A Theory of Business Transfers

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Abstract

In this paper, we develop a new theory of firm dynamics and capital reallocation and use it to study the taxation of private business income, wealth, and capital transfers. Central to the analysis is the idea that certain productive assets—say, customer-bases and trade names—are specific to a business and thus not available in rental markets. Owners grow their businesses by investing in these assets themselves or by purchasing a group of assets that constitute an existing business. Given the specificity of the assets, the reallocation of capital through business transfers takes time and, as a result, theory predicts a dispersion in marginal products, with capital gradually being transferred from owners with low marginal products to those with high marginal products. Despite the gradual reallocation, the allocations we analyze are efficient. Introducing business taxation has implications for business entry, investment, and capital transfers. We find taxes on business income are the least distortive and achieve the highest welfare when compared to taxes on wealth and capital transfers.

Key words: optimal assignment, capital allocation, firm dynamics, capital taxation

1 Introduction

This paper starts with the premise that most of the assets transferred in private business sales are intangible in nature and accumulated over time by the business founders and successors. We are motivated by data on the typical business sale, which includes the transfer of self-created intangibles such as customer bases, trademarks, and going concern value. These are assets that are not commonly leased and are usually sold all at once when the owners relocate, retire, or begin a new venture. Because such transfers are rare events, little is known about the investments that ongoing private businesses make, despite the fact that they generate more than half of all U.S. business income and significant wealth for their owners. To address this, we propose a new theory of business transfers that incorporates an indivisibility in bilateral trades, effectively assuming that businesses are sold as a unit in pairwise meetings. The theory is disciplined by administrative data on business transfers from the Internal Revenue Service and used to study firm dynamics and business taxation.[1]

The main element of the theory is the technology of firms. Firms are collections of three factors: *nontransferable* capital that cannot be bought or sold, *transferable* capital that can be bought and sold, and external factors that are rented on spot markets. The nontransferable capital is in essence the owner's productivity or ability, which can change over time but is inalienable. The transferable capital is the intangible capital recorded on IRS tax forms. The external factors would in practice include employee time, physical capital, and materials. Firms in the model can grow in two ways: through internal investment and through purchases of other businesses, although the opportunities to buy or sell a business are not always available. When they are, potential buyers and sellers engage in bilateral trades. Those that sell can restart another.

Despite the indivisibilities in capital exchange, we show that the allocation of capital is efficient. The model is effectively a neoclassical benchmark in the spirit of Lucas (1978) modified to include competitive factor markets for lumpy transferable capital with terms of trade settled in bilateral meetings. Capital is gradually traded upwards with owners that have a low marginal product of capital selling to those with a high marginal product. Per-unit prices vary across sales and depend on the quantity. The prices are highest for quantities that result in a relatively quick attainment of optimal size and decline for very large transactions because of decreasing returns to scale and thinner markets. Importantly, the price dispersion in this model is not indicative of misallocated resources.

Data from the IRS starting in tax year 2000 is used to discipline the model. Of particular relevance are business asset acquisitions reported on Forms 8594 and 8883 filed by both buyers and sellers. Taxpayers must allocate the business purchase price across different asset

[1] This work is part of a Joint Statistical Research Program Project of the Statistics of Income Division at the IRS investigating tax compliance of intangible-intensive businesses.

categories, including categories of marketable securities, fixed assets, and intangible assets. Included with intangibles are Section 197 assets—customer- and information-based intangibles; non-compete covenants; licenses and permits; franchises, trademarks, and trade names; workforce in place; business books and records; and processes, designs, and patterns—as well as goodwill and going concern value. This information is needed to assess capital gains for sellers and asset bases for buyers. The tax identification numbers from the filings are then linked to business tax forms and each owner's individual tax form. This allows us to construct longitudinal panels over the business and owner life cycles.[2]

We use the data to parameterize the model and then study its predictions for firm dynamics, business wealth, and tax policy impacts. Based on model simulations, we find that roughly 7 percent of transferable capital is traded annually, indivisibly through business sales. Since the reallocation process takes time, our baseline economy will appear to have significant dispersion in marginal products of capital across businesses, with the standard deviation in logs of 46 percent. In fact, if we compare average capital holdings across firms with different productivity levels, we find a relatively flat profile when compared to an economy with divisible capital and a centralized asset market. The flatness in the profile reflects the fact that accumulating non-rentable, non-divisible intangible capital through own investment or purchases takes time.

Because we include both transferable and nontransferable capital when modeling business activity, we consider two familiar but different measures of business wealth and argue that making a distinction between them is relevant for tax policy analysis. The first measure is the value of transferable wealth or an answer to: What is the value of the business if sold today? This value is relevant for analyzing the taxation of realized capital gains following a sale or accrued capital gains if the government annually taxes the assessed value of transferable assets. The second measure is the total value of the ongoing concern that generates a flow of dividends to owners over the business life. The total value includes not only the value of the intangible assets that can be transferred, but also the value reflecting the owners' inalienable productivity. As in the case of accrued gains, taxing the total business value requires an assessment by the tax authority in the absence of transactions.

For our baseline parameterization, the share of wealth that is transferable is large: 25 percent on average and for the median sale. As Bhandari and McGrattan (2021) argue, having a significant transferable share is likely to change the calculus of policy counterfactuals because owners can and will change their investing and trading behaviors following tax reforms. Also relevant for tax analysis is the dispersion in the shares of wealth that are transferable and the income yields from this wealth. If we compare estimates at the 5^{th}

[2] This work is still in progress and unpublished IRS statistics have not yet been cleared for disclosure avoidance. As a result, this draft relies heavily on the Form 8594 purchase price allocations for brokered business transfers that are recorded in the Pratt's Stats database.

and 95^{th} percentiles, we find transferable shares in the range of 0 to 49 percent and income yields in the range of -16 to 11 percent.

The main analysis of the paper compares the impacts of different types of business taxes on firm entry, investment, transfers, and welfare. Our model is sufficiently rich to address a key issue in debates on business taxation: is it better to tax the income of the owners or business wealth and, if the latter, in what way? While most tax revenue from U.S. businesses is generated by taxes on income, advocates of taxing wealth have argued that there may be efficiency gains if wealth taxes result in a better allocation of business capital. The idea is to institute tax policies with a higher incidence on the least productive owners, who would be encouraged to transfer businesses to more productive owners. On the other hand, taxing wealth disincentivizes business owners from investing in their businesses and results in lower levels of intangible capital.

We analyze the effects of raising a fixed amount of revenue per period (in a steady state) with four different taxes: an income tax; a realized capital gains tax on business transfers; an accrual tax on the assessed value of transferable business capital; and a tax on the present value of business wealth accruing to transferable and non-transferable capital. Our main result is that the tax on business income achieves the highest welfare at all levels of revenue achievable, followed by the tax on the present value of wealth, the accrued tax on business capital, and finally the tax on realized capital gains. The ranking mirrors that of investment impacts: taxes on transferable capital, whether levied on realized gains or on accrued capital assessments distort reallocation of capital across firms and significantly lower investment—more so than taxes on business income.

Further intuition can be gleaned from an investigation of the taxes that would have to be paid by each group in the population under the different policy scenarios. With income taxes, businesses with the most productive owners and the highest capital stocks would pay the most. With taxes on the present value of wealth, businesses with the most productive owners have the highest tax bills, regardless of their accumulated capital stocks. Given tax elasticities are higher the higher is productivity or capital, the ranking of income taxes over wealth taxes makes intuitive sense. With taxes on capital paid by all businesses on an accrual basis, business owners with the most capital have the highest tax bills, regardless of their productivity levels. Thus, the same logic applies to a comparison of income taxes versus capital taxes. Finally, in the case of capital gains taxes, those owners that are selling pay all of the taxes. As a result, the tax shuts down trades and amplifies the dispersion of marginal products of capital, which in turn exacerbates the misallocation of capital in the economy.

1.1 Related literature

This paper contributes to several strands of literature. First, our paper is related to the large body of work that studies firm dynamics, productivity, and the allocation of capital. In the seminal paper by Hopenhayn (1992), stochastic productivity drives the demand for capital that is perfectly divisible and competitively traded. Variations of Hopenhayn's framework have been brought to the data by, among others, Hsieh and Klenow (2009, 2014) and Sterk et al. (2021). The findings of these papers about size and productivity differentials across firms have in turn spawned a large literature that focuses on identifying the source of differences as "misallocation" due to regulatory, financial, or informational frictions.[3] We depart from this literature in our focus on intangibles and our modeling of business capital trades. In our model, measured dispersion is an artifact of the market structure whereby all assets in a business are sold as a unit with terms of trade settled in pairwise meetings and is not driven by a misallocation of resources due to surmountable frictions. Furthermore, our emphasis on intangibles in private business makes the existing empirical evidence on firm dynamics less portable since it is largely focused on physical capital in manufacturing. To fill the gap, we shed new light on the lifecycle dynamics of private business by using longitudinal data from annual tax filings along with intermittent business transactions.

A related literature has focused on the transfer of various forms of business capital, taking into account their indivisible nature (Holmes and Schmitz (1990)). These models necessarily make assumptions on the degree of input transferability by studying either the sale of some fixed factor (David (2021)), or capital that can be produced (Ottonello (2014)), or a combination of both, for example, the whole firm (Guntin and Kochen (2020), Gaillard and Kankanamge (2020)). One of the main contributions of our paper is using theory and detailed transaction data to estimate the share of business wealth that is transferable. From a technical perspective, modeling the demand and supply for heterogeneous, indivisible products—in this case, businesses—poses significant challenges in the absence of traditional assumptions adopted when modeling good markets (for example, demand functions with constant elasticity of of substitution). To ensure tractability, previous authors have typically developed models of random search with bargaining—which is well-known to generate inefficient allocations—or directed search with one-sided heterogeneity.[4] We take a different route and model business sales as transactions in a frictionless decentralized market. Build-

[3] See, for example, David and Venkateswaran (2019) and others in the survey article by Restuccia and Rogerson (2017).

[4] Burdett and Mortensen (1998) introduce extensions to the canonical job ladder model to allow for one-sided heterogeneity. Extensions have also allowed for two-sided heterogeneity and random search (Postel-Vinay and Robin (2002) and Bagger and Lentz (2019)) or directed search (Schaal (2017)) but require additional assumptions for tractability. For example, while in Schaal (2017) firms hire a measure of workers, that assumption is not appropriate for modeling trades of indivisible units like businesses.

ing on tools from the matching literature (Choo and Siow (2006), Galichon et al. (2019)), we solve for the equilibrium set of prices and show that it implements an efficient allocation. To the best of our knowledge, we are the first to prove efficiency of a matching model in which the matches that are formed in each period determine the evolution of the distribution of agents' types. We find the efficiency property appealing as it allows us to isolate the dispersion in marginal products that is solely generated by the indivisible nature of capital in private business.

As an application of our framework, we recover measures of business value for both traded and non-traded private firms. Such estimates contribute to a growing literature on the measurement of private business wealth. Most papers in this literature rely either on structural models of entrepreneurship disciplined by survey data such as Cagetti and De Nardi (2006) or non-structural approaches that use administrative data, for instance, the capitalization method used by Saez and Zucman (2016) and Smith et al. (2019). Differently from these papers, our measure leverages theoretically-grounded valuation concepts and primitives that are disciplined by detailed data on business sales merged with the income statements of buyers and sellers.

