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TECHNICAL APPENDIX I: Comment on Gali and Rabanal[†]

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[†] The views expressed herein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

Table of Contents

In this Appendix, I provide some background notes for the model used in my discussion at the NBER macro annual of Gali and Rabanal. My goal is to evaluate whether the Gali-Rabanal SVAR can uncover theoretical impulse responses of a standard RBC model.

Here, I start with a description of the benchmark model and show how to compute a log-linear approximation to its equilibrium. The benchmark model has a geometric trend in growth. I also consider a version of the model with a random walk for technology. Then I show how to estimate the stochastic processes for the shocks using U.S. data on output, investment, hours, and government spending. With the estimates, I can construct time series that are used as "data" for Gali and Rabanal's empirical exercise.

1. The Benchmark Model

1.1. Nomenclature

Below I will use the following notation for our model variables:

N: population ($N_t = (1+g_n)^t$)

c: per-capita consumption

x: per-capita investment

k: per-capita net capital stock

l: per-capita labor input

tr: per-capita government transfers

C: total consumption ($C_t = N_t c_t$)

X: total investment

K: total stock of capital

L: total labor input in production

Z: labor-augmenting technical change ($Z_t = z_t(1+g_z)^t$)

r: rental rate on capital

w: wage rate

τ_v: tax rate on v

$\hat{v}$: detrended, per-capita variable V ($\hat{v}_t = V_t/[N_t(1+g_z)^t]$)

1.2. Maximization problems

Consider an economy with households, firms, and the government. The representative household chooses consumption, investment, and labor to solve the following maximization problem:

$$\max_{\{c_t, x_t, l_t\}} E \sum_{t=0}^{\infty} \beta^t U(c_t, 1-l_t) N_t$$

$$\text{subject to} \quad (1+\tau_{ct})c_t + (1+\tau_{xt})x_t = (1-\tau_{kt})r_t k_t + (1-\tau_{lt})w_t l_t + \tau_{kt}\delta k_t + tr_t$$

$$N_{t+1}k_{t+1} = [(1-\delta)k_t + x_t]N_t$$

$$c_t, x_t \geq 0 \quad \text{in all states}$$

taking processes for the rental rate, wage rate, the tax rates, and transfers as given. The representative firm solves a simple static problem at t:

$$\max_{\{K_t, L_t\}} \; F(K_t, Z_t L_t) - r_t K_t - w_t L_t.$$

The government sets rates of taxes and transfers in such a way that their budget constraint at t, namely,

$$G_t + N_t tr_t = \tau_{kt}(r_t - \delta)N_t k_t + \tau_{lt} w_t l_t N_t + \tau_{ct} N_t c_t + \tau_{xt} N_t x_t$$

is satisfied. In equilibrium, the following conditions must hold:

$$N_t(c_t + x_t) + G_t = F(K_t, Z_t L_t) \tag{1.1}$$

$$N_t k_t = K_t$$

$$N_t l_t = L_t.$$

1.3. First-order conditions

I now derive first-order conditions in this economy. The Lagrangian for the household optimization problem is given by

$$\mathcal{L} = E\sum_t \beta^t N_t \Big\{U(c_t, 1-l_t)$$
$$+\mu_t\Big\{(1-\tau_{kt})r_t k_t + (1-\tau_{lt})w_t l_t + \tau_{kt}\delta k_t + tr_t - (1+\tau_{ct})c_t - (1+\tau_{xt})x_t\Big\}$$
$$+\lambda_t\Big\{(1-\delta)k_t + x_t - (1+g_n)k_{t+1}\Big\}\Big\}$$

In Staff Report 328, we included a penalty function to enforce the nonnegativity constraint on investment. This is especially important for analyzing the Great Depression period. Here, I am considering postwar business cycles and, therefore, assume that the investment decision will be interior.

