002) may be summoned (see Appendix C for details).

Let $v(\mathbf{b})$ denote the value function from the primal, where $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_i)_{i\in I} \in Y$:

$$\begin{aligned} v(\mathbf{b}) &= \inf_{\alpha\geq 0,\beta} \beta \qquad \text{s.t.} \\ (\forall i \in I, a_i \in A_i) \quad \int_{A_i} d\alpha_i(b_i|a_i) &= \mathbf{b}_i, \\ (\forall a \in A) \quad \beta + \sum_{i=1}^{n} \int_{A_i} [v_i(b_i, a_{-i}) - v_i(a)] d\alpha_i(b_i|a_i) &\geq \mathbf{b}_0(a). \end{aligned}$$

Clearly, $v(\mathbf{0}) = 0$. Furthermore, $v(\mathbf{0}_0, \mathbf{1}) = 0$, as the next lemma demonstrates with an extension of the finite dimensional argument in Myerson (1997).

Lemma B.8. *Let* $\mathbf{b}_0 = 0$ *and* $\mathbf{b}_i(a_i) = 1$ *for every* $a_i \in A_i$. *Then* $v(\mathbf{b}) = 0$.

Proof. First notice that $v(\mathbf{b}) \leq 0$, since setting $\beta = 0$ and

$$\alpha_i(B_i|a_i) = \begin{cases} 1 \text{ if } a_i \in B_i \\ 0 \text{ otherwise} \end{cases}$$

is a feasible solution that yields a value of zero. Indeed, if $\{a_i^n\}$ is a sequence in A_i converging to a_i then $\{\alpha_i(\cdot|a_i^n)\}$ is a sequence of measures that converges to $\alpha_i(\cdot|a_i)$ in the weak$*$ topology (although *not in the norm topology*), since for any bounded, continuous f,

$$\int f(b_i) d\alpha(b_i|a_i^n) = f(a_i^n)\mathbf{b}_i(a_i^n) \to f(a_i)\mathbf{b}_i(a_i)$$

because both f and $\mathbf{b}_i$ are continuous and hence so is their product. Therefore, α_i belongs to $C(A_i, M(A_i))$. On the other hand, if α_i has a stationary distribution then $v(\mathbf{b}) \geq 0$. This requires the existence of a positive measure $\sigma_i \in M(A_i)$ such that

$$\int_{A_i} \alpha_i(B_i|a_i) d\sigma_i(a_i) = \sigma_i(B_i)$$

for every $B_i \in \mathcal{B}(A_i)$. To prove that such α_i-stationary distribution exists, define the map $F : \Delta(A_i) \to \Delta(A_i)$ by

$$F(\sigma_i)(B_i) = \int \alpha_i(B_i|a_i) d\sigma_i(a_i).$$

$F(\sigma_i) \in \Delta(A_i)$ because for any sequence of disjoint, Borel subsets $\{B_i^n\} \subset \mathcal{B}(A_i)$,

$$\alpha_i(B_i^n|a_i) \leq \sum_n \alpha_i(B_i^n|a_i) = \alpha_i(\bigcup_n B_i^n|a_i) \leq 1$$

for every $a_i \in A_i$, so by the Bounded Convergence Theorem

$$\sum_n \int \alpha_i(B_i^n|a_i) d\sigma_i(a_i) = \int \sum_n \alpha_i(B_i^n|a_i) d\sigma_i(a_i) = \int \alpha_i(\bigcup_n B_i^n|a_i) d\sigma_i(a_i).$$

Next we show that F is continuous. If $\sigma_i^n \to_{wk*} \sigma_i$ then

$$\int \alpha_i(B_i|a_i)d\sigma_i^n(a_i) \to \int \alpha_i(B_i|a_i)d\sigma_i(a_i)$$

for every B_i, since $\alpha_i(B_i|a_i)$ is continuous in a_i. Therefore, by Schauder's Fixed Point Theorem (Conway, 1990, Theorem V.9.5, page 150), there exists a probability measure $\sigma_i \in \Delta(A_i)$ such that $F(\sigma_i) = \sigma_i$.

Next, we show that the primal constraints, $\beta + \sum_{i=1}^n \int_{A_i}[v_i(b_i, a_{-i}) - v_i(a)]d\alpha_i(b_i|a_i) \geq 0$ for every $a \in A$, have a left-hand side that is continuous in a. By assumption, $v_i(a)$ is continuous, hence it suffices to argue continuity of $\sum \int v_i(b_i, a_{-i})d\alpha_i(b_i|a_i)$.