Finally, we contribute to the public finance literature that studies the consequences of taxation of different sources of income. While a large literature studies the taxation of business income, we focus on the taxation of realized capital gains.[5] In neoclassical settings such as Hopenhayn (1992), if one were to introduce a capital gains tax on private business, there would be no effect unless applied to accrued gains. The reason is that the value of traded physical capital is constant in a stationary equilibrium, and gains in the value of non-traded capital (or owner productivity) are never realized. Our framework is particularly suited for the study of capital gains taxation for private businesses given our explicit modeling of self-created intangibles that are valued and taxed at the time of sale.

The rest of the paper is organized as follows. Section 2 sets up the theoretical environment, including timing of events, descriptions of problems solved by business owners, and a definition of a recursive equilibrium. A characterization of equilibrium and a proof of efficiency are provided in Section 3. In Section 4, we parameterize the model and put it to use to quantify the patterns of trade and resulting firm dynamics. We highlight two measures of business wealth and discuss impacts of taxing wealth that is transferred through business sale. Section 5 concludes.

[5] For studies of business income taxation, see Kitao (2008), Meh (2005), Boar and Midrigan (2019), Bhandari and McGrattan (2021), Bruggemann (2021). Chari et al. (2003) is one of few papers that studies taxation of capital gains by analyzing the issue in the context of the Holmes and Schmitz (1990) model.

2 Model

Entrepreneurs are endowed with a technology that produces consumption goods using three factors: entrepreneurial productivity or skill (z), capital (k), and rentable inputs or labor (n). The factor z is non-transferable and the factors k and n are transferable. Entrepreneurs rent labor, produce, invest in capital, and consume. Entrepreneurs stochastically receive some opportunity to trade capital k with others. Entrepreneurs face an exogenous hazard of exit at some constant rate. Details of these actions are provided next, followed by the entrepreneurs dynamic program and the definition of recursive equilibrium we wish to characterize.

2.1 Environment

Before describing each action, we first introduce some notation. Let Z be the set of productivity levels. We will assume that the productivities $z \in Z$ follow the geometric brownian motion

$$dz = \mu_z z \mathrm{dt} + \sigma_z z \sqrt{\mathrm{dt}} \mathrm{d}\mathcal{W}$$

where $\mathcal{W}$ is a standard Wiener process. We use the set $\mathcal{S} \equiv Z \times K$ to denote the space of productivity levels and potential capital. We let $s \in \mathcal{S}$ denote a pair (z, k) and use $z(s)$ and $k(s)$ to denote the first and the second component of s, respectively. We use $\Delta(\mathcal{S})$ to denote the set of measures over $\mathcal{S}$. For some $\phi \in \Delta(\mathcal{S})$, we use $\phi(ds)$ to denote its density at s.

Trading. An entrepreneur with state s access the market for capital at Poisson rate η. Once in the market, the entrepreneur faces a price-quantity menu denoted by $\{p^m(s, \tilde{s})\}_{\tilde{s} \in S}$ and $\{k^m(s, \tilde{s})\}_{\tilde{s} \in S}$. That is, an entrepreneur that has state s and decides to trade (or match) with an entrepreneur that has state $\tilde{s}$, pays $p^m(s, \tilde{s})$ to the trading partner, and exits the trading stage with capital level $k^m(s, \tilde{s})$. The functions $k^m : \mathcal{S}^2 \to K$ and $p^m : \mathcal{S}^2 \to \mathcal{R}$ are determined as part of an equilibrium that we define later.[6]

We now introduce a few assumptions on the exchange of capital and payments within a match. For the allocation of capital, we impose that entrepreneurs can either sell their entire capital stock, buy the entire capital stock of their trading partner, or trade no capital at all. This assumption amounts to the following restrictions on k^m. For all pairs $(s, \tilde{s}) \in \mathcal{S}^2$,

$$k^m(s, \tilde{s}) \in \{k(s) + k(\tilde{s}), k(s), 0\} \tag{1}$$

$$k^m(\tilde{s}, s) + k^m(s, \tilde{s}) \leq k(s) + k(\tilde{s}) \tag{2}$$

[6] We omit the explicit dependence of individual choices and values on $\{p^m, k^m\}$ when it is clear from the context.

We refer to restriction (1) as *indivisibility*. This restriction captures the key friction in our model, namely, that the reallocation of capital across entrepreneurs in bilateral trades occurs in a "lumpy" fashion.

Production and investment. Decisions concerning goods production, capital investment, and consumption are made. Output is produced using the technology

$$y(s,n) = z(s)\,k(s)^{\alpha}\,n^{\gamma}.$$

The investment technology is modeled as a cost function $c(\theta)$, where c' and c'' are strictly positive. An entrepreneur incurs cost $c(\theta)$ to accumulate capital. Specifically, the change in capital over an interval of length dt is equal to

$$\mathrm{d}k = \theta - \delta_k$$

where δ_k is the depreciation rate of capital.

Entry and exit. Entrepreneurs exit at rate δ. New entrants pay a cost c_e to draw a state $(z,k) \sim G(ds \in S)$. The entry decision $d_e \in \{0,1\}$ is given by

$$\max_{d_e} \int V(s)\,G(ds \in S) - c_e. \tag{3}$$

2.2 Entrepreneurs Dynamic Program

Let $V : S \to \mathcal{R}^{+}$ denote the value of an entrepreneur. Let $W : \mathcal{S} \to \mathcal{R}^{+}$ be the entrepreneur's gains from trade. Given functions $\{p^m, k^m\}$, these value functions solve the following Hamilton-Jacobi-Bellman equation:

$$\begin{aligned}(r+\delta)\,V(s) = \max_{n} z(s)\,k(s)^{\alpha}\,n^{\gamma} - wn + \max_{\theta} \partial_k V(s)\,(\theta - \delta_k) - c(\theta) \\ + \partial_z V(s)\,\mu_z z + \frac{1}{2}\partial_{zz} V(s)\,\sigma_z^2 z + \eta \max_{\lambda(s,\cdot)} W(s;\lambda).\end{aligned} \tag{4}$$

The term on the left-hand side is the annuitized value of being an entrepreneur of type s. The right-hand side includes flow output net of the wage bill, the gain from investment in capital net of the cost of investment, the changes in value induced by the evolution of productivity z, and the expected gains from trade from accessing the market for capital, W. The last term of the HJB is absent from traditional firm dynamics model and it deserves additional clarifications.

Define $v(s,\tilde{s})$ as the (gross) value for firm type s after trade with $\tilde{s}$:

$$v(s,\tilde{s}) \equiv V(z(s), k^m(s,\tilde{s})) - p^m(s,\tilde{s}). \tag{5}$$

Then, for all s, we have the expected gains from trade given by

$$W(s) = \int \max_{\tilde{s}(\boldsymbol{\epsilon})\in S\cup\{o\}} \{v(s,\tilde{s}) - V(s) + \sigma\epsilon(s,\tilde{s}), \sigma\epsilon(s,o)\} F(d\epsilon). \tag{6}$$

The inner maximization represents the optimality for the entrepreneur, who has the price quantity menus $\{p^m(s,\tilde{s}), k^m(s,\tilde{s})\}_{\tilde{s}\in S}$ and, in addition, realizes non-pecuniary match-specific utilities $\{\epsilon(s,\tilde{s})\}_{\tilde{s}\in S} \sim F$. The parameter σ scales the relative importance of the pecuniary versus non-pecuniary benefits from trading. Given $\{p^m(s,\tilde{s}), k^m(s,\tilde{s}), \epsilon(s,\tilde{s})\}_{\tilde{s}\in S}$, the optimal choice of trading partner, $\tilde{s}(\boldsymbol{\epsilon}) \in S\cup\{o\}$ is described by maximization inside integral (6) with $\tilde{s}(\boldsymbol{\epsilon}) = o$ denoting the choice to remain unmatched.

The solution to this problem induces choice probabilities $\lambda : \mathcal{S} \to \Delta(\mathcal{S})$ and $\lambda_o : \mathcal{S} \to [0,1]$ given functions $\{p^m, k^m\}$, so that for all $A \subseteq \mathcal{S}$:

$$\lambda(s,A) \equiv \int \mathbb{I}(\tilde{s}(\boldsymbol{\epsilon}; s) \in A) F(d\boldsymbol{\epsilon})$$

is the probability measure over the event that type s is matched to agents of type $\tilde{s} \in A$, and

$$\lambda_o(s) \equiv \int \mathbb{I}(\tilde{s}(\boldsymbol{\epsilon}) = o; s) F(d\boldsymbol{\epsilon})$$

is the probability measure over the event that $s \in A$ are unmatched. The probability measure λ allow us to express the gains from trade as

$$W(s;\lambda) = \int \{v(s,\tilde{s}) - V(s) + \sigma\epsilon(s,\tilde{s})\} \lambda(s, \mathrm{d}\tilde{s}) + \sigma\epsilon(s,o)\lambda_o(s). \tag{7}$$

2.3 Equilibrium

Let $\phi \in \Delta(\mathcal{S})$ be the measure over entrepreneurs at each point in time. Let $\phi_e \in \Delta(\mathcal{S})$ be the measure of new entrants. The law of motion of the measure over entrepreneurs, ϕ', is described by a function Γ as follows. For all $\hat{A} \equiv \hat{Z} \times \hat{K} \subseteq \mathcal{S}$

$$\dot{\phi}(\hat{A}) \equiv \Gamma(\phi; \lambda, \lambda_o, \theta, \phi_e, k^m)(\hat{A}). \tag{8}$$

We are now ready to define an equilibrium.

Definition 1. A *recursive equilibrium* is given by (i) price quantity menus $p^m : \mathcal{S}^2 \to \mathcal{R}$ and $k^m : \mathcal{S}^2 \to K$; (ii) mass and initial capital for entrants m; (iii) a measure $\phi^* \in \Delta(\mathcal{S})$;

(v) a pair of value functions $V : \mathcal{S} \to \mathcal{R}^+$and $W : \mathcal{S} \to \mathcal{R}^+$; and (vi) choice probabilities $\lambda : \mathcal{S} \to \Delta(\mathcal{S})$ and $\lambda_o : \mathcal{S} \to [0, 1]$ such that:

1. The trading arrangements are feasible, that is, for all pairs $(s, \tilde{s}) \in \mathcal{S}^2$ with each having a positive density under ϕ^* the function k^m satisfies (1)-(2).

2. Given $\{k^m(\cdot), p^m(\cdot)\}$, the value function for incumbent firms and the gains from trade $\{V(\cdot), W(\cdot)\}$ solve the Bellman equations (4)-(6) with optimal choice probabilities $\{\lambda(\cdot), \lambda_o(\cdot)\}$.