The relevant first-order conditions are found by taking derivatives of $\mathcal{L}$ with respect to c_t, l_t, x_t, and k_{t+1}:

$$0 = U_1(c_t, 1-l_t) - \mu_t(1+\tau_{ct})$$
$$0 = -U_2(c_t, 1-l_t) + \mu_t(1-\tau_{lt})w_t$$
$$0 = \mu_t(1+\tau_{xt}) + \lambda_t = 0$$
$$0 = -(1+g_n)\lambda_t + E_t\{\mu_{t+1}[(1-\tau_{kt+1})r_{t+1} + \delta\tau_{kt+1}] + \lambda_{t+1}(1-\delta)\}$$

Eliminating multipliers yields:

$$\frac{U_2(c_t, 1-l_t)}{U_1(c_t, 1-l_t)} = \frac{1-\tau_{lt}}{1+\tau_{ct}} w_t \tag{1.2}$$

$$\frac{1+\tau_{xt}}{1+\tau_{ct}} U_1(c_t, 1-l_t) = \beta E_t\left[\frac{U_1(c_{t+1}, 1-l_{t+1})}{1+\tau_{ct+1}}\{(1-\tau_{kt+1})r_{t+1} + \delta\tau_{kt+1} + (1-\delta)(1+\tau_{xt+1})\}\right]. \tag{1.3}$$

In addition, there are first-order conditions for the firm's static problem. These are

$$r_t = F_1(K_t, Z_t L_t) \tag{1.4}$$

$$w_t = F_2(K_t, Z_t L_t) Z_t. \tag{1.5}$$

Finally, I have a resource constraint given by (1.1).

From here on, I make the following functional form assumptions and auxiliary choices:

$$F(k,l) = k^\theta l^{1-\theta} \tag{1.6}$$

$$U(c, 1-l) = (c(1-l)^\psi)^{1-\sigma}/(1-\sigma) \tag{1.7}$$

$$\tau_{kt} = \tau_{ct} = 0$$

$$s_t = [\log z_t, \tau_{lt}, \tau_{xt}, \log \hat{g}_t]'$$

$$s_{t+1} = P_0 + P s_t + Q\epsilon_{s,t+1}, \quad \epsilon_s \sim N(0_{4\times 1}, I_{4\times 4}). \tag{1.8}$$

I have turned off τ_c since it plays a similar role to τ_n in distorting the labor-leisure choice. Similarly, I have turned off τ_k since it plays a similar role to τ_x in distorting the intertemporal margin.

If I substitute the choices (1.6)-(1.7) into (1.1) and (1.2)-(1.5), then substitute the equilibrium rates r_t and w_t into (1.2) and (1.3), I have:

$$N_t(c_t + g_t) + N_{t+1}k_{t+1} - (1-\delta)N_t k_t = (N_t k_t)^\theta (Z_t N_t l_t)^{1-\theta} \tag{1.9}$$

$$\frac{\psi c_t}{1-l_t} = (1-\tau_{lt})(1-\theta)(N_t k_t)^\theta Z_t^{1-\theta}(N_t l_t)^{-\theta} \tag{1.10}$$

$$\begin{aligned}(1+\tau_{xt})c_t^{-\sigma}(1-l_t)^{\psi(1-\sigma)} \\ = \beta E_t\big[c_{t+1}^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}\big\{(1-\tau_{kt+1})\theta(N_{t+1}k_{t+1})^{\theta-1}(Z_{t+1}N_{t+1}l_{t+1})^{1-\theta} \\ + \delta\tau_{kt+1} + (1-\delta)(1+\tau_{xt+1})\big\}\big].\end{aligned} \tag{1.11}$$

1.4. Log-linear computation

The next big step is to approximate the decision function for capital. Given an approximate function for k_{t+1}, I can use the static equations (1.12) and (1.13) to determine the decisions c_t and l_t.

Log-linearizations are done for a stationary version of the equations (1.9)-(1.11). Thus, before proceeding, I need to normalize variables. Dividing all variables that grow by $(1+g_z)^t$ gives me:

$$\hat{c}_t + \hat{g}_t + (1+g_z)(1+g_n)\hat{k}_{t+1} - (1-\delta)\hat{k}_t = \hat{y}_t = \hat{k}_t^{\theta}(z_t l_t)^{1-\theta} \tag{1.12}$$

$$\frac{\psi \hat{c}_t}{1-l_t} = (1-\tau_{lt})(1-\theta)\hat{k}_t^{\theta} l_t^{-\theta} z_t^{1-\theta} \tag{1.13}$$

$$(1+\tau_{xt})\hat{c}_t^{-\sigma}(1-l_t)^{\psi(1-\sigma)} = \hat{\beta} E_t \hat{c}_{t+1}^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}[\theta \hat{k}_{t+1}^{\theta-1}(z_{t+1}l_{t+1})^{1-\theta} + (1-\delta)(1+\tau_{xt+1})] \tag{1.14}$$

where $\hat{\beta} = \beta(1+g_z)^{-\sigma}$.

