Federal Reserve Bank of Minneapolis
Research Department Staff Report 217

October 1998

NOTES
Sticky Price Models of the Business Cycle: Can the Contract Multiplier Solve the Persistence Problem?

V.V. Chari*

University of Minnesota
and Federal Reserve Bank of Minneapolis

Patrick J. Kehoe*

University of Pennsylvania,
Federal Reserve Bank of Minneapolis,
and National Bureau of Economic Research

Ellen R. McGrattan*

Federal Reserve Bank of Minneapolis

*The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

These notes provide derivations for all of the analytical results of the paper. We have chosen to use capital letters for variables that are not logged and small letters for their logged values. Our notation is such that $z_{i,t} = \log\, Z(i, s^t)$ and $z_t = \log\, Z(s^t)$. Any exceptions will be noted. The main analytical results of the paper are summarized in equations (34), (45), (77), (96), and (116) below. In the last section of the notes, we provide details of the computations for the "all-in-one" model. These computations are the basis for results in Table 2 and Table 3. The codes for generating these results are available at: ftp://toni.mpls.frb.fed.us/pub/research/mcgrattan/sr217

1. Nomenclature

- $C(s^t)$: consumption in state s^t
- $L(s^t)$: labor supply in state s^t
- $Y(s^t)$: aggregate output in state s^t
- $Y(i, s^t)$: intermediate good of type i in state s^t
- $L(i, s^t)$: labor used by monopolist i in state s^t
- $K(i, s^t)$: capital stock of monopolist i in state s^t
- $X(i, s^t)$: investment made by monopolist i in state s^t
- $V(i, s^t)$: unit costs paid by monopolist i in state s^t
- $P(i, s^t)$: price charged by ith monopolist in state s^t
- $P(s^{t-1})$: prices currently being set, which are conditional on state s^{t-1}
- $\bar{P}(s^t)$: aggregate price level in state s^t
- $M(s^t)$: nominal money balances in state s^t

- $\mu(s^t)$: growth in nominal money balances, $t-1$ to t
- $W(s^t)$: wage rate in state s^t
- $Q(s^\tau|s^t)$: price in state s^t of a claim to one dollar in s^τ
- $U(C, L, M/\bar{P})$: utility function
- $F(K, L)$: production function
- $D(P_i/\bar{P}, \{Y_j/Y\})$: input demand functions for intermediate good i
- $\phi(X/K)$: adjustment cost function

2. Model Economy

Since we will use the first order conditions over and over again in these notes, we start with a statement of the optimization problems solved by all of the agents in the economy and the associated first order conditions.

The problem solved by the final goods producers each period is

$$\max_{Y(i,s^t)} \ \bar{P}(s^t) - \int P(i, s^{t-1}) Y(i, s^t)/Y(s^t)\, di \tag{1}$$

subject to

$$\int g\left(\frac{Y(i, s^t)}{Y(s^t)}\right) di = 1 \tag{2}$$

The first order conditions for this problem are

$$P(i, s^{t-1}) = \lambda(s^t) g'\left(\frac{Y(i, s^t)}{Y(s^t)}\right) \tag{3}$$

where λ is the Lagrange multiplier on the constraint (2). The zero-profit condition,

$$\bar{P}(s^t) = \int P(i, s^{t-1}) Y(i, s^t)/Y(s^t)\, di, \tag{4}$$

and the first order condition for P_i imply the following for the relative price

$$(5) \qquad \frac{P(i)}{\bar{P}} = \frac{g'\left(\frac{Y(i)}{Y}\right)}{\int g'\left(\frac{Y(j)}{Y}\right)\frac{Y(j)}{Y}\,dj}.$$

Inverting this equation gives the input demand functions

$$(6) \qquad Y(i,s^t) = D\left(\frac{P(i,s^{t-1})}{\bar{P}(s^t)}\int g'\left(\frac{Y(j,s^t)}{Y(s^t)}\right)\frac{Y(j,s^t)}{Y(s^t)}\,dj\right)Y(s^t)$$

where $D \equiv (g')^{-1}$.

If we assume that $g(y) = y^\theta$, which is the case in most of the paper, then we have:

$$(7) \qquad Y(i,s^t) = \left[\frac{\bar{P}(s^t)}{P(i,s^{t-1})}\right]^{\frac{1}{1-\theta}} Y(s^t)$$

$$(8) \qquad \bar{P}(s^t) = \left[\int P(i,s^{t-1})^{\frac{\theta}{\theta-1}}\,di\right]^{\frac{\theta-1}{\theta}}.$$

The problem solved by consumers is

$$(9) \qquad \max \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi(s^t)\, U(C(s^t), L(s^t), M(s^t)/\bar{P}(s^t)),$$

subject to the sequence of budget constraints

$$(10) \qquad \bar{P}(s^t)C(s^t) + M(s^t) + \sum_{s_{t+1}} Q(s^{t+1}|s^t)B(s^{t+1}) \\ \leq \bar{P}(s^t)W(s^t)L(s^t) + M(s^{t-1}) + B(s^t) + \Pi(s^t) + T(s^t), \quad t = 0, 1, \ldots,$$

and borrowing constraints $B(s^{t+1}) \geq \bar{B}$ for some large negative number $\bar{B}$.

The first order conditions for the consumer are therefore given by the following equations:

$$(11) \qquad -\frac{U_l(s^t)}{U_c(s^t)} = W(s^t)$$

$$(12) \qquad \frac{U_c(s^t)}{\bar{P}(s^t)} = \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t)\frac{U_c(s^{t+1})}{\bar{P}(s^{t+1})} + \frac{U_m(s^t)}{\bar{P}(s^t)}$$

$$(13) \qquad Q(s^\tau|s^t) = \beta^{\tau-t}\pi(s^\tau|s^t)\frac{U_c(s^\tau)}{U_c(s^t)}\frac{\bar{P}(s^t)}{\bar{P}(s^\tau)} \quad \text{for all } \tau > \text{t}$$

where $U(s^t)$ is shorthand for $U(C(s^t), L(s^t), M(s^t)/\bar{P}(s^t))$.

The problem solved by the monopolist adjusting his price is to choose $P(i, s^{t-1})$, $K(i, s^\tau)$, $X(i, s^\tau)$, and $L(i, s^\tau)$ $\tau = t, \ldots, t+N-1$ to maximize

$$\text{(14)} \quad \sum_{\tau=t}^{\infty} \sum_{s^\tau} Q(s^\tau | s^{t-1}) \left[P(i, s^{t-1}) Y(i, s^\tau) - \bar{P}(s^\tau) W(s^\tau) L(i, s^\tau) - \bar{P}(s^\tau) X(i, s^\tau) \right]$$

subject to the demand for good i in (6), the production technology:

$$\text{(15)} \quad Y(i, s^t) = F(K(i, s^{t-1}), L(i, s^t))$$

and the law of motion for capital used in producing i:

$$\text{(16)} \quad K(i, s^t) = (1-\delta) K(i, s^{t-1}) + X(i, s^t) - \phi\left(\frac{X(i, s^t)}{K(i, s^{t-1})}\right) K(i, s^{t-1}).$$