3. The decision to enter for new entrants solves (3). The mass of new entrants is given by
$$\int V(s)\, G(ds \in \mathcal{S}) - c_e \leq 0 \tag{9a}$$
$$m\left[\int V(s)\, G(ds \in \mathcal{S}) - c_e\right] = 0 \tag{9b}$$

4. The measure ϕ is stationary
$$0 = \Gamma\phi^*. \tag{10}$$

3 Characterizing the Equilibrium

In this section, we provide a characterization of the equilibrium and discuss its properties. We compute the equilibrium in two steps. First, we take the value function V and the equilibrium measure of firms ϕ as given and characterize prices p^m, the allocation—that is, the choices of trading partners (λ, λ_o) and capital k^m conditional on trading— and the gains from trade W that are consistent with market clearing. Second, we solve for (V, ϕ) such that households optimize given the menu of prices and terms of trades and ϕ is, in turn, consistent with household decisions. We start with properties of the limiting case as $\sigma \to 0$ that turns off the non-pecuniary preference shocks. In section 3.2, we discuss the characterization of the more general setup with $\sigma > 0$. The environment with preference shocks is motivated by our interest in studying capital gain taxes, which introduce a wedge in the transfer of utility between agents.

3.1 Equilibrium characterization without preference shocks

For the limit $\sigma \to 0$, we leverage results from linear duality to characterize the matching patterns and the terms of trade.[7]

[7] See Galichon (2016) for details on the Monge-Kantorovich transportation problem.

Characterizing prices and allocations given (ϕ, V). As a first step, we introduce an auxiliary problem concerned with finding assignments that maximize the total surplus (as measured using V) by trading capital so as to preserve the measure ϕ. Define the largest surplus from matching for a pair $(s, \tilde{s})$ as follows:

$$X(s,\tilde{s}) = \max\left\{V(z, k+\tilde{k}), V(s)+V(\tilde{s}), V(\tilde{z}, k+\tilde{k})\right\} - \left(V(s)+V(\tilde{s})\right).$$

The three arguments are possible outcomes in a match, namely, type s buys the capital from type $\tilde{s}$, no trade, and type s sells the capital to $\tilde{s}$. If we split the measure ϕ into two measures ϕ^a and ϕ^b such that for $A \subseteq \mathcal{S}$, then

$$\phi^a(A) = \phi^b(A) = \frac{\phi(A)}{2}.$$

For measures $\{\phi^a, \phi^b\}$, an assignment $\left(\pi, \pi_o^a, \pi_o^b\right)$ that maximizes surplus, solves the following maximization problem:

$$Q(\phi, V) = \max_{\pi, \pi_0^a, \pi_0^b \geq 0} \int X(s,\tilde{s})\pi(ds \in \mathcal{S}, d\tilde{s} \in \mathcal{S}) \tag{11}$$

such that for $A \subseteq \mathcal{S}$

$$\int \pi\left(ds \in A, d\tilde{s} \in \mathcal{S}\right) + \pi_o^a\left(ds \in A\right) = \phi^a(A) \tag{12}$$

$$\int \pi(ds \in \mathcal{S}, d\tilde{s} \in A) + \pi_o^b(d\tilde{s} \in A) = \phi^b(A). \tag{13}$$

We label this problem as *P1.* The next theorem shows that we can back out $(p^m, k^m, \lambda, \lambda_o)$ from the solution of *P1.*

Theorem 1. *Let μ^a and μ^b be the Lagrange multipliers on (12) and (13), respectively, in problem P1. Let $\left(\pi, \pi_o^a, \pi_o^b\right)$ be the optimal assignment in problem P1. The functions*

$$k^m(s,\tilde{s}) \in \arg\max\{V(z, k+\tilde{k}), V(s)+V(\tilde{s}), V(\tilde{z}, k+\tilde{k})\} \tag{14}$$

$$p^m(s,\tilde{s}) = V(z, k^m(s,\tilde{s})) - V(s) - \mu^a(s) \tag{15}$$

$$p^m(\tilde{s}, s) = V(z, k^m(\tilde{s}, s)) - V(\tilde{s}) - \mu^b\left(\tilde{s}\right) \tag{16}$$

$$W(s) = \mu^a(s) = \mu^b(s) \tag{17}$$

and measures for all $A, \tilde{A} \subseteq \mathcal{S}$

$$\lambda(A, \tilde{A}) = \frac{\pi(A, \tilde{A}) + \pi(\tilde{A}, A)}{\phi(A)} \tag{18}$$

$$\lambda_o(A) = \frac{\pi_o^a(ds \in A) + \pi_o^b(d\tilde{s} \in A)}{\phi(A)} \tag{19}$$

satisfy (1)-(2).

Theorem (1) states that the assignment from problem *P1* recovers the allocation of capital across entrepreneurs and the shadow prices on constraints (12) and (13) recover the terms of and the gains from trade. More specifically, equation (14) implies that the allocation of capital maximizes the pairwise surplus. Using the envelope theorem, the value of this perturbation to the interim planner who solves problem *P1* is given by $(\mu^a + \mu^b)/2$. In effect, this measures the social gains from having more entrepreneurs of type s and hence it corresponds to the (gross) gains from trade $W(s)$. Given the symmetry of X, it is easy to verify that μ^a equals μ^b and that $\pi(\cdot, \cdot)$ is symmetric. Then, the terms of trade are determined by exploiting the insight that social and private gains from trade are equal at the optimal assignment. To see this, consider a perturbation that changes the measure ϕ at some state s. The private change in value of a type s entrepreneur is given by $V(z(s), k^m(s, \tilde{s})) - p^m(s, \tilde{s})$. Optimality ensures that the private value is equalized across all trading partners $\tilde{s}$ with strictly positive probabilities. Equating the social and private values gives us equation (17) and (16) that pin down the pairwise terms of trade.

Characterizing (ϕ, V) given $(p^m, k^m, \lambda, \lambda_o, W)$. In the second step, we use the outcomes of the first to update value functions and the invariant measure. The characterization in the first step gives us a handy way of solving the Bellman equation. Given the value of $W(s)$ from the solution from problem *P1*, the HJB can be solved using standard methods (e.g., finite differences as in Achdou). The policy functions for investment, θ, and for trades, λ, govern the law of motion of the distribution in equation (8) given ϕ_e. Thus ϕ is given by condition (10). Together step 1 and step 2 characterize the recursive competitive equilibrium as a fixed point. This characterization naturally lends itself to a computational algorithm where we iterate between step 1 and step 2 until convergence.

3.1.1 Properties

The next corollary further sharpens the characterization of the price function p^m.

Corollary 1. *With $\sigma \to 0$, there exists a function $\mathcal{P} : K \to R_+$ such that*

$$p^m(s, \tilde{s}) = \mathcal{P}(k(s)) \quad \textit{for all } k^m(s, \tilde{s}) < k(s).$$

This corollary says that the pairwise prices only depend on the quantity sold. The intuition for this result is straightforward. The sellers value from trade is equal to the price he extracts from the buyer plus the value of starting anew with zero capital and the current level of productivity. The second component is independent of the trading partner. Thus, conditional on selling to multiple buyers, a seller who maximizes the value from trading must necessarily charge the same price to all buyers. A similar argument from the perspective of the buyer shows that the prices will not depend on the sellers productivity. We can then conclude that the prices are only a function of the quantity traded, and summarize this dependence using the function $\mathcal{P}(\cdot)$ function. In Section 4, we will discuss the forces that determine the shape of $\mathcal{P}(\cdot)$.

The second property we highlight is that solving for the equilibrium in the market for capital is equivalent to looking for the set of stable matches between entrepreneurs. A set of matches is stable if there does not exist an alternative price quantity $(\hat{k}^m, \hat{p}^m) \neq (k^m, p^m)$, and a pair $(s, \tilde{s}) \in \mathcal{S}^2$ such that capital allocation $\left\{\hat{k}^m(s, \tilde{s}), \hat{k}^m(\tilde{s}, s)\right\}$ satisfies (1)-(2), $\hat{p}^m(s, \tilde{s}) + \hat{p}^m(\tilde{s}, s) \geq 0$ and

$$\begin{aligned} V\left(z(s), \hat{k}^m(s, \tilde{s})\right) - \hat{p}^m(s, \tilde{s}) &\geq V\left(z(s), k^m(s, \tilde{s})\right) - p^m(s, \tilde{s}) \\ V\left(z(\tilde{s}), \hat{k}^m(\tilde{s}, s)\right) - \hat{p}^m(\tilde{s}, s) &\geq V\left(z(\tilde{s}), k^m(\tilde{s}, s)\right) - p^m(\tilde{s}, s) \end{aligned} \tag{20}$$

with at least one of the last two inequalities being strict.

Corollary 2. *The equilibrium in the market for capital generates a set of stable matches.*

We conclude this section by discussing the efficiency properties of our competitive equilibrium. Given ϕ_0, consider a planner that solves the following maximization problem.

$$P(\phi_0) = \max_{\{\lambda_t, \lambda_{o,t}, \theta_t, k_t^m, m_t\}} \int_0^\infty \exp(-rt) \left[\int \left[y(s) - c\left(\theta_t(s)\right)\right] \phi_t(ds \in S) - c_e m_t\right] \mathrm{dt}$$

such that k_t^m satisfies feasibility conditions (1) and (2), $\lambda_t, \lambda_{o,t}$ satisfy optimality given ϕ_t, $\phi_e(m) = mG$, and

$$\dot{\phi_t}(s) = \Gamma\left(s, \phi_t; \lambda_t, \lambda_{o,t}, \theta_t, \phi_e, k_t^m\right) \quad \forall s \in \mathcal{S}, \tag{21}$$

We label this problem as *P2*. Given linear preferences, maximizing discounted welfare is the same as maximizing discounted output. We denote a solution to *P2* as stationary if $\phi_t = \phi_0$ for all t. In the next theorem, we show that stationary recursive equilibrium is efficient.

Theorem 2. *A stationary recursive equilibrium as defined in Definition (1) with the stationary measure ϕ^* achieves $P(\phi^*)$ in problem* P2. *Furthermore, any stationary solution to* P2 *constitutes a stationary recursive equilibrium with pairwise stability.*

The forces towards efficiency were foreshadowed in the formulation of the problem *P1*. Given (ϕ, V), the optimal assignment maximizes output. Beyond the static assignment, there are two additional features in problem *P2*—entry and investment—that need to be addressed. In the appendix, we show that the value of creating a new firm as well as the value of a new unit of capital to the planner coincides with the private value. Thus, zero profits for new entrants, and firm optimality with respect to θ are sufficient to ensure that the allocation is dynamically efficient.

3.2 Equilibrium characterization with preference shocks

For $\sigma > 0$, we leverage Choo and Siow (2006) and Galichon et al. (2019) framework to characterize the matching patterns and the terms of trade. Like with $\sigma = 0$ limit, the analysis proceeds in two steps.

Characterizing prices and allocations given (ϕ, V). As before, the first step involves recovering the allocations and prices. Recall that $v(s, \tilde{s})$ is the value after trade for firm type s with trading partner $\tilde{s}$ and given by

$$v(s, \tilde{s}) = V(z(s), k^m(s, \tilde{s})) - p^m(s, \tilde{s}). \tag{22}$$

When preference shocks ϵ are extreme value type 1 distributed, the choice probabilities λ have a familiar expression

$$\exp\left(\frac{v(s, \tilde{s}) - V(s)}{\sigma}\right) = \frac{\lambda(s, d\tilde{s})}{\lambda_o(s)}. \tag{23}$$

Given (ϕ, V), the optimal assignment λ, λ_o is solution to following set of equations

$$\sigma \ln \frac{\lambda(ds, d\tilde{s})}{\lambda_o(ds)} + \sigma \ln \frac{\lambda(d\tilde{s}, ds)}{\lambda_o(d\tilde{s})} = V(z(s), k^m(s, \tilde{s})) + V(z(\tilde{s}), k^m(\tilde{s}, s)) - V(s) - V(\tilde{s})$$

$$\int \lambda(ds, d\tilde{s} \in \mathcal{S}) + \lambda_o(ds) = 1.$$

The allocation of capital given pair $(s, \tilde{s})$ is still given by (14), and payments $p^m(s, \tilde{s})$ can be backed out from equation (22) and (23). This completes the counterpart of the first step from Section 3.1.