To do the log-linear approximation, I will also need the steady state values of the variables in (1.12)-(1.14) (assuming constant values for z, the taxes, and government spending):

$$\hat{k}/l = \left(\frac{(1+\tau_x)(1-\hat{\beta}(1-\delta))}{\hat{\beta}\theta z^{1-\theta}} \right)^{1/(\theta-1)}$$

$$\hat{c} = \left[(\hat{k}/l)^{\theta-1} z^{1-\theta} - (1+g_z)(1+g_n) + 1 - \delta \right] \hat{k} - \hat{g} = \xi_1 \hat{k} - \hat{g}$$

$$\hat{c} = \left[(1-\tau_l)(1-\theta)(\hat{k}/l)^{\theta} z^{1-\theta}/\psi \right] (1 - 1/(\hat{k}/l)\ \hat{k}) = \xi_2 - \xi_3 \hat{k}$$

where the last 2 equations imply $\hat{k} = (\xi_2 + \hat{g})/(\xi_1 + \xi_3)$, $\hat{c} = \xi_1 \hat{k} - \hat{g}$, $l = (1/(\hat{k}/l))\hat{k}$.

Assume that the solution for the capital decision takes the form:

$$\log \hat{k}_{t+1} = \gamma_k \log \hat{k}_t + \gamma \,[\, \log z_t \quad \tau_{lt} \quad \tau_{xt} \quad \log \hat{g}_t \,]' + \text{constant}, \tag{1.15}$$

where γ_k is a scalar and γ is 1×4 and equal to $[\gamma_z, \gamma_l, \gamma_x, \gamma_g]$. Assume the residual from the dynamic first-order condition (1.14) can be written (after substitutions from (1.12) and (1.13)):

$$f(E_t \log \hat{k}_{t+2}, \log \hat{k}_{t+1}, \log \hat{k}_t, \log z_{t+1}, \log z_t, \tau_{lt+1}, \tau_{lt}, \tau_{xt+1}, \tau_{xt}, \log \hat{g}_{t+1}, \log \hat{g}_t)$$
$$\approx a_0 E_t \log \hat{k}_{t+2} + a_1 \log \hat{k}_{t+1} + a_2 \log \hat{k}_t + b_0 E_t s_{t+1} + b_1 s_t.$$

Then the general solution algorithm is to find γ_k that solves the quadratic equation

$$a_0 \gamma_k^2 + a_1 \gamma_k + a_2 = 0,$$

and γ that solves the linear equations:

$$a_0 \gamma_k \gamma + a_0 \gamma P + a_1 \gamma + b_0 P + b_1 = 0_{1\times 4}.$$

Note that this implies:

$$\gamma = -[(a_0 a + a_1) I_{4\times 4} + a_0 P']^{-1} (b_0 P + b_1 I_{4\times 4})'.$$

Once I have values for the the coefficients γ_k and γ, I can use (1.12) and (1.13) to back out c_t and l_t (either nonlinearly or by way of a log-linear approximation).

2. A Version of the Model with Random Walk Technology

2.1. Nomenclature

The only changes relative to the benchmark model described in Section 1 are:

Z: labor-augmenting technical change ($Z_t = Z_{t-1} z_t$)

z: the innovation to technology

$\hat{v}$: detrended, per-capita variable V ($\hat{v}_t = V_t/[N_t Z_t]$) with the exception of k

$\hat{k}$: detrended, per-capita capital, $\hat{k}_t = K_t/[N_t Z_{t-1}]$

2.2. Maximization problems

The maximization problems are the same as those in Section 1 except that households in this version assume $Z_t = Z_{t-1} z_t$ with the process for $\log z_t$ assumed to be autoregressive.