The first order conditions for the case with $F(K, L) = K^{\alpha_1} L^{\alpha_2}$ are given by

$$\text{(17)} \quad \sum_{\tau} \sum_{s^\tau} Q(s^\tau | s^{t-1}) \Bigg\{ Y(i, s^\tau) + Y(s^\tau) \left[1 - \bar{P}(s^\tau) V(i, s^\tau) / P(i, s^{t-1}) \right]$$
$$D'\left(\frac{P(i, s^{t-1})}{\bar{P}(s^\tau)} \int g'\left(\frac{Y(j, s^\tau)}{Y(s^\tau)}\right) \frac{Y(j, s^\tau)}{Y(s^\tau)}\, dj \right) g'\left(\frac{Y(i, s^\tau)}{Y(s^\tau)}\right) \Bigg\} = 0$$

$$\text{(18)} \quad V(i, s^t) = W(s^t) / F_l(i, s^t)$$

$$\text{(19)} \quad \frac{1}{1 - \phi'(i, s^t)} = \sum_{s_{t+1}} \frac{Q(s^{t+1} | s^{t-1}) \bar{P}(s^{t+1})}{Q(s^t | s^{t-1}) \bar{P}(s^t)} \Bigg[V(i, s^{t+1}) F_k(i, s^{t+1})$$
$$+ \frac{1}{1 - \phi'(i, s^{t+1})} \left\{ 1 - \delta - \phi(i, s^{t+1}) + \phi'(i, s^{t+1}) \frac{X(i, s^{t+1})}{K(i, s^t)} \right\} \Bigg]$$

where $F(i, s^t)$ and $\phi(i, s^t)$ are shorthand for $F(K(i, s^{t-1}), L(i, s^t))$ and $\phi(X(i, s^t)/K(i, s^{t-1}))$, respectively. The monopolists not setting prices will still maximize with respect to labor, investment, and capital. Therefore, there will be one pricing equation and N Euler equations for capital. The first order conditions for those monopolists not setting prices depend on the prices that they last set.

Note that if the technology of the final goods producer is given by $g(y) = y^\theta$, then the first order condition in (17) can be written more simply as follows:

$$(20) \quad P_i = \frac{1}{\theta} \frac{\sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1}) Y(s^\tau) \bar{P}(s^\tau)^{\frac{2-\theta}{1-\theta}} V(i, s^\tau)}{\sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1}) Y(s^\tau) \bar{P}(s^\tau)^{\frac{1}{1-\theta}}}$$

If adjustment costs are equal to zero, then the Euler equation for capital (19) can be written more simply as follows:

$$(21) \quad U_c(s^t) = \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t)\, U_c(s^{t+1}) \left[V(i, s^{t+1}) F_k(i, s^{t+1}) + 1 - \delta \right]$$

Finally, the following equilibrium constraints must hold:

$$\begin{aligned}
(22) \quad M(s^t) &= \mu(s^t) M(s^{t-1}) \\
(23) \quad T(s^t) &= M(s^t) - M(s^{t-1}) \\
(24) \quad L(s^t) &= \int L(i, s^t)\, di \\
(25) \quad Y(s^t) &= C(s^t) + \int X(i, s^t)\, di.
\end{aligned}$$

To summarize, we have equations (6)-(4) from the final goods producers, equations (11)-(13) from the consumers, equations (16)-(19) from the intermediate goods producers, and equations (22)-(15) that must hold in equilibrium.

3. Benchmark Economy

In this section, we derive analytical results for our benchmark economy with $N = 2$, $g(y) = y^\theta$, $F(K, L) = L$, and the more simple money demand equation:

$$(26) \quad \frac{M(s^t)}{\bar{P}(s^t)} = C(s^t)$$

in place of (12). This assumes an interest elasticity of 0. If we linearize (26), we get

$$(27) \quad m_t - \bar{p}_t = c_t.$$

Our derivations use the money demand equation, the pricing equation for the monopolists, the resource constraint, and the first order condition for consumers relating the wage rate to $-U_l/U_c$. The pricing equation for the group of monopolists (named i) that is currently adjusting prices is:

$$(28) \quad P(i,s^{t-1}) = \frac{\sum_{s^t} Q(s^t|s^{t-1})\{Y(s^t)\bar{P}(s^t)^{\frac{2-\theta}{1-\theta}}W(s^t) + Q(s^{t+1}|s^t)Y(s^{t+1})\bar{P}(s^{t+1})^{\frac{2-\theta}{1-\theta}}W(s^{t+1})\}}{\theta\sum_{s^t} Q(s^t|s^{t-1})\{Y(s^t)\bar{P}(s^t)^{\frac{1}{1-\theta}} + Q(s^{t+1}|s^t)Y(s^{t+1})\bar{P}(s^{t+1})^{\frac{1}{1-\theta}}\}}$$

which follows from (20) with $N = 2$ and the unit cost equal to the wage rate, $V(i,s^t) = W(s^t)$. If we assume $\beta \approx 1$, then the linearized pricing equation is equal to

$$(29) \quad p_{i,t-1} = \frac{1}{2}E_{t-1}\{\bar{p}_t + \bar{p}_{t+1} + w_t + w_{t+1}\}.$$

where $E_t \equiv E[\cdot|s^t]$.

The resource constraint is simple since there is no investment. It is given by

$$\begin{aligned}(30) \quad C(s^t) &= Y(s^t) \\ &= L(s^t)\left[\frac{1}{2}\sum_{i=1}^{2}P(s^{t-i})^{\frac{\theta}{\theta-1}}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\sum_{i=1}^{2}P(s^{t-i})^{\frac{1}{\theta-1}}\right]^{-1}.\end{aligned}$$

The definition of aggregate output in (30) can be derived simply by noting that $L(s^t) = \int L(i,s^t)\,di = \int Y(i,s^t)\,di$ and using the demand functions for the $Y(i,s^t)$. Linearizing the resource constraint gives the following

$$(31) \quad c_t = y_t = l_t.$$

Let γ denote the response of monopolist i's price to expected changes in current and future output. In the benchmark economy, γ is equal to the elasticity of the wage with respect to consumption (i.e., $d(-U_l/U_c)/dC \times (-CU_c/U_l)$ evaluated at $C = Y$, $L = Y$, $M/\bar{P} = Y$). (Below we discuss restrictions on preferences that imply that $M/\bar{P} = Y$.) With this assumption, the first order condition

$$(32) \quad W(s^t) = -U_l(C(s^t), L(s^t), M(s^t)/\bar{P}(s^t))/U_c(C(s^t), L(s^t), M(s^t)/\bar{P}(s^t))$$

can be linearized, with the result given by

$$w_t = \gamma y_t. \tag{33}$$

We can now derive the equation reported in the paper. Substitute (33) into (29) to get

$$p_{i,t-1} = \frac{1}{2} E_{t-1} \left[\bar{p}_t + \bar{p}_{t+1} + \gamma (y_t + y_{t+1}) \right]. \tag{34}$$

Let $P(s^{t-1})$ denote the price set by monopolists adjusting their price currently. In this economy the aggregate price is

$$\bar{P}(s^t) = \left[\frac{1}{2} P(s^{t-1})^{\frac{\theta}{\theta-1}} + P(s^{t-2})^{\frac{\theta}{\theta-1}} \right]^{\frac{\theta-1}{\theta}} \tag{35}$$

which, when linearized, is given by

$$\bar{p}_t = \frac{1}{2} p_{t-1} + \frac{1}{2} p_{t-2}. \tag{36}$$

Because equation (37) assumes that the ith group of monopolists is changing prices, it must be the case that $p_{i,t-1} = p_{t-1}$. Therefore, it follows from (34) and (36) that

$$p_{t-1} = \frac{1}{2} E_{t-1} \left[\frac{1}{2} p_{t-1} + \frac{1}{2} p_{t-2} + \frac{1}{2} p_t + \frac{1}{2} p_{t-1} + \gamma (y_t + y_{t+1}) \right]. \tag{37}$$

Manipulating (37) yields

$$p_{t-1} = \frac{1}{2} p_{t-2} + \frac{1}{2} E_{t-1} p_t + \gamma E_{t-1} \left[y_t + y_{t+1} \right]. \tag{38}$$

If we use the money demand equation to eliminate output in (38), we get

$$p_{t-1} = \frac{1}{2} p_{t-2} + \frac{1}{2} E_{t-1} p_t + \gamma E_{t-1} \left[m_t - \frac{1}{2} p_{t-1} - \frac{1}{2} p_{t-2} + m_{t+1} - \frac{1}{2} p_t - \frac{1}{2} p_{t-1} \right] \tag{39}$$

or with some rearranging,

$$E_{t-1} p_t - 2 \frac{1+\gamma}{1-\gamma} p_{t-1} + p_{t-2} = - \frac{2\gamma}{1-\gamma} E_{t-1} (m_t + m_{t+1}). \tag{40}$$