Given a sequence $\{a^n\} \subset A$ converging to some $a \in A$, weak$*$ continuity of α and continuity of v_i imply that

$$\int v_i(b_i, a_{-i})d\alpha_i(b_i|a_i^n) \;\to\; \int v_i(b_i, a_{-i})d\alpha_i(b_i|a_i)$$

for every i, as required.

Since continuity implies Borel measurability, for any positive measure $\mu \in M(A)$ on A, it must be the case that

$$\int_A \beta + \sum_{i=1}^n \int_{A_i}[v_i(b_i, a_{-i}) - v_i(a)]d\alpha_i(b_i|a_i)d\mu(a) \;\geq\; 0.$$

But letting $\mu = \prod_i \sigma_i$, where σ_i is α_i-stationary, implies (by Fubini's Theorem) that

$$\int_A \int_{A_i}[v_i(b_i, a_{-i}) - v_i(a)]d\alpha_i(b_i|a_i)d\mu(a) \;=\; 0$$

for every i, therefore $\beta \geq 0$, hence $v(\mathbf{0}_0, \mathbf{1}_i) \geq 0$, and finally $v(\mathbf{0}_0, \mathbf{1}_i) = 0$. □

Next, we prove the following lemma.

Lemma B.9. $v(\mathbf{b}) = \sup\{\mathbf{b}_0(a) : a \in A\} = \max\{\mathbf{b}_0(a) : a \in A\}$.

Proof. The second equality follows by Weierstrass' Theorem. If $\mathbf{b}_i \equiv 1$ then by the proof of Lemma B.8, it follows immediately that $v(\mathbf{b}) = \sup\{\mathbf{b}_0(a) : a \in A\}$, since any feasible primal variable β must satisfy $\beta \geq \mathbf{b}_0(a)$ for every $a \in A$. Otherwise, assuming $\mathbf{b}_i(a_i) > 0$ for every a_i without loss of generality, letting $\widehat{\alpha}_i = \alpha_i/\mathbf{b}_i$ we are back in the environment of Lemma B.8 where there is a stationary distribution $\sigma_i \in \Delta(A_i)$ for $\widehat{\alpha}_i$. Defining $d\mu = \prod_i \mathbf{b}_i d\sigma_i$, it again follows that $\beta \geq \sup\{\mathbf{b}_0(a) : a \in A\}$. On the other hand, $v(\mathbf{b}) \leq \sup\{\mathbf{b}_0(a) : a \in A\}$, since $\beta = \sup\{\mathbf{b}_0(a) : a \in A\}$ and

$$\alpha_i(B_i|a_i) \;=\; \begin{cases} \mathbf{b}_i(a_i) \text{ if } a_i \in B_i \\ 0 \text{ otherwise} \end{cases}$$

is a feasible solution. □

Proposition B.10. *$v(\mathbf{b})$ is subdifferentiable at $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_i)$ for $\mathbf{b}_i > 0$.*

Proof. By Lemma B.9, $|v(\mathbf{b}) - v(y)| = |\sup\{\mathbf{b}_i\} - \sup\{y_i\}| \leq \sup|\mathbf{b}_i - y_i| = \|\mathbf{b}_i - y_i\|$. Therefore, by the third condition of Theorem C.2, the result follows. □

This leads us to the final result of this appendix.

Theorem B.11. *Correlated equilibrium exists in continuous games.*

Proof. By Proposition B.10, v is subdifferentiable at $\mathbf{b} = (\mathbf{1}, \mathbf{0})$ as defined in Lemma B.8, so by Theorem C.1, there exists a dual solution and there is no duality gap. A dual solution at $\mathbf{b}$ incluides a correlated equilibrium. □

C Linear Programming

We recall basic results on linear programming from Gretsky et al.(2002). Let X and Y be ordered, locally convex, topological vector spaces with dual spaces X^* and Y^*, respectively. A *linear program* is a triple (A, b, c^*) such that $A : X \to Y$ is a continuous linear operator, $b \in Y$, and $c^* \in X^*$. The *adjoint* of A is denoted $A^* : Y^* \to X^*$.