2.3. First-order conditions

The first-order conditions are the same as in Section 1.

2.4. Log-linear computation

The main difference between the benchmark model and the version with random-walk technology is the step taken to normalize variables In this version, the normalized variables are:

$$\hat{c}_t = c_t/Z_t, \hat{x}_t = x_t/Z_t, \hat{g}_t = g_t/Z_t, \hat{y}_t = y_t/Z_t, \hat{k}_t = k_t/Z_{t-1}.$$

Using the functional forms for F and U in (1.6) and (1.7), respectively, the equilibrium rental and wage rates are:

$$r_t = \theta K_t^{\theta-1}(Z_t L_t)^{1-\theta} = \theta \hat{k}_t^{\theta-1}(z_t l_t)^{1-\theta}$$
$$w_t = (1-\theta) K_t^{\theta}(Z_t L_t)^{-\theta} Z_t = (1-\theta)\hat{k}_t^{\theta}(z_t l_t)^{-\theta} Z_t.$$

This implies the following first-order conditions

$$\hat{c}_t + \hat{g}_t + (1+g_n)\hat{k}_{t+1} - (1-\delta) z_t^{-1}\hat{k}_t = \hat{y}_t = \hat{k}_t^{\theta} l_t^{1-\theta} z_t^{-\theta} \tag{2.1}$$
$$\frac{\psi \hat{c}_t}{1-l_t} = (1-\tau_{lt})(1-\theta)\hat{k}_t^{\theta}(z_t l_t)^{-\theta} \tag{2.2}$$
$$\begin{aligned}&(1+\tau_{xt})\hat{c}_t^{-\sigma}(1-l_t)^{\psi(1-\sigma)} \\ &\quad = \beta z_{t+1}^{-\sigma} E_t \hat{c}_{t+1}^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}[\theta \hat{k}_{t+1}^{\theta-1}(z_{t+1} l_{t+1})^{1-\theta} + (1-\delta)(1+\tau_{xt+1})].\end{aligned} \tag{2.3}$$

Next, I compute the steady state of the system for constant values for z, the taxes, and government spending:

$$\hat{k}/l = \left(\frac{(1+\tau_x)(1-\beta z^{-\sigma}(1-\delta))}{\beta z^{-\sigma} \theta z^{1-\theta}} \right)^{1/(\theta-1)}$$

$$\hat{c} = \left[(\hat{k}/l)^{\theta-1} z^{-\theta} - (1+g_n) + (1-\delta) z^{-1} \right] \hat{k} - \hat{g} = \xi_1 \hat{k} - \hat{g}$$

$$\hat{c} = \left[(1-\tau_l)(1-\theta)(\hat{k}/l)^{\theta} z^{-\theta}/\psi \right] (1 - 1/(\hat{k}/l)\ \hat{k}) = \xi_2 - \xi_3 \hat{k}$$

where the last 2 equations imply $\hat{k} = (\xi_2 + \hat{g})/(\xi_1 + \xi_3)$, $\hat{c} = \xi_1 \hat{k} - \hat{g}$, $l = (1/(\hat{k}/l))\hat{k}$.

The form of the solution and the procedure for computing it is the same as in the benchmark case.

3. U.S. Data

The national account data are taken from the *Survey of Current Business* NIPA tables available at www.bea.gov. Population and hours data are taken from Edward Prescott and Alexander Ueberfeldt, "U.S. Hours and Productivity Behavior using CPS Hours Worked Data: 1959:I to 2003:II." I use the Matlab file setupdata.m to convert the raw data into input files for maximum likelihood estimation.

4. MLE Estimation

I now describe the general method I use to estimate the processes governing the four exogenous variables in s_t with the data described above.