This difference equation has the following stable solution

$$p_{t-1} = a p_{t-2} + \frac{2a\gamma}{1-\gamma} E_{t-1} \sum_{i=0}^{\infty} a^i (m_{t+i} + m_{t+1+i}) \tag{41}$$

where a is the root of $a^2 - \lambda a + 1$, $\lambda = 2(1+\gamma)/(1-\gamma)$, inside the unit circle. This root is equal to

$$(42) \quad a = \frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}.$$

If we assume that m_t is a random walk then

$$(43) \quad \begin{aligned} p_{t-1} &= a p_{t-2} + \frac{2a\gamma}{1-\gamma}\frac{2}{1-a} m_{t-1} \\ &= a p_{t-2} + (1-a) m_{t-1} \end{aligned}$$

which follows from the definition of a. If we take an average of p_{t-1} and p_{t-2}, then we have the aggregate price, namely

$$(44) \quad \bar{p}_t = a\bar{p}_{t-1} + \frac{1}{2}(1-a)(m_{t-1} + m_{t-2}).$$

Using the money demand equation as well, we have the main result:

$$(45) \quad \begin{aligned} y_t &= a y_{t-1} + m_t - a m_{t-1} - \bar{p}_t + a\bar{p}_{t-1} \\ &= a y_{t-1} + m_t - a m_{t-1} - \frac{1}{2}(1-a)(m_{t-1} + m_{t-2}) \\ &= a y_{t-1} + m_t - m_{t-1} + (1-a) m_{t-1} - \frac{1}{2}(1-a)(m_{t-1} + m_{t-2}) \\ &= a y_{t-1} + m_t - m_{t-1} + \frac{1}{2}(1-a)(m_{t-1} - m_{t-2}). \end{aligned}$$

Persistent movements in output y require a value of a near 1 or, equivalently, a value of γ near 0. The value of γ depends on the choice of preferences.

For the benchmark economy, we assumed

$$(46) \quad U(C, L, \bar{M}) = \frac{1}{1-\sigma}\left[\left(\omega\, C^{\frac{\eta-1}{\eta}} + (1-\omega)\, \bar{M}^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}} (1-L)^{\psi}\right]^{1-\sigma} \equiv \frac{1}{1-\sigma}\Phi(C, L, \bar{M})^{1-\sigma}$$

where

$$(47) \quad \Phi(C, L, \bar{M}) = \Psi(C, M)^{\frac{\eta}{\eta-1}} (1-L)^{\psi}$$

where $\bar{M}$ denotes real balances $(M/\bar{P})$. The first and second partial derivatives of this function are as follows

$$(48)\qquad U_c = \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}\omega C^{-\frac{1}{\eta}}(1-L)^{\psi}$$

$$(49)\qquad U_l = -\Phi^{-\sigma}\Psi^{\frac{\eta}{\eta-1}}\psi(1-L)^{\psi-1}$$

$$(50)\qquad U_m = \Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}(1-\omega)\bar{M}^{-\frac{1}{\eta}}(1-L)^{\psi}$$

$$\begin{aligned}(51)\qquad U_{cc} &= -\sigma\Phi^{-\sigma-1}(\Psi^{\frac{1}{\eta-1}}\omega C^{-\frac{1}{\eta}}(1-L)^{\psi})^2 \\ &\quad +\Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}-1}\omega^2/\eta C^{-\frac{2}{\eta}}(1-L)^{\psi} \\ &\quad -\Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}\omega/\eta C^{-\frac{1}{\eta}-1}(1-L)^{\psi}\end{aligned}$$

$$\begin{aligned}(52)\qquad U_{cl} &= \sigma\Phi^{-\sigma-1}\Psi^{\frac{\eta}{\eta-1}}\psi(1-L)^{\psi-1}\Psi^{\frac{1}{\eta-1}}\omega C^{-\frac{1}{\eta}}(1-L)^{\psi} \\ &\quad -\Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}\omega C^{-\frac{1}{\eta}}\psi(1-L)^{\psi-1}\end{aligned}$$

$$\begin{aligned}(53)\qquad U_{lm} &= \sigma\Phi^{-\sigma-1}\Psi^{\frac{\eta}{\eta-1}}\psi(1-L)^{\psi-1}\Psi^{\frac{1}{\eta-1}}(1-\omega)\bar{M}^{-\frac{1}{\eta}}(1-L)^{\psi} \\ &\quad -\Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}(1-\omega)\bar{M}^{-\frac{1}{\eta}}\psi(1-L)^{\psi-1}\end{aligned}$$

$$(54)\qquad U_{ll} = -\sigma\Phi^{-\sigma-1}(\psi(1-L)^{\psi-1}\Psi^{\frac{\eta}{\eta-1}})^2+\Phi^{-\sigma}\Psi^{\frac{\eta}{\eta-1}}\psi(\psi-1)(1-L)^{\psi-2}$$

$$\begin{aligned}(55)\qquad U_{cm} &= -\sigma\Phi^{-\sigma-1}(\Psi^{\frac{1}{\eta-1}}\omega C^{-\frac{1}{\eta}}(1-L)^{\psi})(\Psi^{\frac{1}{\eta-1}}(1-\omega)\bar{M}^{-\frac{1}{\eta}}(1-L)^{\psi}) \\ &\quad +\Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}-1}\omega(1-\omega)/\eta C^{-\frac{1}{\eta}}\bar{M}^{-\frac{1}{\eta}}(1-L)^{\psi}\end{aligned}$$

$$\begin{aligned}(56)\qquad U_{mm} &= -\sigma\Phi^{-\sigma-1}(\Psi^{\frac{1}{\eta-1}}(1-\omega)\bar{M}^{-\frac{1}{\eta}}(1-L)^{\psi})^2 \\ &\quad +\Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}-1}\omega^2/\eta\bar{M}^{-\frac{2}{\eta}}(1-L)^{\psi} \\ &\quad -\Phi^{-\sigma}\Psi^{\frac{1}{\eta-1}}(1-\omega)/\eta\bar{M}^{-\frac{1}{\eta}-1}(1-L)^{\psi}\end{aligned}$$

where the arguments of the utility function and its derivatives have been dropped for convenience.

The main result depends on γ which is the elasticity of $-U_l/U_c$ with respect to C. In general if we linearize the first order condition (11), then we get the following equation:

$$W w_t = \frac{(-U_{cl}U_c + U_{cc}U_l)C}{U_c^2}c_t + \frac{(-U_{ll}U_c + U_{cl}U_l)L}{U_c^2}l_t + \frac{(-U_{lm}U_c + U_{cm}U_l)\bar{M}}{U_c^2}(m_t - \bar{p}_t) \tag{57}$$

where the coefficients on w_t, c_t, l_t and $m_t - \bar{p}_t$ are steady state values. This can also be rewritten more succinctly as

$$w_t = \left(\frac{U_{cl}C}{U_l} - \frac{U_{cc}C}{U_c}\right)c_t + \left(\frac{U_{ll}L}{U_l} - \frac{U_{cl}L}{U_c}\right)l_t + \left(\frac{U_{lm}\bar{M}}{U_l} - \frac{U_{cm}\bar{M}}{U_c}\right)(m_t - \bar{p}_t). \tag{58}$$

For the utility function in (46), the linearized wage equation is therefore

$$w_t = \left(\frac{\omega C^{\frac{\eta-1}{\eta}}}{\Psi}\left(\frac{\eta-1}{\eta}\right) + \frac{1}{\eta}\right)c_t + \left(\frac{L}{1-L}\right)l_t + \left(\frac{(1-\omega)\bar{M}^{\frac{\eta-1}{\eta}}}{\Psi}\left(\frac{\eta-1}{\eta}\right)\right)(m_t - \bar{p}_t). \tag{59}$$

In the example we are working with, we assume that $y_t = c_t = l_t = m_t - \bar{p}_t$ and $C = L = \bar{M}$. Therefore we can sum up the three coefficients in (58) to get

$$\begin{aligned} \gamma &= 1/(1-C) \\ &= 1 + \theta\omega/\psi. \end{aligned} \tag{60}$$

Alternatively we could have set $L = C$ and $M/\bar{P} = C$ in the utility function and then taken the total derivative of -U_l/U_c with respect to C and multiplied the result by $C/(-U_l/U_c)$ to get the elasticity. To derive the value of γ we have used the fact that $C = \theta\omega/(\psi + \theta\omega)$ in the steady state (i.e., $W = \theta$ and $W = -U_l(C,C,C)/U_c(C,C,C)$). In this case $\gamma > 1$ (and therefore $a = (1-\sqrt{\gamma})/(1+\sqrt{\gamma}) < 0$). Therefore, the persistence of output is nil.