Characterizing (ϕ, V) given $(p^m, k^m, \lambda, \lambda_o)$. In the second step, we again use the outcomes of the first to update value functions and the invariant measure. As before, we can obtain a succinct expression for the net gains from trade and use that to update the value function V. In the appendix, we show that

$$W(s) = -\sigma \ln(\lambda_o(s)) \tag{24}$$

Expression (24) states that larger gains from trade are associated with a lower probability of not unmatched. Expression (24) and the Bellman equation (4) update V. The update for ϕ is also same as before and uses the mapping Γ defined in equation (8).

4 Results

In this section, we parameterize the model and use it as a laboratory to study patterns of trade in our environment with capital indivisibilities and bilateral trades. The results are compared to an ideal analogue with divisible capital and centralized markets. We then estimate business wealth and the impact of taxing this wealth on welfare and firm dynamics. We consider taxation of the total value of the business, $V(s)$ and the taxation of transferable capital, $k(s)$—on an accrual basis or on realized gains after a business has been sold. The results are compared to those of an economy with taxation of business incomes $y(s)$.

4.1 Model Parameters

In Table 1, we report our baseline parameter estimates. The values are chosen so that the model generates realistic growth in profits by business age, investment rates, and relative sizes for businesses that buy transferable capital versus those that sell.

In the first two rows of Table 1, we report the production shares of transferable capital and rentable inputs. The values we use are $\alpha = 0.1$ and $\gamma = 0.7$, respectively. The moments that are most sensitive to these estimates are the sale price (normalized by the size of the seller) and the relative size of buyer to seller. The discount rate r is set to 0.06 to be consistent with U.S. returns to capital. The death rate δ is set to 10 percent, which is intended to capture the rate of business exit. The depreciation rate of transferable capital is set equal to 5.8 percent to be consistent with U.S. rates.

Next in the table is the investment cost function, which is chosen to be quadratic with a coefficient A of 25 necessary to generate a plausible doubling in firm size within five years and a tripling within ten years. Another parameter that is relevant to the buy versus build choice is the trading rate η. This parameter is set equal to 1, which generates time to sale duration that is consistent with U.S. business sales.

The final two rows in Table 1 are parameter values that govern the productivity processes. The first is the probability distribution for entrants. Here, we assume that there is a mass point initially with $k = 1$. After that, the productivity process is geometric Brownian

TABLE 1. BASELINE PARAMETERS

Parameter	**Value**
Share of transferable capital	$\alpha = 0.1$
Share of rentable input	$\gamma = 0.70$
Discount rate	$r = 0.06$
Death rate	$\delta = 0.1$
Depreciation rate	$\delta_k = 0.058$
Investment cost, $C(\theta) = A\theta^\rho$	$A = 25, \rho = 2$
Trading rate	$\eta = 1$
Entry distribution, G	mass point at $z = z_0, k = 1$
Productivity process	$\mu_z = 0, \sigma_z = 0.25$

motion without drift a a local standard deviation parameter equal 0.25. The productivity processes, along with choices related to investment and entry and exit generate plausible growth rates in profits over the lifecycle when the model is compared to observations on private businesses.

4.2 Firm Dynamics

Next, we explore the patterns of trade that this model generates by comparing characteristics of buyers and sellers and by comparing the results of the economy with indivisible capital and bilateral trades to one with divisible capital and centralized markets.

In Figure 1, we plot a "bubble" map showing the frequency of trades between buyers and sellers according to their levels of productivity. On the x-axis, we report the sellers' productivity levels, and on the y-axis, we report the buyers' productivity levels. The main pattern to notice is that capital is moving up in a marginal product of capital sense, that is, from sellers with low productivity to buyers with high productivity. In Figure 2, we plot the quantity sold. There are a large number of multi-unit sales that move firms close to their optimal size quickly. The higher the seller's productivity, the larger the sale.

Because building businesses takes time, whether owners invest or purchase, the marginal products of capital are not equated across firms. In Figure 3, we plot the distribution of marginal products in our baseline model, which has a standard deviation in logs around 46 percent. In addition to a significant dispersion in marginal products of capital, we find