4.1. State-space form in the general case

I assume that X is a vector of state variables from the model and Y are observables. The state-space form then is

$$X_{t+1} = AX_t + B\epsilon_{t+1}$$

$$Y_t = CX_t + \omega_t$$

$$\omega_t = D\omega_{t-1} + \eta_t$$

where D is equal to parameters governing serial correlation of measurement error. Assume that $E\eta_t\eta_t' = R$, $E\epsilon_t\eta_s' = 0$ for all periods t and s. Define $\bar{Y}_t \equiv Y_{t+1} - DY_t$. Then I can rewrite the system as:

$$X_{t+1} = AX_t + B\epsilon_{t+1}$$

$$\bar{Y}_t = \bar{C}X_t + CB\epsilon_{t+1} + \eta_{t+1}$$

4.2. Log-likelihood function

The log-likehlihood function is

$$L(\Theta) = \sum_{t=0}^{T-1} \left\{ \log|\Omega_t| + \text{trace}(\Omega_t^{-1}u_t u_t') - \log|\partial f(Z_t, \Theta)/\partial Z_t| \right\} \tag{4.1}$$

where the parameters to be estimated are stacked in vector Θ, the innvation vector is u_t, and its covariance is Ω_t. The last term in (4.1) is nonzero if the Y are not the raw series but depend on the raw series Z plus the parameter vector. For example, if I estimate g_z and use per-capita values as our raw data, then Z is per-capita data and Y is detrended, per-capita data.

The innovation vector u_t and its covariance Ω_t are defined as follows:

$$\begin{aligned} u_t &= \bar{Y}_t - \hat{E}[\bar{Y}_t|\bar{Y}_{t-1}, \bar{Y}_{t-2}, \ldots, \bar{Y}_0, \hat{X}_0] \\ &= Y_{t+1} - \hat{E}[Y_{t+1}|Y_t, Y_{t-1}, \ldots, Y_0, \hat{X}_0] \\ &= Y_{t+1} - DY_t - \bar{C}\hat{X}_t \end{aligned}$$

$$\Omega_t = Eu_t u_t' = \bar{C}\Sigma_t\bar{C}' + R + CBB'C'.$$

which in turn depends on the predicted state $\hat{X}_t$:

$$\hat{X}_t = \hat{E}[X_t|Y_t, Y_t, \ldots, Y_0, \hat{X}_0].$$

The predicted state evolves according to

$$\hat{X}_{t+1} = A\hat{X}_t + K_t u_t$$

where K_t is the Kalman gain,

$$K_t = (BB'C' + A\Sigma_t\bar{C}')\Omega_t^{-1}$$

$$\Sigma_{t+1} = A\Sigma_t A' + BB' - (BB'C' + A\Sigma_t\bar{C}')\Omega_t^{-1}(\bar{C}\Sigma_t A' + CBB')$$

with state covariance Σ_t.

4.3. MLE in the Benchmark Case

In the benchmark case, I have $X_t = [\log \hat{k}_t, \log z_t, \tau_{lt}, \tau_{xt}, \log \hat{g}_t, 1]'$, $Y_t = [\log \hat{y}_t, \log \hat{x}_t, \log l_t, \log \hat{g}_t]$, and

$$A = \begin{bmatrix} \gamma_k & \gamma_z \quad \gamma_l \quad \gamma_x \quad \gamma_g & \gamma_0 \\ 0_{4\times 1} & P & P_0 \\ 0 & 0_{1\times 4} & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0_{1\times 4} \\ Q \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \phi_{yk} & \phi_{yz} & \phi_{yl} & 0 & \phi_{yg} & \phi_{y0} \\ \phi_{xk} & 0 & 0 & 0 & 0 & \phi_{x0} \\ \phi_{lk} & \phi_{lz} & \phi_{ll} & 0 & \phi_{lg} & \phi_{l0} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} \phi_{yk'} \\ \phi_{xk'} \\ \phi_{lk'} \\ 0 \end{bmatrix} \begin{bmatrix} \gamma_k & \gamma_z & \gamma_l & \gamma_x & \gamma_g & 0 \end{bmatrix}. \quad (4.2)$$