For the specific utility function in (46), we can choose $\omega = 1/(2-\beta)$ and $\eta \approx 0$ and, in this case, there is no need to impose the simple money demand relation in (26) arbitrarily. With $\omega = 1/(2-\beta)$ and $\eta \approx 0$, it is the case that the interest elasticity is about 0 and $C = M/\bar{P}$ in a steady state. To see this, note that $U_m = U_c(1-\beta)$ in the steady state. For the utility function in (46), this yields consumption equals real balances if $\omega = 1/(2-\beta)$.

Now consider the money demand equation in (12) which is derived as a first order condition for consumer optimization. That equation can be rewritten as

$$
\begin{aligned}
(61) \qquad \frac{U_m(s^t)}{U_c(s^t)} &= 1 - \sum_{s^{t+1}} \beta\pi(s^{t+1}|s^t)\frac{U_c(s^{t+1})}{U_c(s^t)}\frac{\bar{P}(s^t)}{\bar{P}(s^{t+1})} \\
&\equiv 1 - 1/R(s^t).
\end{aligned}
$$

Note that the sum in (61) is the inverse of the gross interest rate $R(s^t)$. Log-linearizing this equation yields the following

$$
(62) \qquad \left(\frac{U_{cm}C}{U_m} - \frac{U_{cc}C}{U_c}\right)c_t + \left(\frac{U_{lm}L}{U_m} - \frac{U_{cl}L}{U_c}\right)l_t + \left(\frac{U_{mm}\bar{M}}{U_m} - \frac{U_{cm}\bar{M}}{U_c}\right)(m_t - \bar{p}_t) = \log\frac{R(s^t)-1}{R(s^t)}.
$$

For the utility function in (46), this simplifies to

$$
(63) \qquad \frac{1}{\eta}c_t - \frac{1}{\eta}(m_t - \bar{p}_t) = \log\frac{R(s^t)-1}{R(s^t)}
$$

(without assuming that $C = L = \bar{M}$) and therefore

$$
(64) \qquad m_t - \bar{p}_t = c_t - \eta\log\frac{R(s^t)-1}{R(s^t)}.
$$

For η small, the interest rate term can be ignored. Therefore, with $\omega = 1/(2-\beta)$ and $\eta \approx 0$, the impulse response function for output is given by (45), which is our main result.

4. Zero Income Effects Economy

The only change here from the benchmark economy is our choice of preferences. Let

$$
(65) \qquad U(C, L, \bar{M}) = \frac{1}{1-\sigma}\left(\frac{C^{1-\kappa}}{1-\kappa} - \omega_1\frac{L^{1+\xi}}{1+\xi} + \omega_2\frac{\bar{M}^{1-\nu}}{1-\nu}\right)^{1-\sigma} \equiv \frac{1}{1-\sigma}\Phi(C, L, \bar{M})^{1-\sigma}.
$$

The first and second partial derivatives are therefore given by

$$
(66) \qquad U_c = \Phi^{-\sigma}C^{-\kappa}
$$

$$U_l = \Phi^{-\sigma}(-\omega_1)L^{\xi} \tag{67}$$

$$U_m = \Phi^{-\sigma}\omega_2\bar{M}^{-\nu} \tag{68}$$

$$U_{cc} = -\sigma\Phi^{-\sigma-1}C^{-\kappa}C^{-\kappa} - \Phi^{-\sigma}\kappa C^{-\kappa-1} \tag{69}$$

$$U_{cl} = -\sigma\Phi^{-\sigma-1}(-\omega_1)C^{-\kappa}L^{\xi} \tag{70}$$

$$U_{cm} = -\sigma\Phi^{-\sigma-1}\omega_2 C^{-\kappa}M^{-\nu} \tag{71}$$

$$U_{ll} = -\sigma\Phi^{-\sigma-1}\omega_1^2 L^{\xi}L^{\xi} - \Phi^{-\sigma}\omega_1\xi L^{\xi-1} \tag{72}$$

$$U_{lm} = -\sigma\Phi^{-\sigma-1}(-\omega_1)\omega_2 L^{\xi}\bar{M}^{-\nu} \tag{73}$$

$$U_{mm} = -\sigma\Phi^{-\sigma-1}(-\omega_1)\omega_2 L^{\xi}\bar{M}^{-\nu} \tag{74}$$

$$U_{mm} = -\sigma\Phi^{-\sigma-1}\omega_2^2\bar{M}^{-\nu}\bar{M}^{-\nu} - \Phi^{-\sigma}\omega_2\nu\bar{M}^{-\nu-1} \tag{75}$$

Using these derivatives in (58) the linearized wage equation is given by

$$w_t = \kappa c_t + \xi l_t, \tag{76}$$

which does not depend on $m_t - \bar{p}_t$ because $U_{lm}/U_l = U_{cm}/U_c$ in the steady state. If we assume only labor in production, then $c = l = y$. We can now compute the impulse response for output exactly as before except. In this case, however, $\gamma = \kappa + \xi$ and the pricing equation is

$$p_{i,t-1} = \frac{1}{2}E_{t-1}\left[\bar{p}_t + \bar{p}_{t+1} + (\kappa + \xi)(y_t + y_{t+1})\right]. \tag{77}$$

Using this definition for γ in equation (34), we can derive the impulse responses for output exactly as in the benchmark economy. To get persistent output responses, we need to choose small (but positive) values for both κ and ξ. In the technical appendix, we show that even with κ and ξ small, when capital is turned on, there is no longer any persistence.

We should point out one difficulty with this set of preferences: it is not consistent with the simpler form of the money demand that we assumed in our derivations in the paper. Recall that we assumed the simple money demand relation (26) when deriving the impulse responses for output. For the preferences in (65), there is no choice for κ, ξ, and ν that gives both a small value of γ and a small value for the interest elasticity (which we assumed in the derivations). The linearized money demand equation is

$$\kappa c_t - \nu(m_t - \bar{p}_t) = log\,\frac{R(s^t)-1}{R(s^t)}. \tag{78}$$

or, when rewritten,

$$m_t - \bar{p}_t = \frac{\kappa}{\nu}c_t - \frac{1}{\nu}log\,\frac{R(s^t)-1}{R(s^t)}. \tag{79}$$

If $\kappa = \nu$ and κ is a small number, then the interest elasticity is huge.