Figure 1. Predicted Pattern of Trade: Productivities

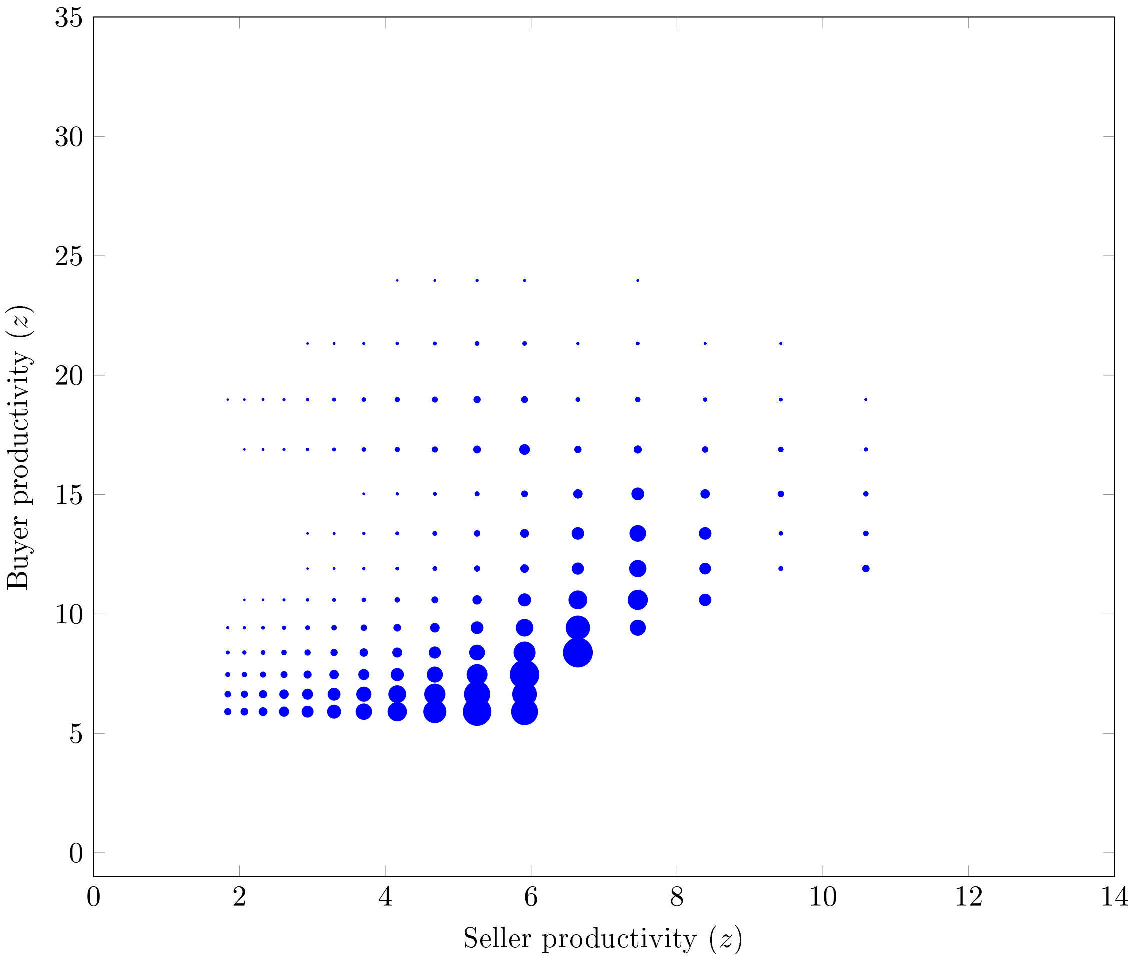


Figure 2. Predicted Pattern of Trade: Quantity Sold

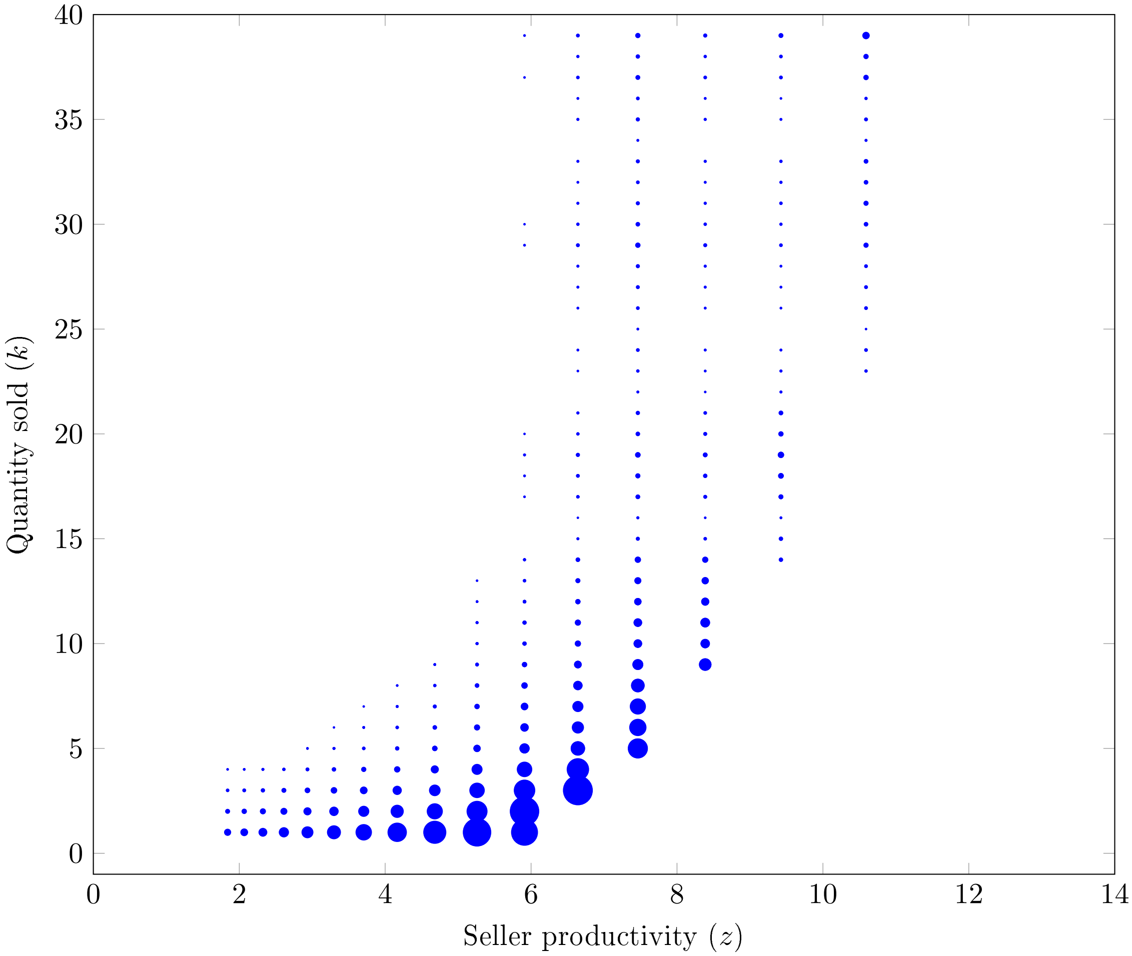
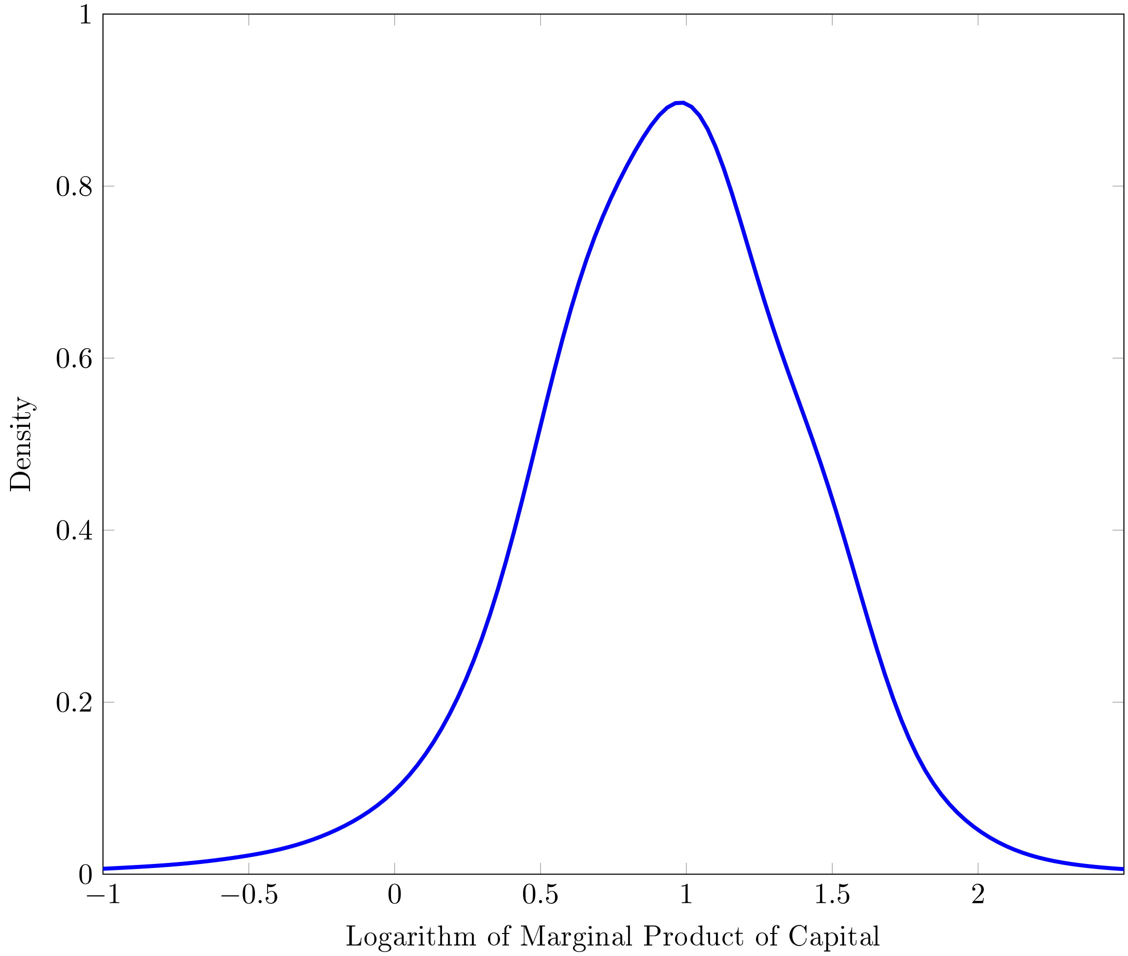


Figure 3. Predicted Dispersion in Marginal Product of Capital



significant dispersion in per-unit prices when the businesses are sold. Figure 4 plots these prices as a function of the quantity of capital in the businesses. As the figure shows, the per-unit prices rise quickly when the quantity of capital falls below five units. The price schedule flattens out at intermediate-sized businesses but then falls off after k of 25. At that point, there are few buyers looking for large quantities of capital.

4.3 Business Wealth

In this section, we report on the model's predictions for two common measures of business wealth. The first is the present discounted value of owner dividends, $V(s)$, which captures returns to both transferable capital k and non-transferable capital z. More familiarly, this value can be interpreted as the private-business counterpart of a stock price for shares if it were a publicly-traded company. We use these values to estimate variation in business

Figure 4. Predicted Per-Unit Prices by Quantity Sold

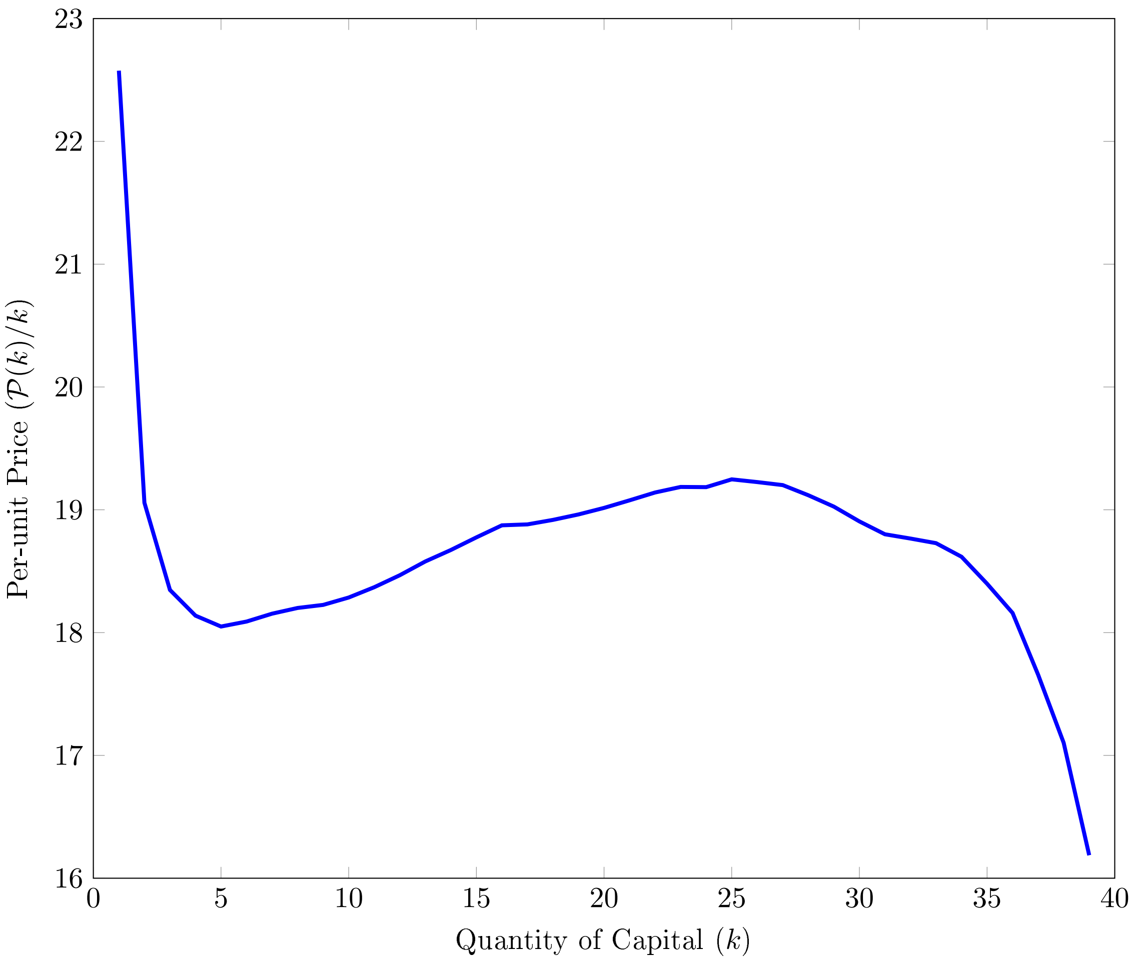


Table 2. Predicted Income Yields and Transferable Shares

Percentile	**Income Yield** $\frac{y(s)-C(\theta(s))}{V(s)}$	**Transferable Share** $\frac{\mathcal{P}(k(s))}{V(s)}$
5	-0.16	0.00
25	0.07	0.16
50	0.09	0.25
75	0.10	0.36
95	0.11	0.49
99	0.13	0.64

returns and as inputs when comparing the effects of taxing business income versus business wealth. The second measure of business wealth is often reported in surveys of consumer finances that ask respondents to estimate the price of the business if it were sold today. This measure in our model is the price of transferable capital, $\mathcal{P}(k(s))$. We use these values to estimate variation in transferable shares of private business wealth and as inputs when comparing the effects of taxing business income versus business capital or capital gain.

In Table 2, we report distributional statistics for income yields and transferable shares. The *income yield* is a common measure of the return to business and is given by the ratio of owner income $y(s) - C(\theta(s))$ to business value $V(s)$. The predicted returns again highlight the heterogeneity across business outcomes, with estimates ranging from -16 percent at the 5^{th} percentile of the distribution to 11 percent at the 95^{th}. The median business has a return of 9 percent, which is slightly higher than stock returns for U.S. publicly-traded companies.

The third column of Table 2 shows the *transferable share*, which is the ratio of transferable capital to the total value. At the bottom of the distribution are businesses that have just sold their businesses but have not yet accumulated any new capital. The median business has a transferable share equal to 25 percent and the 95^{th}-percentile business has a transferable share of 49 percent.

We turn next to evaluating different forms of business taxation, comparing in particular impacts on entry, investment, trading, and welfare.

4.4 Business Taxation

Given our two notions of business wealth, we revisit a central question in public finance, which is how to optimally raise a certain amount of revenues if the government can tax either

Table 3. Impacts of Tax Experiments

% Changes in	Income $\tau_b y$	Value $\tau_v V$	Capital $\tau_k \mathcal{P}(k)$
Steady-state aggregates			
Entry	-23	-26	-23
Investment	-25	-26	-63
Capital	-28	-26	-55
Transactions			
Fraction traded	0	-3	2
Relative buyer to seller size	11	1	12
Price to seller wage bill	-23	-18	-66

incomes or some measure of business wealth. In the case of wealth, we consider taxing either stocks—for example, business capital, $k(s)$, or business value, $V(s)$—or gains subsequent to a transfer, $\mathcal{P}(k(s))$.[8] Our measure of welfare is the steady-state value at entry, which is proportional to wages given agents are indifferent between working for someone else and working for themselves when starting a business. We find that the policy prescription for most levels of desired revenue is to raise it exclusively with a tax on incomes.