The coefficients ϕ are derived by log-linearizing (1.13) after substituting in for consumption from (1.12):

$$\begin{aligned} 0 \approx \psi\big\{&\hat{k}^\theta(zl)^{1-\theta}[\theta \log \hat{k}_t + (1-\theta)(\log z_t + \log l_t)] \\ &- (1+g_z)(1+g_n)\hat{k}\log \hat{k}_{t+1} + (1-\delta)\hat{k}\log \hat{k}_t - \hat{g}\log \hat{g}_t\big\} \\ &+ (1-\theta)(1-\tau_l)\hat{k}^\theta l^{-\theta} z^{1-\theta}(1-l)\big\{1/(1-\tau_l)\,\tau_{lt} \\ &- \theta \log \hat{k}_t + \theta \log l_t - (1-\theta)\log z_t + l/(1-l)\log l_t\big\} \end{aligned}$$

which I write succinctly as

$$\log l_t = \phi_{lk} \log \hat{k}_t + \phi_{lz} \log z_t + \phi_{ll} \tau_{lt} + \phi_{lg} \log \hat{g}_t + \phi_{lk'} \log \hat{k}_{t+1}. \tag{4.3}$$

Using this equation for $\log l$, I use the production relation and the capital accumulation equation to write $\log \hat{y}$ and $\log \hat{x}$ as follows:

$$\begin{aligned}
\log \hat{y}_t &= (\theta + (1-\theta)\phi_{lk}) \log \hat{k}_t + (1-\theta)(1+\phi_{lz}) \log z_t \\
&\qquad + (1-\theta)[\phi_{ll}\tau_{lt} + \phi_{lg}\hat{g}_t + \phi_{lk'} \log \hat{k}_{t+1}] \\
&\equiv \phi_{yk} \log \hat{k}_t + \phi_{yz} \log z_t + \phi_{yl}\tau_{lt} + \phi_{yg} \log \hat{g}_t + \phi_{yk'} \log \hat{k}_{t+1} && (4.4) \\
\log \hat{x}_t &= (1+g_z)(1+g_n)\hat{k}/\hat{x} \log \hat{k}_{t+1} - (1-\delta)\hat{k}/\hat{x} \log \hat{k}_t \\
&\equiv \phi_{xk} \log \hat{k}_t + \phi_{xk'} \log \hat{k}_{t+1}. && (4.5)
\end{aligned}$$

I fixed parameters of preferences, production, and growth and estimated the processes for the shocks. The parameters that were fixed were: $\psi = 2.24$, $\sigma = 1$, $\beta = .9722$, $\theta = .35$, $\delta = .0464$, $g_n = 1.5\%$, and $g_z = 1.6\%$. I also set $D = 0_{4\times 4}$ and $R = .0001 \times I_{4\times 4}$. The parameters that were estimated were elements of P_0, P, and Q.

4.4. MLE in the Random Walk Case

In the case of random-walk technology, the settings are slightly different. In this case, I have $X_{st} = [\log \hat{k}_t, \log z_t, \tau_{lt}, \tau_{xt}, \log \hat{g}_t, 1]'$, $X_t = [X_{st}, X_{st-1}]'$, and $Y_t = [\log y_t - \log y_{t-1}, \log x_t - \log x_{t-1}, \log l_t, \log g_t - \log g_{t-1}]$. I can write the growth rates in Y_t as elements of X_t as follows:

$$\begin{aligned}
\log y_t - \log y_{t-1} &= \log(\hat{y}_t Z_t) - \log(\hat{y}_{t-1} Z_{t-1}) \\
&= \log(\hat{y}_t) - \log(\hat{y}_{t-1}) + \log z_t \\
&= \phi_{yk}(\log \hat{k}_t - \log \hat{k}_{t-1}) + (1+\phi_{yz}) \log z_t - \phi_{yz} \log z_{t-1} \\
&\quad + \phi_{yl}(\tau_{lt} - \tau_{lt-1}) + \phi_{yg}(\log \hat{g}_t - \log \hat{g}_{t-1}) + \phi_{yk'}(\log \hat{k}_{t+1} - \log \hat{k}_t)
\end{aligned}$$

Similarly I can write the growth rates for x_t and g_t in terms of the elements of X_t.