5. Economy with Convex Demand

The economy with convex demand is the same as the benchmark economy except that the assumption $g(y) = y^\theta$ is dropped. As a result the pricing equation for the monopolist is now given by

$$\begin{aligned}\sum_\tau \sum_{s^\tau} Q(s^\tau|s^{t-1})\Big\{Y(i,s^\tau) + Y(s^\tau)\left[1 - \bar{P}(s^\tau)W(s^\tau)/P(i,s^{t-1})\right] \\ D'\left(\frac{P(i,s^{t-1}}{\bar{P}(s^\tau)}\int g'\left(\frac{Y(j,s^\tau)}{Y(s^\tau)}\right)\frac{Y(j,s^\tau)}{Y(s^\tau)}\,dj\right) g'\left(\frac{Y(i,s^\tau)}{Y(s^\tau)}\right)\Big\} = 0\end{aligned} \tag{80}$$

which follows from (17) with the unit cost equal to the wage rate, $V(i,s^t) = W(s^t)$. If we assume $\beta \approx 1$, then the linearized pricing equation is given by

$$p_{i,t-1} = \frac{1}{2}E_{t-1}\{\bar{p}_t + \bar{p}_{t+1} + \varphi(w_t + w_{t+1})\}. \tag{81}$$

where

$$\varphi = \frac{1-1/\epsilon}{2-\chi/\epsilon} \tag{82}$$

and $\epsilon = -D'(g'(1))g'(1)$, $\chi = -D''(g'(1))g'(1)/D'(g'(1))$. When linearizing the pricing equation, we use the fact that $g''(1) = 1/D'(g'(1))$ by the definition of D. To simplify the coefficients, we use the fact that $W = (\epsilon - 1)/\epsilon$ in a steady state. Finally, we use the relation

$$y_{i,\tau} - y_\tau = \frac{g'(1)}{g''(1)}(p_i - \bar{p}_\tau) \tag{83}$$

to eliminate y_i's from the linearized equation.

Note that the benchmark economy is a nested case of this one. If we set $g(y) = y^\theta$, then the following are true

$$\begin{aligned}
D(x) &= \left(\frac{x}{\theta}\right)^{\frac{1}{\theta-1}} && (84)\\
D'(x) &= \frac{1}{\theta(\theta-1)}\left(\frac{x}{\theta}\right)^{\frac{2-\theta}{\theta-1}} && (85)\\
D''(x) &= \frac{2-\theta}{\theta^2(1-\theta)^2}\left(\frac{x}{\theta}\right)^{\frac{3-2\theta}{\theta-1}} && (86)\\
g'(y) &= \theta y^{\theta-1} && (87)\\
g''(y) &= \theta(\theta-1)y^{\theta-2} && (88)
\end{aligned}$$

and therefore

$$\begin{aligned}
D'(g'(1)) &= D'(\theta) = \frac{1}{\theta(\theta-1)} && (89)\\
D''(g'(1)) &= D''(\theta) = \frac{2-\theta}{\theta^2(1-\theta)^2} && (90)\\
\epsilon &= -D'(\theta)\theta = \frac{1}{1-\theta} && (91)\\
\chi/\epsilon &= (2-\theta) && (92)\\
\varphi &= \frac{1-1/\epsilon}{2-\chi/\epsilon} = 1. && (93)
\end{aligned}$$

Again, the resource constraint is simple since there is no investment. It is given by

$$
\begin{aligned}
(94)\quad C(s^t) &= Y(s^t) \\
&= L(s^t)\left[\frac{1}{2}\sum_i D\left(\frac{P(s^{t-i})}{\bar{P}(s^\tau)}\frac{1}{2}\sum_j g'\left(\frac{Y(j,s^\tau)}{Y(s^\tau)}\right)\frac{Y(j,s^\tau)}{Y(s^\tau)}\right)\right]^{-1}
\end{aligned}
$$

The definition of aggregate output in (94) can be derived simply by noting that $L(s^t) = \int L(i,s^t)\,di$ $= \int Y(i,s^t)\,di$ and using the demand functions for the $Y(i,s^t)$. Linearizing the resource constraint gives the following

$$(95)\quad c_t = y_t = l_t.$$

All of the calculations for the benchmark model carry through except that now we have $\gamma = \varphi(1+\theta\omega/\psi)$ and, therefore,

$$(96)\quad p_{i,t-1} = \frac{1}{2}E_{t-1}\left[\bar{p}_t + \bar{p}_{t+1} + \varphi(1+\theta\omega/\psi)(y_t + y_{t+1})\right].$$

Using this definition for γ in equation (34), we can derive the impulse responses for output exactly as in the benchmark economy. To get persistent output, we need very convex demand (i.e., φ small).

Discussions of the elasticities of demand in the paper use the following linearized elasticity equations. Let $\epsilon(P_i/P)$ be the elasticity of demand so that

$$
\begin{aligned}
(97)\quad \epsilon(P_i/P) &= -\frac{d(Y_i/Y)}{d(P_i/\bar{P})}\frac{P_i/\bar{P}}{Y_i/Y} \\
&= -\frac{D'\left(\frac{P_i}{P}\int g'\left(\frac{Y_j}{Y}\right)\left(\frac{Y_j}{Y}\right)dj\right)\left(\int g'\left(\frac{Y_j}{Y}\right)\left(\frac{Y_j}{Y}\right)dj\right)\left(\frac{P_i}{P}\right)}{D\left(\frac{P_i}{P}\int g'\left(\frac{Y_j}{Y}\right)\left(\frac{Y_j}{Y}\right)dj\right)}
\end{aligned}
$$

If we write this in terms of relative demands then we can use Kimball's (1995) parameters to parameterize φ. Using the second equality in (97), we have

$$(98)\quad \epsilon(Y_i/Y) = -\frac{D'\left(g'\left(\frac{Y_i}{Y}\right)\right)g'\left(\frac{Y_i}{Y}\right)}{Y_i/Y}$$

Expanding (98) around $Y_i/Y = 1$, we get the following approximation

$$
\begin{aligned}
(99) \quad \epsilon(z) &\approx -D'(g'(1))g'(1) - \left[D''(g'(1))g''(1)g'(1) + D'(g'(1))g''(1) - D'(g'(1))g'(1)\right](z-1) \\
&= \epsilon(1) - \left[-\chi + 1 + \epsilon(1)\right](z-1).
\end{aligned}
$$

Kimball (1995) uses parameter values such that $\epsilon(1) = 11$ and $\epsilon(1.01) = 8$. This implies that if the demand for good i increases 1 percent above average, then the elasticity falls from 11 to 8. This occurs if $\chi = -288$. In the examples in the paper we have $\epsilon = 10$ and $\chi = -289$. This implies that the elasticity goes from 10 to 7 if the demand is 1 percent above average.

If we expand the demand function around the relative price then we can show what the choices of elasticities imply for the fall in demand when the relative price rises. The demand is given approximately by

$$
\begin{aligned}
(100) \quad D\left(\frac{P_i}{\bar{P}} \int g'\left(\frac{Y_j}{Y}\right)\left(\frac{Y_j}{Y}\right) dj\right) &\approx D(g'(1)) + D'(g'(1))g'(1)\left(\frac{P_i}{\bar{P}} - 1\right) \\
&\quad + \frac{1}{2} D''(g'(1))g'(1)^2 \left(\frac{P_i}{\bar{P}} - 1\right)^2 \\
&= 1 - 11\left(\frac{P_i}{\bar{P}} - 1\right) - \frac{3168}{2}\left(\frac{P_i}{\bar{P}} - 1\right)^2
\end{aligned}
$$

if $\epsilon = 11$ and $\chi = -288$, or

$$
(101) \quad D\left(\frac{P_i}{\bar{P}} \int g'\left(\frac{Y_j}{Y}\right)\left(\frac{Y_j}{Y}\right) dj\right) \approx 1 - 10\left(\frac{P_i}{\bar{P}} - 1\right) - \frac{2890}{2}\left(\frac{P_i}{\bar{P}} - 1\right)^2
$$

if $\epsilon = 10$ and $\chi = -289$. In the latter case, if $P_i/P = 1.02$ then $D \approx 0.22$. If $P_i/P = 1.023$, then $D \approx 0$.

