TECHNICAL APPENDIX

Solving the Stochastic Growth Model with a Finite Element Method

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Details of the calculation of approximate consumption functions for the stochastic growth model are provided below.

1. *Initialization.*

   i. specify parameters of utility and production: β, δ, τ, ρ, α, and σ_ϵ;

   ii. specify the grid on capital: $\vec{k} = [k_1, k_2, k_3, \ldots, k_{n_k}]$;

   iii. specify the grid on technology: $\vec{\theta} = [\theta_1, \theta_2, \ldots, \theta_{n_\theta}]$;

   iv. specify an initial guess for consumption: $c(k_i, \theta_j)$ for each k_i, θ_j pair where k_i, $i = 1, \ldots n_k$, is an element of $\vec{k}$ and θ_j, $j = 1, \ldots n_\theta$, is an element of θ_j; let c_a denote consumption at grid point (or *node*) a and $\vec{c}$ is the vector with elements c_a, $a = 1, \ldots, nnodes$, $nnodes = n_k n_\theta$;

   v. specify parameters for quadrature:

      a. for integration with respect to ϵ, specify $[\underline{\epsilon}, \overline{\epsilon}]$ and m_ϵ;

      b. for integration with respect to k on interval $[k_i, k_{i+1}]$, specify $m_{k,i}$;

      c. for integration with respect to θ on interval $[\theta_i, \theta_{i+1}]$, specify $m_{\theta,i}$;

   vi. specify the maximum number of iterations of the Newton-Raphson algorithm and a criterion for stopping iterations (i.e., a norm $||\cdot||$ and a tolerance parameter).

2. *Calculation of abscissas and weights for quadrature.*

   i. given parameters of step $1(v)$, calculate abscissas and weights for integration done in step 3. (See Press, et.al. (1986) for details.)

3. *Main loop.*

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do t = 1 to maximum number of iterations
    set H = 0, J = 0 where H is nnodes × 1 and J is nnodes × nnodes
    do e = 1 to number of elements (=(n_k − 1)(n_θ − 1))
       determine current intervals for k, θ in element e
       determine the 4 global nodes for element e (call them nodes 1-4)
       do j = 1 to number of quadrature points for θ on element e
          do i = 1 to number of quadrature points for k on element e
             calculate consumption c (for (i, j)th quadrature point in element e)
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e.g., if (k,θ) are the current coordinates and element e is given by $[\underline{k}_e, \overline{k}_e] \times [\underline{\theta}_e, \overline{\theta}_e]$, then

$$\begin{aligned} c = & c_{\text{node1}} \frac{\overline{k}_e - k}{\overline{k}_e - \underline{k}_e} \cdot \frac{\overline{\theta}_e - \theta}{\overline{\theta}_e - \underline{\theta}_e} + c_{\text{node2}} \frac{k - \underline{k}_e}{\overline{k}_e - \underline{k}_e} \cdot \frac{\overline{\theta}_e - \theta}{\overline{\theta}_e - \underline{\theta}_e} + \\ & c_{\text{node3}} \frac{k - \underline{k}_e}{\overline{k}_e - \underline{k}_e} \cdot \frac{\theta - \underline{\theta}_e}{\overline{\theta}_e - \underline{\theta}_e} + c_{\text{node4}} \frac{\overline{k}_e - k}{\overline{k}_e - \underline{k}_e} \cdot \frac{\theta - \underline{\theta}_e}{\overline{\theta}_e - \underline{\theta}_e} \end{aligned}$$

calculate next period capital, $K = \theta k^{\alpha} - c + (1-\delta)k$

calculate next period's marginal product of capital $mp_K = \alpha K^{\alpha - 1}$

set $s = 0$

do $l = 1$ to m_ϵ

set ϵ equal to the lth quadrature point (calculated in step 1(v)a)

calculate the next period shock, $\Theta = \theta^{\rho} \exp(\epsilon)$

determine the element that includes point (K, Θ) and call it E

determine the 4 global nodes for element E (denote by NODEs 1-4)

calculate next period consumption C,

e.g., if (K, Θ) are the coordinates and E is the element, then

$$\begin{aligned} C = & c_{\text{NODE1}} \frac{\overline{k}_E - K}{\overline{k}_E - \underline{k}_E} \cdot \frac{\overline{\theta}_E - \Theta}{\overline{\theta}_E - \underline{\theta}_E} + c_{\text{NODE2}} \frac{K - \underline{k}_E}{\overline{k}_E - \underline{k}_E} \cdot \frac{\overline{\theta}_E - \Theta}{\overline{\theta}_E - \underline{\theta}_E} + \\ & c_{\text{NODE3}} \frac{K - \underline{k}_E}{\overline{k}_E - \underline{k}_E} \cdot \frac{\Theta - \underline{\theta}_E}{\overline{\theta}_E - \underline{\theta}_E} + c_{\text{NODE4}} \frac{\overline{k}_E - K}{\overline{k}_E - \underline{k}_E} \cdot \frac{\Theta - \underline{\theta}_E}{\overline{\theta}_E - \underline{\theta}_E} \end{aligned}$$

calculate next period's marginal utility ($mu = \beta C^{-\tau}$)

add ($mu \times \Theta mp_K + 1 - \delta \times$ f(ϵ) $\times$ lth quadrature weight) to s

calculate derivatives of s with respect to c_a for all nodes a

end loop over l

compute residual: $R = s - c^{-\tau}$

update elements of H associated with global nodes for element e,

e.g., if (k,θ) are the current coordinates, if element e is given by $[\underline{k}_e, \overline{k}_e] \times [\underline{\theta}_e, \overline{\theta}_e]$, and if $\omega_{i,j}$ are the (i,j) quadrature weights, then

$$H(\text{node } 1) = H(\text{node } 1) + \omega_{i,j} \times R \times \frac{\overline{k}_e - k}{\overline{k}_e - \underline{k}_e} \cdot \frac{\overline{\theta}_e - \theta}{\overline{\theta}_e - \underline{\theta}_e}$$

$$H(\text{node } 2) = H(\text{node } 2) + \omega_{i,j} \times R \times \frac{k-\underline{k}_e}{\overline{k}_e-\underline{k}_e} \cdot \frac{\overline{\theta}_e-\theta}{\overline{\theta}_e-\underline{\theta}_e}$$
$$H(\text{node } 3) = H(\text{node } 3) + \omega_{i,j} \times R \times \frac{k-\underline{k}_e}{\overline{k}_e-\underline{k}_e} \cdot \frac{\theta-\underline{\theta}_e}{\overline{\theta}_e-\underline{\theta}_e}$$
$$H(\text{node } 4) = H(\text{node } 4) + \omega_{i,j} \times R \times \frac{\overline{k}_e-k}{\overline{k}_e-\underline{k}_e} \cdot \frac{\theta-\underline{\theta}_e}{\overline{\theta}_e-\underline{\theta}_e}$$

calculate derivatives of $H(\text{node } l)$, $l = 1, 2, 3, 4$ with respect to ...

unknowns c_a for all nodes a and put the answer in the node lth ...

row of J (this calculation uses the derivatives of s and $c^{-\tau}$)

end loops over i, j

end loop over e

if $k_1 = 0$, enforce $c(0, \theta) = 0$, i.e., eliminate elements for $k = 0$ nodes from H and J

solve $Jx = H$ for x and update the consumption vector: $\vec{c} = \vec{c}$ - x

if $||x|| <$ tolerance, go to step 4

end loop over t

4. *Print results.*