Federal Reserve Bank of Minneapolis
Research Department Staff Report

June 2004

TECHNICAL APPENDIX II: Comment on Gali and Rabanal[†]

Ellen R. McGrattan
Federal Reserve Bank of Minneapolis
and University of Minnesota

[†] The views expressed herein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

Table of Contents

These notes contain derivations of expressions and results reported in my comments on Gali and Rabanal for the NBER Macro Annual, 2004.

1. The Model

The model is a business cycle model with staggered price and wage setting. It has many of the same elements as the models in Chari, Kehoe, and McGrattan (2000, 2001). In order to compare my results to those of Gali and Rabanal, I also allow for habit persistence in preferences.

1.1. Uncertainty

In each period t, the economy experiences one of finitely many events s_t. I denote by $s^t = (s_0, \ldots, s_t)$ the history of events up through and including period t. The probability, as of period zero, of any particular history s^t is $\pi(s^t)$. The initial realization s_0 is given.

1.2. The Final Goods Producers

Final goods producers behave competitively and solve a static profit-maximization problem. In each period producers choose inputs $y(i)$ for $i \in [0, 1]$ and output y to maximize profits given by

$$\max \; Py - \int_0^1 P(i)y(i)\, di \tag{1.1}$$

subject to

$$y = \left(\int_0^1 y(i)^\theta di\right)^{\frac{1}{\theta}} \tag{1.2}$$

where y is the final good, P is the price of the final good, $y(i)$ are intermediate goods, and $P(i)$ are the prices of the intermediate goods.

I can rewrite the first-order condition with respect to $y(i)$ to get the input demand function:

$$y(i) = \left[\frac{P}{P(i)}\right]^{\frac{1}{1-\theta}} y \tag{1.3}$$

To get the price of the final good, we use the zero-profit condition which implies that

$$P = \left[\int_0^1 P(i)^{\frac{\theta}{\theta-1}} \, di \right]^{\frac{\theta-1}{\theta}} . \tag{1.4}$$

1.3. Consumer problem

Consider next the problem faced by consumers. One can think of the economy organized into a continuum of unions indexed by j. Each union j consists of all the consumers in the economy with labor of type j. This union realizes that it faces a downward sloping demand curve for its type of labor. It sets nominal wages for $\mathcal{N}$ periods at t, $t+\mathcal{N}$, $t+2\mathcal{N}$, and so on. Thus, it faces constraints

$$\begin{aligned} W(j, s^{t-1}) &= W(j, s^t) = \ldots = W(j, s^{t+\mathcal{N}-1}) \\ W(j, s^{t+\mathcal{N}}) &= W(j, s^{t+\mathcal{N}+1}) = \ldots = W(j, s^{t+2\mathcal{N}-1}) \end{aligned}$$

and so on in addition to the ones below.

The problem solved by a union of type j is to maximize utility:

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) \; U\left(c(j, s^t), c(j, s^{t-1}), L^s(j, s^t), M^d(j, s^t)/P(s^t); \varphi_t\right),$$

which allows for habit persistence and preference shocks (φ), subject to the sequence of budget constraints, the definition of labor supply, and the labor demands of the firms:

$$\begin{aligned} & P(s^t)c(j, s^t) + M^d(j, s^t) + \sum_{s_{t+1}} Q(s^{t+1}|s^t)B(j, s^{t+1}) \\ & \qquad \leq W(j, s^{t-1})L^s(j, s^t) + M^d(j, s^{t-1}) + B(j, s^t) + \Pi(s^t) + T(s^t) \qquad (1.5) \\ & L^s(j, s^t) = \int l(i, j, s^t) \, di \\ & l(i, j, s^t) = \left(\frac{\bar{W}(s^t)}{W(j, s^{t-1})} \right)^{\frac{1}{1-v}} L^d(i, s^t), \quad \text{for all } i. \end{aligned}$$

There are also borrowing constraints $B(s^{t+1}) \geq -P(s^t)\bar{b}$. M and B are their holdings of money and contingent claims, Q is the price of the claims, $W(j, s^{t-1})$ is the nominal wage chosen by one cohort of unions, Π are profits, and T are government transfers. In this economy, the union chooses the wage but agrees to supply whatever is demanded at that wage.

The Lagrangian in this case is

$$\mathcal{L} = \dots \beta^t \sum_{s^t} \pi(s^t) \Bigg\{ U\left(c(j,s^t), c(j,s^{t-1}), \bar{W}(s^t)^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{1}{v-1}} L^d(s^t), \frac{M^d(j,s^t)}{P(s^t)} \right)$$
$$+ \zeta(j,s^t) \{ \bar{W}(s^t)^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{v}{v-1}} L^d(s^t)/P(s^t) + M^d(j,s^{t-1})/P(s^t)$$
$$+ B(j,s^t)/P(s^t) + \Pi(s^t)/P(s^t) + T(s^t)/P(s^t)$$
$$- c(j,s^t) - M^d(j,s^t)/P(s^t) - \sum_{s_{t+1}} Q(s^{t+1}|s^t) B(j,s^{t+1})/P(s^t) \}$$
$$+ \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t) \Bigg[U\left(c(j,s^{t+1}), c(j,s^t), \bar{W}(s^{t+1})^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{1}{v-1}} L^d(s^{t+1}), \frac{M^d(j,s^{t+1})}{P(s^{t+1})} \right)$$
$$+ \zeta(j,s^{t+1}) \{ \bar{W}(s^{t+1})^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{v}{v-1}} L^d(s^{t+1})/P(s^{t+1}) + M^d(j,s^t)/P(s^{t+1})$$
$$+ B(j,s^{t+1})/P(s^{t+1}) + \Pi(s^{t+1})/P(s^{t+1}) + T(s^{t+1})/P(s^{t+1})$$
$$- c(j,s^{t+1}) - M^d(j,s^{t+1})/P(s^{t+1}) - \sum_{s_{t+2}} Q(s^{t+2}|s^{t+1}) B(j,s^{t+2})/P(s^{t+1}) \} \dots$$

where $L^d(s^t) = \int L^d(i,s^t)\, di$ and, thus,

$$L^s(j,s^t) = \bar{W}(s^t)^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{1}{v-1}} L^d(s^t). \tag{1.6}$$