6. Economy with Specific Factors

The specific factors economy allows for decreasing or increasing returns to scale in production. In this section, we derive reduced form equations for the specific factors economy with $N = 2$ and $F(K, L) = L^{1-\alpha}$. As before, we assume the simple form of the money demand equation so that, when linearized, we use

$$(102)\quad m_t - \bar{p}_t = c_t.$$

Our derivations use the money demand equation, the pricing equation for the monopolists, the resource constraint, and the first order condition for consumers relating the wage rate to $-U_l/U_c$. The pricing equation for the group of monopolists (named i) that is currently adjusting prices:

$$(103)\quad P(i, s^{t-1}) = \frac{\sum_{s^t} Q(s^t|s^{t-1})\{Y(s^t)\bar{P}(s^t)^{\frac{2-\theta}{1-\theta}}V(i,s^t) + Q(s^{t+1}|s^t)Y(s^{t+1})\bar{P}(s^{t+1})^{\frac{2-\theta}{1-\theta}}V(i,s^{t+1})\}}{\theta\sum_{s^t} Q(s^t|s^{t-1})Y(s^t)\{\bar{P}(s^t)^{\frac{1}{1-\theta}} + Q(s^{t+1}|s^t)Y(s^{t+1})\bar{P}(s^{t+1})^{\frac{1}{1-\theta}}\}}$$

where the unit cost $V(i, s^t)$ is

$$(104)\quad V(i, s^t) = \frac{1}{1-\alpha}W(s^t)L(i, s^t)^{\alpha}.$$

The linearized pricing equation is equal to

$$(105)\quad p_{i,t-1} = \frac{1}{2}E_{t-1}\left\{\bar{p}_t + \bar{p}_{t+1} + v_{i,t} + v_{i,t+1}\right\}$$

and the linearized unit costs are

$$\begin{aligned}(106)\quad v_{i,t} &= w_t + \alpha l_{i,t} \\ &= w_t + \frac{\alpha}{1-\alpha}y_{i,t} \\ &= w_t + \frac{\alpha}{(1-\alpha)(1-\theta)}(\bar{p}_t - p_{i,t-1}) + \frac{\alpha}{1-\alpha}y_t\end{aligned}$$

and

$$(107)\quad v_{i,t+1} = w_{t+1} + \frac{\alpha}{(1-\alpha)(1-\theta)}(\bar{p}_{t+1} - p_{i,t-1}) + \frac{\alpha}{1-\alpha}y_{t+1}.$$

Notice that $p_{i,t-1}$ appears in both equations since the price remains fixed for 2 periods. If we substitute the cost functions into (105) then we have

$$(108)\quad p_{i,t-1} = \frac{1}{2} E_{t-1}\left\{(1+A\varepsilon)\bar{p}_t + \bar{p}_{t+1} + w_t + w_{t+1} + A(y_t + y_{t+1})\right\} - A\varepsilon p_{i,t-1}$$

where $A = \alpha/(1-\alpha)$ and $\varepsilon = 1/(1-\theta)$. Solving for $p_{i,t-1}$, we have

$$(109)\quad p_{i,t-1} = \frac{1}{2} E_{t-1}\left\{\bar{p}_t + \bar{p}_{t+1} + \frac{1}{1+A\varepsilon}(w_t + w_{t+1}) + \frac{A}{1+A\varepsilon}(y_t + y_{t+1})\right\}.$$

The resource constraint is simple since there is no investment. It is given by

$$\begin{aligned}(110)\quad C(s^t) &= Y(s^t) \\ &= L(s^t)^{1-\alpha}\left[\frac{1}{2}\sum_{i=1}^{2} P(s^{t-i})^{\frac{\theta}{\theta-1}}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\sum_{i=1}^{2} P(s^{t-i})^{\frac{1}{(\theta-1)(1-\alpha)}}\right]^{\alpha-1}.\end{aligned}$$

The definition of aggregate output in (110) can be derived simply by noting that $L(s^t) = \int Y(i,s^t)^{\frac{1}{1-\alpha}}\,di$ and using the demand functions for the $Y(i,s^t)$. Linearizing the resource constraint gives the following

$$(111)\quad c_t = y_t = (1-\alpha)l_t.$$

We now derive the elasticity of the wage with respect to consumption so that we have a relation between w_t and y_t. Assuming our benchmark utility, the linearized wage equation is given by (59). If we assume that $\omega = 1/(2-\beta)$ and that the interest elasticity of money demand is approximately 0, then $m_t = \bar{p}_t = c_t$ and $\bar{M} = C$. In this case,

$$(112)\quad w_t = c_t + \frac{L}{1-L} l_t.$$

Using the fact that $C = Y$, $L = Y^{\frac{1}{1-\alpha}}$, $c_t = y_t$ and $l_t = y_t/(1-\alpha)$, the wage equation further simplifies to

$$(113)\quad w_t = \left(1 + \frac{\frac{1}{1-\alpha} Y^{\frac{1}{1-\alpha}}}{1 - Y^{\frac{1}{1-\alpha}}}\right) y_t.$$

In a steady state, the unit cost equals θ. Using the fact that $W = -U_l/U_c$, we have

$$\theta = -\frac{1}{1-\alpha} Y^{\frac{\alpha}{1-\alpha}} \frac{U_l(Y, Y^{\frac{1}{1-\alpha}}, Y)}{U_c(Y, Y^{\frac{1}{1-\alpha}}, Y)} \tag{114}$$

in a steady state and therefore

$$1 + \frac{\frac{1}{1-\alpha} Y^{\frac{1}{1-\alpha}}}{1 - Y^{\frac{1}{1-\alpha}}} = 1 + \frac{\theta\omega}{\psi}. \tag{115}$$

Thus, the linearized wage equation in this economy is the same as in the benchmark economy even though the production function is different.

If we substitute $w_t = (1 + \theta\omega/\psi) y_t$ into (109) we have $\gamma = (1 + \theta\omega/\psi + A)/(1 + A\varepsilon)$ and

$$p_{i,t-1} = \frac{1}{2} E_{t-1} \left[\bar{p}_t + \bar{p}_{t+1} + \left(\frac{1 + \theta\omega/\psi + A}{1 + A\varepsilon} \right) (y_t + y_{t+1}) \right]. \tag{116}$$

Using this definition for γ in equation (34), we can derive the impulse responses for output exactly as in the benchmark economy.

Suppose instead we had assumed that $U(C, L, \bar{M}) = \log(C) - \frac{1}{1+s} L^{1+s} + G(M)$ as in Kiley (1997). In this case the elasticity of $-U_l/U_c$ with respect to C is $1 + s/(1-\alpha)$ and therefore

$$p_{i,t-1} = \frac{1}{2} E_{t-1} \left[\bar{p}_t + \bar{p}_{t+1} + \left(\frac{1 + s/(1-\alpha) + A}{1 + A\varepsilon} \right) (y_t + y_{t+1}) \right]. \tag{117}$$

Kiley reports a different value for γ (which he calls b on page 5.) He reports $\gamma = (1+s)/(1-\alpha)$. In other words, the denominator $1 + A\varepsilon$ in (117) doesn't appear. The reason for this is that he assumes $y_{i,t} \approx y_t$ when deriving the pricing equation (so that the term $\alpha(p_t - p_{i,t-1})/[(1-\alpha)(1-\theta)]$ does not appear in (107) or (107)). Obviously this makes a big difference if $A\varepsilon$ is not close to 0.

Kiley (1997) is interested in examples with increasing returns ($\alpha < 0$). For most parameterizations with $\alpha < 0$, γ is negative and the dynamic system displays cycles (2 imaginary roots).