In Table 3, we compare aggregate statistics for three taxed economies relative to a no-tax baseline. In the first, we tax income at a rate of $\tau_b = 25$ percent to achieve a pre-specified level of revenue, namely $\tau_b y(s) = 20$. In the second, we tax the firm value at a rate $\tau_v = 3.8$ percent to achieve the same level of revenue. In the third, we tax capital at a rate of $\tau_k = 38$ percent.[9] When comparing steady-state aggregates, we find the different tax experiments have roughly the same impact on the entry decision. In all cases, entry is down on the order of 23 to 26 percent relative to the no-tax baseline. However, for investment and capital, there is a much larger negative impact for the tax on capital as compared to taxing business income or value. For example, investment falls 63 percent when we increase the tax on

[8]Introducing capital gains taxation takes us out of the transferable utility framework. The gains from trade depend on which firm is a buyer and which is a seller in the match. Relative to the model of labor income taxation in Dupuy et al. (2020), the requirements of pair-wise stability need to be augmented to include deviations with respect to which side of the market–buying or selling–any firm would optimally want to be. In the appendix, we show how we address this issue by applying the preference shock formulation in Galichon et al. (2019) to our setting.

[9]As we show later, the revenue raised in these experiments was not feasible with a tax on capital gains only.

capital as compared to a drop of 25 percent when we increase the tax on income.

Table 3 also reports statistics based on the business transfers in the tax and no-tax economies. Across tax regimes other than a capital gains tax, we find a relatively small change in the fraction of business traded. On the other hand, we find large differences when computing relative size of buyer to seller or the ratio of price to wage bill. Taxing value does not affect the relative size of buyers and sellers and has the smallest effect on the ratio of price to wage bill. On the other hand, a tax on capital has a large effect on the ratio of price to wage bill. This is consistent with the fact that the price scales close to linearly with the quantity sold for all but the very smallest sales.

We turn next to the impact of business taxation on welfare. In Figure 5, we plot welfare as we vary the revenue raised from four different sources. We normalize all estimates by the no-tax baseline welfare and thus report a level of 1 in all cases when tax revenues are 0. Revenues rise with the tax rates up to a point that differs by source. The first source of revenue that we consider is a tax on income, $\tau_b y(s)$. With all other taxes equal to zero, this tax raises a maximal revenue at a tax rate of $\tau_b = 41$ percent and a revenue-to-output ratio of 25 percent.

As we see from Figure 5, the income tax yields higher welfare than any other tax. Consider first the alternative policy of taxing the firm's value, $\tau_v V(s)$. The value tax can raise a maximal revenue of 42, which is consistent with a tax rate of $\tau_v = 55$ percent and a revenue-to-output ratio of 25 percent. Relative to the income tax, however, welfare is lower at all levels of revenue. The intuition for this result is straightforward: a tax on $V(s)$ is a tax on the option value of later selling the capital accumulated in the business, which leads to lower investment in the business.

We find that taxing capital is also worse than taxing income—regardless of whether capital is taxed on an accrual basis or upon sale of the business. As we showed above, the per-unit price of transferable capital $\mathcal{P}(k(s))$ depends only on the transferable capital stock and can be used to value assets owned by any business, regardless of whether it is an ongoing concern or transferred. We analyze both cases. In the case that a tax is assessed on all businesses, we assume the tax rate is τ_k. In the case that a tax is assessed only when a business is sold, we assume the tax rate is τ_{cg}. The results are shown in Figure 5. In both cases, we find lower welfare at every level of revenue raised when compared to the income tax. Results for the capital gains are only displayed for revenue levels in the range of 0 to 3 because trades do not occur at higher levels. Close to 99 percent tax rate, the revenue-to-output ratio is only 1 percent. In the case of a tax on owned capital for all businesses, the highest achievable tax rate is 74 percent, which yields a revenue-to-output ratio of 15 percent.

If the tax on capital is assessed on all businesses, then the welfare does not fall off as fast as it does in the case of a tax on realized gains. But given that business transfers are not discouraged with a tax assessed on all businesses, one could ask why the tax on capital, $\tau_k \mathcal{P}(k(s))$, does worse than the tax on income, $\tau_b y(s)$? One easy—although incomplete—answer to this is they have differential impacts on investment. In Figure 6, we plot the predicted investment levels for the four different tax policies and again normalize levels for the no-tax case. We see a similar picture as in Figure 5, although investment levels fall off more quickly when we tax capital either on an accrual basis or by way of realized capital gains. This happens in large part because of differences in incidence across businesses that differ in their levels of productivity z and capital k.

To investigate this point further, we compute the tax bills that would have to be paid for each owner indexed by s across the population under the different tax regimes. Take, for example, the economy with only a tax on business incomes, $\tau_b y(s)$. Figure 7 shows a heat-map with colors indicating the log ratio of tax bills from the income-tax-only economy and the capital-tax-only economy, that is, $\log(\tau_b y(s)) - \log(\tau_k \mathcal{P}(k(s)))$. This is plotted for all possible levels of productivity z and capital. As the figure shows, owners with high productivity and low capital have the highest tax bill when revenues are financed by income taxes, while owners with low productivity and high capital have the highest tax bill when revenues are financed by capital taxes. The tax on incomes yields higher welfare and higher investment because tax elasticities fall steeply with productivity. The most productive owners are the most inelastic in response to tax changes. Although not shown, if we repeat the exercise for taxes on firm value, we would find similar tax bills regardless of capital and thus more inelastic owners bearing more of the tax burden in the case that revenues come from taxing $V(s)$ as compared to $\mathcal{P}(k(s))$.

Finally, we ask how firm dynamics change across tax policy regimes. As we noted earlier, an important statistic is a dispersion in the marginal products of capital, which gives us some sense of the misallocation of capital across owners. For each level of revenue and in each tax regime, we recalculate the standard deviation of the log of the marginal product of capital. This statistic remains relatively constant in all tax regimes except that with taxes on capital gains on owners that sell their businesses. As tax rates on capital gains rise, the number of trades declines quickly, and the misallocation of capital is amplified significantly. For example, as the tax rate approaches 100 percent, the standard deviation of log marginal products of capital almost doubles to 78 percent as compared to 46 percent in the baseline no-tax case.