Every linear program has two "sides:" the *primal* (P) and the *dual* (D).

$$\sup_{x \in X} \{c^*(x) : Ax \leq b, x \geq 0\} \tag{P}$$

$$\inf_{y^* \in Y^*} \{y^*(b) : A^*y^* \geq c^*, y^* \geq 0\} \tag{D}$$

We say *there is no gap* if the value of the primal equals the value of the dual. Fixing the operator A and the functional c^*, and viewing the constraint b as a variable, define the value function of the primal to be $v : Y \to \mathbb{R} \cup \{\pm\infty\}$ as

$$v(b) \;=\; \sup_{x \in X} \{c^*(x) : Ax \leq b \text{ and } x \geq 0\}$$

The *subdifferential* of a concave function $v : Y \to \mathbb{R} \cup \{\pm\infty\}$ at a point $b \in \operatorname{dom} v$ is

$$\partial v(b) \;=\; \{y^* \in Y^* : y^*(y - b) \geq v(y) - v(b) \text{ for all } y \in Y\},$$

where $\operatorname{dom} v = \{b : v(b) < +\infty\}$. We say v is subdifferentiable at b if $\partial v(b)$ is non-empty.

Theorem C.1 (Theorem 1, Gretsky et al.(2002))**.** *Consider any linear program (A, b, c^*). Both the dual has a solution and there is no gap if and only if v is subdifferentiable at b.*

Suppose $f : Y \to \mathbb{R} \cup \{+\infty\}$ is proper, concave, and Y is a normed, linear space.

Theorem C.2 (Section 5, Gretsky et al.(2002))**.** *Each of the following conditions implies the next and the last is equivalent to the subdifferentiability of f at $b \in \mathrm{dom}\ f$.*

1. *f is upper semicontinuous and b is an interior point of dom f;*
2. *f is locally Lipschitz at the point b, i.e., there exists $\delta > 0$ such that f is Lipschitz on dom $f \cap B(b, \delta)$;*
3. *f has bounded steepness at the point b, i.e., $\frac{f(b)-f(y)}{\|y-b\|}$ is bounded above.*

For the value function of a linear program, upper semicontinuity is commonplace.

Lemma C.3 (Lemma 1, Gretsky et al.(2002))**.** *If the value function v for a linear program on ordered normed linear spaces is proper, then v is an upper semicontinuous extended real-valued (concave and homogeneous) function.*

Proposition C.4. *Given a linear program (A, b, c^*) without duality gap, if $\overline{x}$ is feasible for the primal and $\overline{y}^*$ is feasible for the dual then the following are equivalent assertions.*

1. *$\overline{x}$ solves the primal and $\overline{y}$ solves the dual.*
2. *$c^*(\overline{x}) = \overline{y}^*(A\overline{x}) = \overline{x}(A^*y^*) = \overline{y}^*(b)$.*
3. *$\overline{x}$ solves $\sup\{c^*(x) - \overline{y}^*(Ax) : x \geq 0\}$ and $\overline{y}^*$ solves $\inf\{y^*(b) - \overline{x}(A^*y^*) : y^* \geq 0\}$.*

Proof. (1) $\Leftrightarrow$ (2): For any feasible x and y^*, we must have $c^*(x) \leq y^*(Ax) = x(A^*y^*) \leq y^*(b)$. Since there is no duality gap, $c^*(\overline{x}) = \overline{y}^*(b)$ if and only if $\overline{x}$ solves the primal and $\overline{y}^*$ solves the dual. (2) $\Rightarrow$ (3): Since $c^*(x) - \overline{y}^*Ax \leq 0$ for any $x \geq 0$ (because $c^* \leq \overline{y}^*A$) and $y^*(b) - y^*A\overline{x} \geq 0$ for any $y^* \geq 0$ (because $A\overline{x} \leq b$). (3) $\Rightarrow$ (2): Notice that $\sup\{c^*(x) - \overline{y}^*(Ax) : x \geq 0\} = 0$, since $x = 0$ is feasible and $\overline{y}^*$ satisfies $c^* \leq \overline{y}^*A$. Similarly, $\inf\{y^*(b) - \overline{x}(A^*y^*) : y^* \geq 0\} = 0$, too. By definition of the adjoint, $\overline{y}^*(A\overline{x}) = \overline{x}(A^*\overline{y}^*)$, therefore $c^*(\overline{x}) = \overline{y}^*(A\overline{x}) = \overline{x}(A^*\overline{y}^*) = \overline{y}^*(b)$. $\square$