7. The General Model

In this section, we describe in some detail the numerical algorithm used to solve the full-blown model of Section 2. The solution takes the form

$$Z_t = AZ_{t-1} + BS_t \tag{118}$$

where $Z_t = [z_t', \ldots, z_{t-N+2}']'$ is a $(N+2)(N-1) \times 1$ vector and

$$z_t = [p_{t-1} - m_{t-1}, k_{1,t}, \ldots, k_{N,t}, y_t]' \tag{119}$$

$$S_t = [\mu_t, \mu_{t-1}, \ldots, \mu_{t-N+1}]'. \tag{120}$$

The matrices A and B are chosen to satisfy the first order conditions which can be written generally as follows

$$\mathcal{E}_t \left[a_0 Z_{t+N-1} + a_1 Z_{t+N-2} + \ldots a_N Z_{t-1} + b_0 S_{t+N-1} + b_1 S_{t+N-2} + \ldots b_{N-1} S_t\right] = 0 \tag{121}$$

where $\mathcal{E}_t \equiv E[\cdot|s^{t-1}]$ for the first residual (the pricing equation) and $\mathcal{E}_t = E[\cdot|s^t]$ for all other residuals.

Writing the residuals as in (121) makes the notation simpler but actually implies lots of duplication. For example, lagged prices appear multiple times. Our subroutine uses a smaller set of variables when constructing residuals of the first order conditions. In particular, there are two inputs: the vector of parameters appearing in the first order conditions and the following vector of variables:

$$\begin{aligned} \mathcal{Z} \equiv\ & [z_{t+N-1}, z_{t+N-2}, \ldots, z_t, p_{t-2} - m_{t-2}, \ldots, p_{t-N} - m_{t-N}, k_{1,t-1}, \ldots, k_{N,t-1}, \\ & \mu_{t+N-1}, \ldots, \mu_{t-N+1}]'. \end{aligned} \tag{122}$$

We show later that all other variables can be constructed once we know those in $\mathcal{Z}$.

Above we assumed that $\beta \approx 1$ when deriving the linearized pricing equation. In writing the code, we will not make this assumption. If we linearize (17) we get

$$
\begin{aligned}
(123)\quad p_{t-1} \;=\; & \frac{1}{(N-1)\sum_{i=0}^{N-1}\beta^i} E_{t-1}\Big[p_{t-N} + (1+\beta)p_{t-N+1} + \ldots (1+\beta+\ldots+\beta^{N-2})p_{t-2} \\
& +(\beta+\ldots+\beta^{N-1})p_t + \ldots + \beta^{N-1}p_{t+N-2} \\
& +\varphi N\left\{v_{i,t} + \beta v_{i,t+1} + \ldots + \beta^{N-1}v_{i,t+N-1}\right\}\Big]
\end{aligned}
$$

where the constant terms have been ignored. It turns out that it is most convenient to write the residuals by first normalizing the prices: we divide them by the money supply. If we do this, then the pricing equation in (123) is equivalent to

$$
\begin{aligned}
(124)\quad p_{t-1} \;-\; & m_{t-1} \\
= & \frac{1}{(N-1)\sum_{i=0}^{N-1}\beta^i} E_{t-1}\Big[(p_{t-N} - m_{t-N}) + \ldots (1+\beta+\ldots+\beta^{N-2})(p_{t-2} - m_{t-2}) \\
& +(\beta+\ldots+\beta^{N-1})(p_t - m_t) + \ldots + \beta^{N-1}(p_{t+N-2} - m_{t+N-2}) \\
& +\varphi N\left\{v_{i,t} + \beta v_{i,t+1} + \ldots + \beta^{N-1}v_{i,t+N-1}\right\} \\
& +[(\beta+\ldots+\beta^{N-1}) + (\beta^2+\ldots+\beta^{N-1}) + \ldots + \beta^{N-1}]\mu_t \\
& +[(\beta^2+\ldots+\beta^{N-1}) + \ldots + \beta^{N-1}]\mu_{t+1} \\
& +\ldots+\beta^{N-1}\mu_{t+N-2} \\
& -\mu_{t-N+1} - [1+(1+\beta)]\mu_{t-N+2} - [1+(1+\beta)+(1+\beta+\beta^2)]\mu_{t-N+3} \\
& -\ldots-[1+(1+\beta)+\ldots+(1+\beta+\ldots+\beta^{N-2})]\mu_{t-1}\Big].
\end{aligned}
$$

We had to write the pricing equation as above because we do not have explicit functional forms for $g(\cdot)$ (and hence the demand function $D(\cdot)$ in (17)). The other residuals can either be linearized by hand or numerically. For ease of reading the code, we chose to linearize them numerically.

In addition to the pricing equation we have the the money demand equation and N Euler equations for capital:

$$\text{(125)} \quad \frac{U_c(s^t)}{\bar{P}(s^t)} = \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t) \frac{U_c(s^{t+1})}{\bar{P}(s^{t+1})} + \frac{U_m(s^t)}{\bar{P}(s^t)}$$

$$\begin{aligned} \text{(126)} \quad U_c(s^t) &= [1-\phi'(i,s^t)]\beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t)\, U_c(s^{t+1}) \Bigg[V(i,s^{t+1}) F_k(i,s^{t+1}) \\ &\quad + \frac{1}{1-\phi'(i,s^{t+1})} \left\{ 1-\delta-\phi(i,s^{t+1}) + \phi'(i,s^{t+1}) \frac{X(i,s^{t+1})}{K(i,s^t)} \right\} \Bigg] \end{aligned}$$

In writing the residuals, we will use the following convention for naming the cohorts (which is different than that used above). we will assume that monopolists named i are those that set their prices i periods ago. For example, in t, group 1 charges p_{t-1}, group 2 charges p_{t-2}, and so on. Note that the particular assignment is not important. To evaluate the pricing equation we need unit costs of group 1 for t, $t+1$, ..., $t+N-1$. For these costs, we will use the notation: $v_{1,t}$, $v_{2,t+1}$, ..., $v_{N,t+N-1}$ where

$$\text{(127)} \quad v_{1,t} = w_t + (1-\alpha_2) l_{1,t} - \alpha_1 k_{N,t-1} - \log(\alpha_2)$$

$$\text{(128)} \quad v_{2,t+1} = w_{t+1} + (1-\alpha_2) l_{2,t+1} - \alpha_1 k_{1,t} - \log(\alpha_2)$$

$$\vdots$$

$$\text{(129)} \quad v_{N,t+N-1} = w_{t+N-1} + (1-\alpha_2) l_{N,t+N-1} - \alpha_1 k_{N-1,t+N-2} - \log(\alpha_2)$$

if $F(K,L) = K^{\alpha_1} L^{\alpha_2}$. The capital stocks are included in $\mathcal{Z}$. The labor inputs are given by

$$\text{(130)} \quad l_{1,t} = \frac{\epsilon}{\alpha_2}(\bar{p}_t - p_{t-1}) - \frac{\alpha_1}{\alpha_2} k_{N,t-1} + \frac{1}{\alpha_2} y_t$$

$$\text{(131)} \quad l_{2,t+1} = \frac{\epsilon}{\alpha_2}(\bar{p}_{t+1} - p_{t-1}) - \frac{\alpha_1}{\alpha_2} k_{1,t} + \frac{1}{\alpha_2} y_{t+1}$$

$$\vdots$$

$$(132)\quad l_{N,t+N-1} = \frac{\epsilon}{\alpha_2}(\bar{p}_{t+N-1} - p_{t-1}) - \frac{\alpha_1}{\alpha_2}k_{N-1,t+N-2} + \frac{1}{\alpha_2}y_{t+N-1}$$

which follows from $y_{i,t} - y_t = -\epsilon(p_{i,t-1} - \bar{p}_t)$ and $\exp(y_{i,t}) = F(\exp(k_{i-1,t-1}), \exp(l_{i,t}))$. Aggregate output is given in $\mathcal{Z}$. For the relative prices, we write