To obtain the ϕ coefficients, I log-linearize (2.2) after substituting in for consumption from (2.1):

$$\begin{aligned}0 \approx \psi\big\{\hat{k}^{\theta} l^{1-\theta} z^{-\theta}[\theta(\log \hat{k}_t - \log z_t) + (1-\theta)\log l_t] \\ - (1+g_n)\hat{k}\log \hat{k}_{t+1} + (1-\delta)z^{-1}\hat{k}(\log \hat{k}_t - \log z_t) - \hat{g}\log \hat{g}_t\big\} \\ + (1-\theta)(1-\tau_l)\hat{k}^{\theta}(zl)^{-\theta}(1-l)\big\{1/(1-\tau_l)\,\tau_{lt} \\ - \theta \log \hat{k}_t + \theta(\log l_t + \log z_t) + l/(1-l)\log l_t\big\}\end{aligned}$$

which again I write succinctly as I did in (4.3). Using the equation for $\log l$, I use the production relation and the capital accumulation equation to write $\log \hat{y}$ and $\log \hat{x}$ as follows:

$$\begin{aligned}\log \hat{y}_t &= (\theta + (1-\theta)\phi_{lk})\log \hat{k}_t + ((1-\theta)\phi_{lz} - \theta)\log z_t \\ &\quad + (1-\theta)[\phi_{ll}\tau_{lt} + \phi_{lg}\hat{g}_t + \phi_{lk'}\log \hat{k}_{t+1}] \\ &\equiv \phi_{yk}\log \hat{k}_t + \phi_{yz}\log z_t + \phi_{yl}\tau_{lt} + \phi_{yg}\log \hat{g}_t + \phi_{yk'}\log \hat{k}_{t+1} \qquad (4.6) \\ \log \hat{x}_t &= (1+g_n)\hat{k}/\hat{x}\log \hat{k}_{t+1} - (1-\delta)z^{-1}\hat{k}/\hat{x}(\log \hat{k}_t - \log z_t) \\ &\equiv \phi_{xk}\log \hat{k}_t + \phi_{xz}\log z_t + \phi_{xk'}\log \hat{k}_{t+1}. \qquad (4.7)\end{aligned}$$

The matrices in the state space form are

$$A = \begin{bmatrix} A_s & 0 \\ I & 0 \end{bmatrix} \qquad B = \begin{bmatrix} B_s \\ 0 \end{bmatrix}$$

where

$$A_s = \begin{bmatrix} \gamma_k & \gamma_z \quad \gamma_l \quad \gamma_x \quad \gamma_g & \gamma_0 \\ 0_{4\times 1} & P & P_0 \\ 0 & 0_{1\times 4} & 1 \end{bmatrix}$$

$$B_s = \begin{bmatrix} 0_{1\times 4} \\ Q \\ 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} \phi_{yk} - \phi_{yk'} & 1+\phi_{yz} & \phi_{yl} & 0 & \phi_{yg} & \phi_{y0} & -\phi_{yk} & -\phi_{yz} & -\phi_{yl} & 0 & -\phi_{yg} & -\phi_{y0} \\ \phi_{xk} - \phi_{xk'} & 1+\phi_{xz} & 0 & 0 & 0 & \phi_{x0} & -\phi_{xk} & -\phi_{xz} & 0 & 0 & 0 & -\phi_{x0} \\ \phi_{lk} & \phi_{lz} & \phi_{ll} & 0 & \phi_{lg} & \phi_{l0} & & & & & & \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \phi_{yk'} \\ \phi_{xk'} \\ \phi_{lk'} \\ 0 \end{bmatrix} [\,\gamma_k \quad \gamma_z \quad \gamma_l \quad \gamma_x \quad \gamma_g \quad 0\,]\,. \tag{4.8}$$

5. Simulating Data from the Models

I draw 1000 sequences $\{\epsilon_{s,t}\}$. Given MLE estimates for P_0, P, Q, and initial conditions for s, I can use (1.8) to derive sequences for technology, tax rates, and spending. Given an initial condition for the capital stock k_0, I can use (1.8) to derive the time path for a sequence $\{k_t\}$. With technology, tax rates, spending, and capital, I have the entire state vector X_t period by period. I then use $Y_t = CX_t$ (since I have assumed negligible measurement error) for my observable vector where C is (4.2) in the benchmark case and (4.8) in the random-walk case.