Figure 5. Predicted Welfare Gains, Varying Business Tax Rates

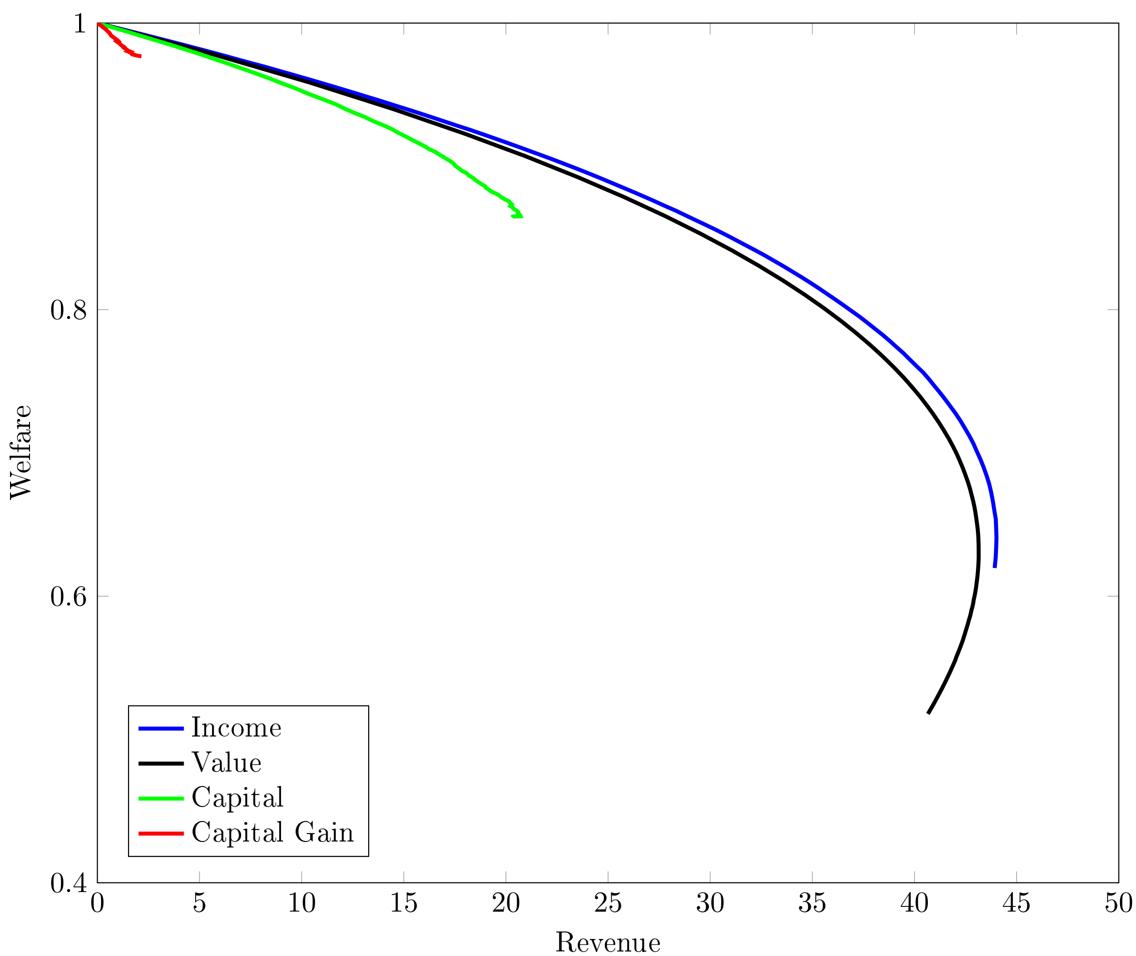


Figure 6. Predicted Investment, Varying Business Tax Rates

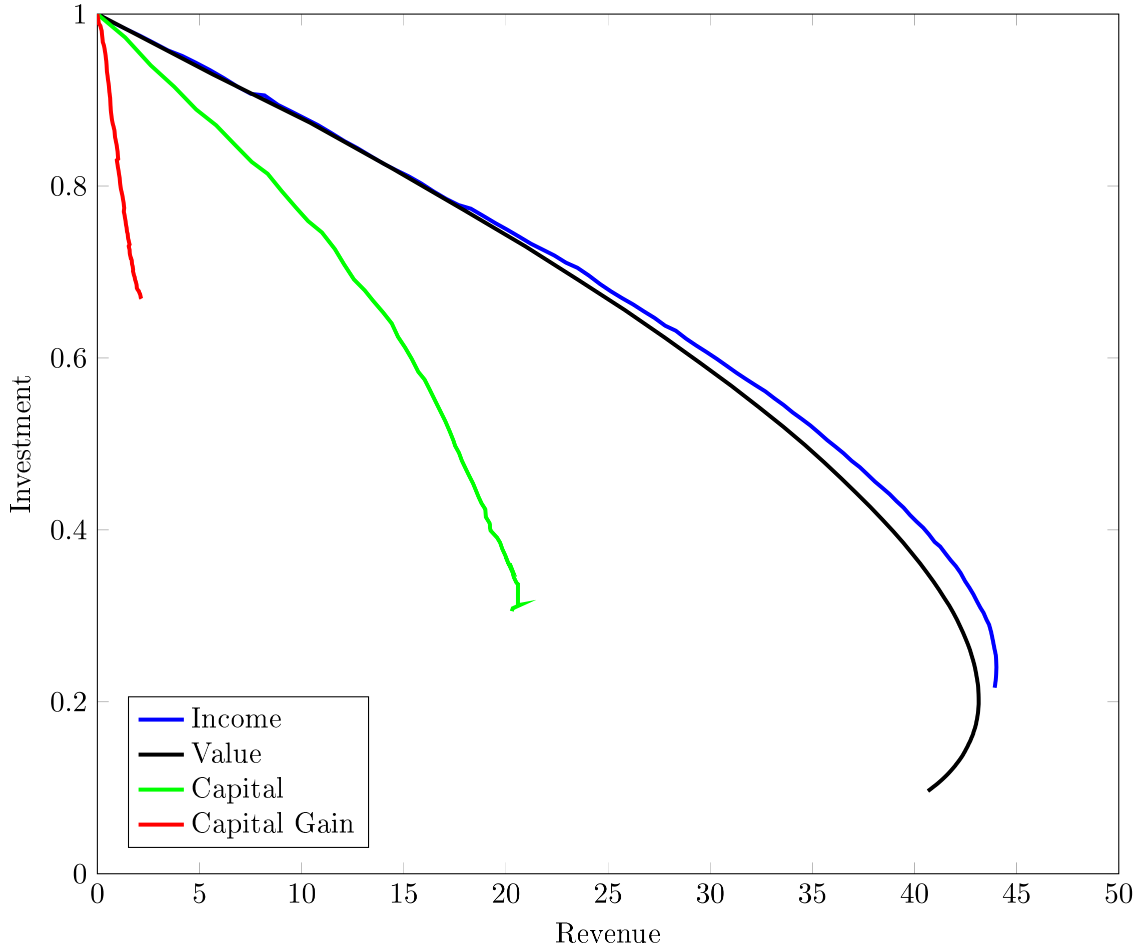
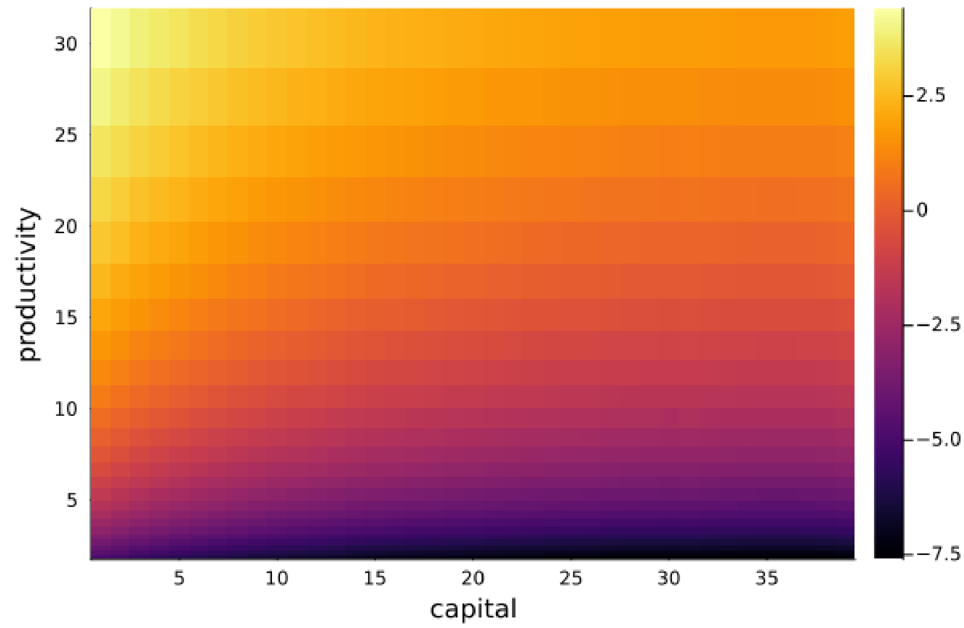


Figure 7. Log Ratio of Tax Bills: Income vs Capital



5 Conclusion

Theory has been developed to study the reallocation of capital through business sales. The capital we modeled is neither divisible nor typically sold in centralized markets, but constitutes most capital transferred in private business sales in the United States. We used the theory to study firm dynamics, business wealth, and business taxation. With parameters disciplined by administrative tax data, the theory predicts significant dispersion in marginal products of capital, returns to business wealth, and heterogeneity of transferable capital shares. Comparisons of taxes on business incomes versus different measures of wealth reveal a clear welfare ranking of income taxes over wealth taxes.

In order to keep the mathematics and numerics as transparent as possible, we made certain assumptions that can be relaxed in future work. We used quasi-linear preferences to exploit tools from the matching literature and prove efficiency. We assumed capital is indivisible but otherwise homogeneous for tractability. These choices, among others, must ultimately be disciplined by additional observations from the data.

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Appendix

Proof of Theorem 1

We prove the theorem by using duality to cast the Monge-Kantorovich problem in a form that highlights the properties of the allocation, in particular the feasibility of capital and prices and the stability of the equilibrium. Consider problem $P1$,

$$\max_{\pi\geq 0,\pi_o^a\geq 0,\pi_o^b\geq 0} \quad \Sigma_{s,\tilde{s}} X\left(s,\tilde{s}\right)\pi\left(s,\tilde{s}\right)+\Sigma_s V\left(s\right)\pi_o^a\left(s\right)+\Sigma_{\tilde{s}} V\left(\tilde{s}\right)\pi_o^b\left(\tilde{s}\right)$$

$$\begin{aligned} s.t. \quad & \Sigma_{\tilde{s}}\pi\left(s,\tilde{s}\right)+\pi_o^a\left(s\right)=\phi\left(s\right)/2 \\ & \Sigma_s\pi\left(s,\tilde{s}\right)+\pi_o^b\left(\tilde{s}\right)=\phi\left(\tilde{s}\right)/2 \end{aligned}$$

Formulate the Lagrangian,

$$\begin{aligned} &\max_{\pi\geq 0,\pi_o^a\geq 0,\pi_o^b\geq 0} \Sigma_{s,\tilde{s}} X\left(s,\tilde{s}\right)\pi\left(s,\tilde{s}\right)+\Sigma_s V\left(s\right)\pi_o^a\left(s\right)+\Sigma_{\tilde{s}} V\left(\tilde{s}\right)\pi_o^b\left(\tilde{s}\right) \\ &+\min_{\mu^a,\mu^b}\Sigma_s\mu^a\left(s\right)\left[\phi\left(s\right)/2-\Sigma_{\tilde{s}}\pi\left(s,\tilde{s}\right)-\pi_o^a\left(s\right)\right]+\Sigma_{\tilde{s}}\mu^b\left(\tilde{s}\right)\left[\phi\left(\tilde{s}\right)/2-\Sigma_s\pi\left(s,\tilde{s}\right)-\pi_o^b\left(\tilde{s}\right)\right] \end{aligned}$$

Using the minimax theorem,

$$\begin{aligned} &\min_{\mu^a,\mu^b} \quad \Sigma_s\mu^a\left(s\right)\phi\left(s\right)/2+\Sigma_{\tilde{s}}\mu^b\left(\tilde{s}\right)\phi\left(\tilde{s}\right)/2 \\ &+\max_{\pi\geq 0} \quad \Sigma_{s,\tilde{s}}\left[X\left(s,\tilde{s}\right)-\mu^a\left(s\right)-\mu^b\left(\tilde{s}\right)\right]\pi\left(s,\tilde{s}\right) \\ &+\max_{\pi^a\geq 0,\pi^b\geq 0}\Sigma_s\pi_o^a\left(s\right)\left[V\left(s\right)-\mu^a\left(s\right)\right]+\Sigma_{\tilde{s}}\pi_o^b\left(\tilde{s}\right)\left[V\left(\tilde{s}\right)-\mu^b\left(\tilde{s}\right)\right] \end{aligned}$$

or equivalently,

$$\begin{aligned} &\min_{\mu^a,\mu^b}\Sigma_s\mu^a\left(s\right)\phi\left(s\right)/2+\Sigma_{\tilde{s}}\mu^b\left(\tilde{s}\right)\phi\left(\tilde{s}\right)/2 \\ &s.t.\mu^a\left(s\right)+\mu^b\left(\tilde{s}\right)\geq X\left(s,\tilde{s}\right) \\ &\mu^a\left(s\right)\geq V\left(s\right) \\ &\mu^b\left(\tilde{s}\right)\geq V\left(\tilde{s}\right) \end{aligned}$$

We denote the latter problem the dual of $P1$. Observe that the dual problem is invariant to swapping the labels a and b on μ, which implies that at an optimal solution $\mu^a=\mu^b$. We conclude the proof of the theorem in two steps.

First, it is easy to see that conditional on a match, the choice of capital is feasible by definition of X. Hence conditions (1) and (2) are satisfied. In addition, for matches that are formed in equilibrium, that is, $\pi(s,\tilde{s})>0$, $\mu^a(s)+\mu^b(\tilde{s})=X(s,\tilde{s})$. This result follows from complementary slackness of the dual problem. Summing up (17) and (18) guarantees that (3) is satisfied (with equality). It also immediately follows that (19) and (20) satisfy the restrictions (9a) and (9b) on the measures.

Second, we show that the pair (p^m,k^m) satisfies pairwise stability given V. Suppose, by contradiction, that it is not the case. That is, there exists a pair $(s,\tilde{s})$, feasible capital allocation $k^m(s,\tilde{s})$ and prices $\hat{p}$, such that

$$V\left(z,\hat{k}^m(s,\tilde{s})\right)-\hat{p}(s,\tilde{s})\geq\mu(s)$$
$$V\left(\tilde{z},\hat{k}^m(s,\tilde{s})\right)-\hat{p}(\tilde{s},s)\geq\mu(\tilde{s})$$

with at least one inequality being strict, and

$$\hat{p}(s,\tilde{s})+\hat{p}(\tilde{s},s)\geq 0.$$

Without loss of generality, we consider the capital allocation that would maximize the sum of the values of the deviating pair. Summing up the values from deviating we get

$$X(s,\tilde{s})-(\hat{p}(s,\tilde{s})+\hat{p}(\tilde{s},s))>\mu(s)+\mu(\tilde{s}).$$

Using the first constraint in the dual problem,

$$\mu(s)+\mu(\tilde{s})\geq X(s,\tilde{s})$$

which implies $\hat{p}(s,\tilde{s})+\hat{p}(\tilde{s},s)<0$, a contradiction that concludes the proof.