$$\begin{aligned}
(133)\quad \bar{p}_t - p_{t-1} &= \frac{1}{N}[(p_{t-1} - m_{t-1}) + \ldots + (p_{t-N} - m_{t-N})] - (p_{t-1} - m_{t-1}) \\
&\quad - \frac{N-1}{N}\mu_{t-1} - \frac{N-2}{N}\mu_{t-2} - \ldots - \frac{1}{N}\mu_{t-N+1} \\
(134)\quad \bar{p}_{t+1} - p_{t-1} &= \frac{1}{N}[(p_t - m_t) + \ldots + (p_{t-N+1} - m_{t-N+1})] - (p_{t-1} - m_{t-1}) \\
&\quad + \frac{1}{N}\mu_t - \frac{N-2}{N}\mu_{t-1} - \ldots - \frac{1}{N}\mu_{t-N+2} \\
(135)\quad \bar{p}_{t+2} - p_{t-1} &= \frac{1}{N}[(p_{t+1} - m_{t+1}) + \ldots + (p_{t-N+2} - m_{t-N+2})] - (p_{t-1} - m_{t-1}) \\
&\quad + \frac{1}{N}\mu_{t+1} + \frac{2}{N}\mu_t - \frac{N-3}{N}\mu_{t-1} - \ldots - \frac{1}{N}\mu_{t-N+3} \\
&\vdots \\
(136)\quad \bar{p}_{t+N-1} - p_{t-1} &= \frac{1}{N}[(p_{t+N-1} - m_{t+N-1}) + \ldots + (p_{t-1} - m_{t-1})] - (p_{t-1} - m_{t-1}) \\
&\quad + \frac{1}{N}\mu_{t+N-2} + \frac{2}{N}\mu_{t+N-3} + \ldots + \frac{N-1}{N}\mu_t
\end{aligned}$$

which depend on terms in $\mathcal{Z}$.

The wage rate appears in the equation for unit costs. To construct wage rates we need $C(s^t)$, $L(s^t)$, and $M(s^t)/\bar{P}(s^t)$. For aggregate consumption, we need aggregate output and the individual investments:

$$\begin{aligned}
(137)\quad C(s^t) &= Y(s^t) - \frac{1}{N}\sum_i X(i, s^t) \\
(138)\quad X(i, s^t) &= \frac{1}{b}\left(1 + b\delta - \sqrt{1 + 2b\delta - 2b(K(i, s^t)/K(i-1, s^{t-1}) - 1 + \delta)}\right) K(i-1, s^{t-1})
\end{aligned}$$

where the capital stocks and output are in $\mathcal{Z}$. When linearized, these equations look like

$$
\begin{aligned}
(139) \quad c_t &= (Yy_t - X\sum x_{i,t}/N)/C \\
(140) \quad k_{i,t} &= (1-\delta)k_{i-1,t-1} + \delta x_{i,t}
\end{aligned}
$$

where constant terms have been ignored. Notice that the monopolists with capital stocks $K(i-1, s^{t-1})$ in $t-1$ are the same monopolists with capital $K(i, s^t)$ using our new naming convention. Monopolists named N this period are named 1 next period since they are the next to change prices. Aggregate labor is given by

$$(141) \quad L(s^t) = \frac{1}{N}\sum_i L(i, s^t)$$

or, in logs, by

$$(142) \quad l_t = \frac{1}{N}\sum_i l_{i,t}.$$

Finally, logged real balances are given by

$$
\begin{aligned}
(143) \quad m_t - \bar{p}_t &= m_t - \frac{1}{N}(p_{t-1} + \ldots + p_{t-N}) \\
&= \frac{1}{N}\{(m_{t-1} + \mu_t) + (m_{t-2} + \mu_t + \mu_{t-1}) + \ldots + (m_{t-N} + \mu_t + \mu_{t-1} + \ldots \mu_{t-N+1})\} \\
&\quad - \frac{1}{N}(p_{t-1} + \ldots + p_{t-N}) \\
&= -\frac{1}{N}\sum_{i=1}^{N}(p_{t-i} - m_{t-i}) + \frac{1}{N}(N\mu_t + (N-1)\mu_{t-1} + \ldots \mu_{t-N+1}).
\end{aligned}
$$

For the pricing equation, we need real balances in $t, t+1, \ldots, t+N-1$ so we need to know the sequences $\{p_{t-N} - m_{t-N}, \ldots\ p_{t+N-2} - m_{t+N-2}\}$. and $\{\mu_{t-N+1}, \ldots,\ \mu_{t+N-1}\}$. These are in $\mathcal{Z}$. The formulas for the relative prices $\bar{p}_\tau - p_{\tau-i}$ can be found in (133)-(136). All of the variables appearing in the money demand equation (126) have at this point been constructed.

A. Solution to A

There are two steps to solving the system of equations in (118). We start with the first step: computing A. We use standard methods to solve the deterministic solution $Z_t = AZ_{t-1}$.

Define $\mathcal{X}_t$ to be the following vector of state variables:

$$\mathcal{X}_t = [p_{t-2} - m_{t-2}, \ldots, p_{t-N} - m_{t-N}, k_{1,t-1}, \ldots, k_{N,t-1}]'. \tag{144}$$

Using the definition of $\mathcal{X}$ in (144), the residuals (dropping terms with μ) can be written as

$$A_1 \begin{bmatrix} \mathcal{X}_{t+1} \\ \\ Z_{t+N-1} \end{bmatrix} + A_2 \begin{bmatrix} \mathcal{X}_t \\ \\ Z_{t+N-2} \end{bmatrix} + \begin{pmatrix} \text{shock} \\ \text{terms} \end{pmatrix} = 0 \tag{145}$$

where elements of A_1 and A_2 are either coefficients of linearized residuals or 1's and 0's used to associate variables with their lagged values. To compute A, we find generalized eigenvalues Λ (and associated eigenvectors D) such that $A_2 D = -A_1 D \Lambda$. For a unique stationary equilibrium, we need $2N - 1$ roots inside the unit circle. Note that $\mathcal{X}$ has length $2N - 1$. If we sort the eigenvalues and eigenvectors so that the roots inside one are ordered first, then we have

$$\begin{aligned} \mathcal{X}_{t+1} &= D_{11} \Lambda_1 D_{11}^{-1} \mathcal{X}_t && (146) \\ Z_{t+N-2} &= D_{21} D_{11}^{-1} \mathcal{X}_t && (147) \end{aligned}$$

where D_{11} is the upper left partition of D and is $2N - 1 \times 2N - 1$, D_{21} is the lower left partition of D and has dimension $(N+2)(N-2) \times 2N - 1$, and Λ_1 is the upper left partition of the matrix of eigenvalues. Recall that $Z_{t+N-2} = [z'_{t+N-2}, \ldots, z'_t]'$. Recall also that all of the elements in $\mathcal{X}_t$ are also in Z_{t-1}. Therefore, we can use the solutions above to fill in the elements of A.

B. Solution to B

Given A, solving for B involves solving a linear system of equations. The law of motion for the shocks is given by

(148) $S_t = \varrho S_{t-1} + \epsilon_{t+1}$

where the (1,1) of ϱ is ρ and the remaining elements are 1's or 0's and the first element of ϵ_t is nonzero and all other elements are zero. We plug this law of motion and the law of motion for Z_t into (121) using recursion to write the equation in terms of Z_{t-1} and S_t. The coefficients on these variables are both set equal to zero. Setting the coefficients on S_t equal to zero gives us the equations we need for solving the elements of B. The problem then is to find B such that $\mathcal{E}_t\{FBG + H\}S_t = 0$ where the elements of matrices F, G, and H are functions of the parameters and the computed elements of A. If $\mathcal{E}_t \equiv E[\cdot|\mu_t, \mu_{t-1}, \ldots]$, then B is given by

(149) $\text{vec}(B) = -(G \otimes F')\text{vec}(H)$

where vec(B) is a vector with the columns of B stacked one after another.

Note, however, that the first residual equation has $\mathcal{E}_t \equiv E[\cdot|\mu_{t-1}, \mu_{t-2}, \ldots]$. Therefore, we have to treat it slightly differently from the others. Even so, the solution procedure is an application of undetermined coefficients, and all elements of B can be found by solving a system of linear equations.