Proof of Theorem 2

We prove efficiency under an assumption that productivity space is discrete. This implies a discrete space of agent types, $\mathcal{S}=\{s_1,...,s_N\}$. We do so to keep notation simple, but the result extends naturally to a continuum of types. Accordingly, let $g(s)$ be the probability mass function of entrants of type s. Given ϕ_0, consider a planner that solves the following maximization problem.

$$P_t(\phi_0)=\max_{\{\lambda_t,\lambda_{o,t},\theta_t,k_t^m,n,m_t\}}\int_t^\infty \exp(-r(\tau-t))\left\{\Sigma_{s\in S}\left[y(s,n)-c(\theta_t(s))\right]\phi_t(s)-n_0m_t\right\}\mathrm{d}\tau$$

subject to

$$\dot{\phi}_t(s)=\Gamma\left(s,\phi_t;\lambda_t,\lambda_{o,t},\theta_t,\phi_e,k_t^m\right)\quad\forall s\in\mathcal{S},$$

feasibility of k^m and $\phi_e(s,m) = mg(s)$ for all $s \in \mathcal{S}$ and

$$N_t^w = \int n_t(s)\phi_t(s) + n_0 m_t$$

$$\dot{N}_t^w = \delta(N - N_t^w) - m_t$$

Set-up

The recursive formulation of the planner's problem is

$$rP(\phi_t, N_t^w) = \max_{\{\lambda_t, \lambda_{o,t}, \theta_t, k_t^m, n, m_t\}} \quad \Sigma_s [y(s,n) - c(\theta_t(s))]\phi_t(s) + \sum_{\hat{s}} \frac{\partial P(\phi_t)}{\partial \phi_t(\hat{s})} \Gamma(\hat{s}, \phi_t; \lambda_t, \lambda_{o,t}, \theta_t, k_t^m, m_t)$$
$$+ \frac{\partial P(\phi_t, N_t^w)}{\partial N_t^w} \delta(N - m_t - N_t^w).$$

In what follows, we omit some arguments of the functions Γ for brevity. The optimality conditions are

$$\begin{aligned} c'(\theta_t(s))\phi_t(s) &= \Sigma_{\hat{s}} \frac{\partial P(\phi_t)}{\partial \phi_t(\hat{s})} \frac{\partial \Gamma(\hat{s})}{\partial \theta_t(s)} \\ \xi n_0 + \frac{\partial P(\phi_t, N_t^w)}{\partial N_t^w} &= \Sigma_{\hat{s}} \frac{\partial P(\phi_t)}{\partial \phi_t(\hat{s})} \frac{\partial \Gamma(\hat{s})}{\partial \phi_{e,t}(\hat{s})} \frac{\partial \phi_{e,t}(\hat{s})}{\partial m_t} \\ \gamma z k^\alpha n^{\gamma-1} &= \xi. \end{aligned}$$

By the envelope theorem,

$$\begin{aligned} r\frac{\partial P(\phi_t)}{\partial \phi_t(s)} &= y(s) - c(\theta_t(s)) + \Sigma_{\hat{s}} \frac{\partial P(\phi_t)}{\partial \phi_t(\hat{s})} \frac{\partial \Gamma(\hat{s})}{\partial \phi_t(s)} + \Sigma_{\hat{s}} \frac{\partial^2 P(\phi_t)}{\partial \phi_t(s)\partial \phi_t(\hat{s})} \Gamma(\hat{s}) \\ r\frac{\partial P(\phi_t)}{\partial N_t^w(s)} &= \xi + \frac{\partial^2 P(\phi_t)}{\partial^2 N_t^w(s)} \dot{N}_t^w - \delta \frac{\partial P(\phi_t)}{\partial N_t^w(s)}. \end{aligned}$$

We define the marginal value to the planner of an additional agents of type s at time t,

$$\tilde{V}(s;\phi_t) = \frac{\partial P(\phi_t)}{\partial \phi_t(s)}.$$

We can formulate the envelope condition above as

$$r\tilde{V}(s;\phi_t) = y(s) - c(\theta_t(s)) + \Sigma_{\hat{s}} \tilde{V}(\hat{s};\phi_t) \frac{\partial \Gamma(\hat{s})}{\partial \phi(s)} + \Sigma_{\hat{s}} \frac{\partial \tilde{V}(s;\phi_t)}{\partial \phi(\hat{s})} \Gamma(\hat{s}).$$

We also define the marginal value along the optimal trajectory

$$V_t(s) = \tilde{V}(s; \phi_t),$$

and obtain the time-derivative

$$\frac{\partial V_t(s)}{\partial t} = \Sigma_{\hat{s}} \frac{\partial \tilde{V}(s, \phi_t)}{\partial \phi_t(\hat{s})} \Gamma(\hat{s}).$$

The envelope condition can be further simplified to

$$rV_t(s) = y(s) - c(\theta_t(s)) + \Sigma_{\hat{s}} V_t(\hat{s}) \frac{\partial \Gamma(\hat{s})}{\partial \phi(s)} + \frac{\partial V_t(s)}{\partial t}.$$

We focus on a stationary planner's problem, which allows us to drop the time subscript and the time derivative from the problem above. Hence,

$$rV(s) = y(s) - c(\theta(s)) + \Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma(\hat{s})}{\partial \phi(s)} \tag{25}$$

$$= y(s) - ... \tag{26}$$

$$+ (V(z, k+1) - V(z, k))(\theta - \delta_k) - C(\theta(s)) \tag{27}$$

$$+ (V(z+1, k) - V(z, k)) \tilde{\mu}^+(z) + (V(z-1, k) - V(z, k)) \tilde{\mu}^-(z) \tag{28}$$

$$+ \Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma_\lambda(\hat{s})}{\partial \phi(s)}. \tag{29}$$

The FOCs with respect to investment and entry become

$$c'(\theta_t(s)) \phi_t(s) = \Sigma_{\hat{s}} \frac{\partial P(\phi_t)}{\partial \phi_t(\hat{s})} \frac{\partial \Gamma(\hat{s})}{\partial \theta_t(s)} = \Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma(\hat{s})}{\partial \theta(s)} = [V(z, k+1) - V(z, k)] \phi_t(s) \tag{30}$$

and

$$c_e = \Sigma_{\hat{s}} \frac{\partial P(\phi_t)}{\partial \phi_t(\hat{s})} \frac{\partial \Gamma(\hat{s})}{\partial \phi_{e,t}(\hat{s})} \frac{\partial \phi_{e,t}(\hat{s})}{\partial m_t} = \Sigma_{\hat{s}} V(\hat{s}) g(\hat{s}). \tag{31}$$

Next, we turn to a linear programming problem in which we solve for the optimal set of matches and capital allocations $(\lambda, \lambda_0, k^m)$. We also show that the last term in (29) is equal to the multiplier associated to the constraints of the same linear programming problem.

Optimal Matching

We set up the following linear problem

$$\begin{aligned}
&\max_{\lambda\geq 0,\lambda_0\geq 0,k^m} \Sigma_s V(s)\,\Gamma_\lambda(s,\phi;\lambda,\lambda_0,k^m)\\
s.t.\quad &\Sigma_{\tilde{s}}\lambda(s,\tilde{s})+\lambda_0(s)=1 \quad \forall s\\
&\Sigma_s\lambda(s,\tilde{s})\,\phi(s)+\lambda_0(\tilde{s})\,\phi(\tilde{s})=\phi(\tilde{s}) \quad \forall\tilde{s}
\end{aligned}$$

To make progress, we re-arrange the objective function using the definition of Γ_λ,

$$\begin{aligned}
&\Sigma_s V(s)\left[\lambda_0(s)\,\phi(s)+\Sigma_{s',s''}\lambda\left(s',s''\right)\mathbb{I}\left\{k^m\left(s',s''\right)=k(s),z\left(s'\right)=z(s)\right\}\phi\left(s'\right)\right]\\
=&\Sigma_s V(s)\left[\lambda_0(s)\,\phi(s)+\Sigma_{s',s''}\left(\frac{\lambda(s',s'')}{2}\mathbb{I}\left\{k^m\left(s',s''\right)=k(s),z\left(s'\right)=z(s)\right\}\phi\left(s'\right)\right.\right.\\
+&\left.\left.\frac{\lambda(s'',s')}{2}\mathbb{I}\left\{k^m\left(s',s''\right)=k(s),z\left(s''\right)=z(s)\right\}\phi\left(s''\right)\right)\right]\\
=&\Sigma_s V(s)\left[\lambda_0(s)\,\phi(s)+\Sigma_{s',s''}\left(\frac{\lambda(s',s'')}{2}\phi\left(s'\right)\Sigma_s V(s)\,\mathbb{I}\left\{k^m\left(s',s''\right)=k(s),z\left(s'\right)=z(s)\right\}\right.\right.\\
+&\left.\left.\frac{\lambda(s'',s')}{2}\phi\left(s''\right)\Sigma_s V(s)\,\mathbb{I}\left\{k^m\left(s',s''\right)=k(s),z\left(s''\right)=z(s)\right\}\right)\right]
\end{aligned}$$

Imposing feasibility of k^m amounts to restricting the indicators above to be such that either s' is a buyer, or s'' is, or neither. The choice of k^m is equivalent to solving the problem

$$X\left(s',s''\right)=\max\left\{V\left(z',k'+k''\right)+V\left(z'',0\right),V\left(s'\right)+V\left(s''\right),V\left(z',0\right)+V\left(z'',k'+k''\right)\right\}.$$

The objective function thus simplifies to

$$\Sigma_s V(s)\,\lambda_0(s)\,\phi(s)+\Sigma_{s',s''}\frac{\lambda(s',s'')}{2}\phi\left(s'\right)X\left(s',s''\right)$$

Let $\pi(s,\tilde{s})=\frac{\lambda(s,\tilde{s})}{2}\phi(s)$ and $\pi_0(s)=\frac{\lambda_0(s)}{2}\phi(s)$.

We label the value to the matching problem as Q.

$$\begin{aligned}
Q(\phi)=\max_{\pi\geq 0,\pi_0\geq 0}&\Sigma_{s,\tilde{s}}\pi(s,\tilde{s})\,X(s,\tilde{s})+\Sigma_s V(s)\,\pi_0(s)+\Sigma_{\tilde{s}}V(\tilde{s})\,\pi_0(\tilde{s})\\
&s.t.\Sigma_{\tilde{s}}\pi(s,\tilde{s})+\pi_0(s)=\frac{\phi(s)}{2}\\
&s.t.\Sigma_s\pi(s,\tilde{s})+\pi_0(\tilde{s})=\frac{\phi(\tilde{s})}{2}
\end{aligned}\tag{32}$$

Notice that this formulation of the matching problem is analogous to the one in the competitive equilibrium. Let $\mu^a(s)$ and $\mu^b(s)$ be the multipliers attached to the constraints of (32). From the envelope theorem,

$$\frac{\partial Q}{\partial \phi(s)} = \frac{\mu^a(s) + \mu^b(s)}{2}$$

and by the symmetry of $X(\cdot,\cdot)$, $\mu^a(s) = \mu^b(s) = \mu(s)$. Since at the solution,

$$Q(\phi) = \Sigma_s V(s) \Gamma_\lambda (s, \phi; \lambda^*, \lambda_0^*, k^{m,*}),$$

is satisfied at all ϕ, we can differentiate both sides to obtain

$$\Sigma_{\hat{s}} V(\hat{s}) \frac{\partial \Gamma_{\lambda^*}(\hat{s})}{\partial \phi(s)} = \mu(s).$$

Characterization

Notice that the Bellman equation, the optimality condition for θ, and the static matching problem are identical to those in the competitive equilibrium. It immediately follows that the competitive equilibrium solves the planner's problem and the equilibrium value and policy functions are the same as the planner's. Last, let ϕ^* be the stationary distribution associated with the planner's problem. The condition $\phi_0 = \phi^*$ guarantees that the economy is stationary.