Taking the derivative of $\mathcal{L}$ with respect to $W(j,s^{t-1})$ I have

$$0 = \sum_{s^t} \pi(s^t) \Big\{ \frac{1}{v-1} \bar{W}(s^t)^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{2-v}{v-1}} L^d(s^t) U_l(j,s^t)$$
$$+ \frac{v}{v-1} \zeta(j,s^t) \bar{W}(s^t)^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{1}{v-1}} L^d(s^t)/P(s^t)$$
$$+ \frac{1}{v-1} \sum_{s_{t+1}} \beta \pi(s^{t+1}|s^t) \bar{W}(s^{t+1})^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{2-v}{v-1}} L^d(s^{t+1}) U_l(j,s^{t+1})$$
$$+ \frac{v}{v-1} \sum_{s_{t+1}} \beta \pi(s^{t+1}|s^t) \zeta(j,s^{t+1}) \bar{W}(s^{t+1})^{\frac{1}{1-v}} W(j,s^{t-1})^{\frac{1}{v-1}} L^d(s^{t+1})/P(s^{t+1}) \dots \tag{1.7}$$

Rewriting this in terms of $W(j,s^{t-1})$ gives me:

$$W(j,s^{t-1}) = -\frac{\sum_{\tau=t}^{t+\mathcal{N}-1} \sum_{s^\tau} \beta^{\tau-t+1} \pi(s^\tau|s^{t-1}) \bar{W}(s^\tau)^{\frac{1}{1-v}} L^d(s^\tau) U_l(j,s^\tau)}{v \sum_{\tau=t}^{t+\mathcal{N}-1} \sum_{s^\tau} \beta^{\tau-t+1} \pi(s^\tau|s^{t-1}) \zeta(j,s^\tau) \bar{W}(s^\tau)^{\frac{1}{1-v}} L^d(s^\tau)/P(s^\tau)}. \tag{1.8}$$

The first-order condition with respect to consumption is given by:

$$\frac{\partial U(j,s^t)}{\partial c(j,s^t)} - \zeta(j,s^t) + \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t) \frac{\partial U(j,s^{t+1})}{\partial c(j,s^t)} = 0 \tag{1.9}$$

The first-order condition with respect to money demand is given by:

$$\frac{U_m(j,s^t)}{P(s^t)} - \frac{\zeta(j,s^t)}{P(s^t)} + \beta \sum_{s_{t+1}} \pi(s^{t+1}|s^t) \frac{\zeta(j,s^{t+1})}{P(s^{t+1})} = 0 \tag{1.10}$$

The equilibrium bond price is found by manipulating the first-order condition found by taking the derivative of $\mathcal{L}$ with respect to $B(j,s^{t+1}$, that is,

$$Q(s^{t+1}|s^t) = \beta\pi(s^{t+1}|s^t) \frac{\zeta(j,s^{t+1})P(s^t)}{\zeta(j,s^t)P(s^{t+1})}. \tag{1.11}$$

Let $R(s^t)$ and $r(s^t)$ be the gross and net nominal interest rates, respectively; they are defined as follows:

$$\frac{1}{R(s^t)} = \frac{1}{1+r(s^t)} = \sum_{s_{t+1}} Q(s^{t+1}|s^t) \tag{1.12}$$

with $r(s^t) = R(s^t) - 1$. Using the definition for r, the money demand equations can be written statically as follows:

$$\frac{U_m(j,s^t)}{\zeta(j,s^t)} = \frac{r(s^t)}{1+r(s^t)}.$$

1.4. Intermediate goods producers

Intermediate goods producers are monopolistically competitive. They set prices for their goods, but they most hold them fixed for N periods. We assume that price-setting is done in a staggered fashion so that $1/N$ of the firms are setting in a particular period. I compute a symmetric equilibrium so we assume that all firms $i \in [0, 1/N]$ behave the same way and all firms $i \in [1/N, 2/N]$ behave the same way, and so on.

More specifically, the problem solved by the intermediate goods producers setting prices is to choose sequences of prices $P(i)$, capital stocks $k(i)$, investments $x(i)$, and labor inputs $l(i,j)$, $j = 1, \ldots, \mathcal{N}$ to maximize

$$\sum_{\tau=0}^{\infty} \sum_{s^\tau} \tilde{Q}(s^\tau) \big[P(i,s^\tau)y(i,s^\tau) - \int W(j,s^{\tau-1})l(i,j,s^\tau)\,dj - P(s^\tau)x(i,s^\tau)\big] \tag{1.13}$$

subject to the input demand (1.3), the production technology:

$$y(i,s^t) = F(k(i,s^{t-1}), A(s^t)L^d(i,s^t)) \tag{1.14}$$

the constraint on labor

$$L^d(i,s^t) \leq \left[\int l(i,j,s^t)^v \, dj\right]^{\frac{1}{v}}, \tag{1.15}$$

the law of motion for capital used in producing good i

$$k(i,s^t) = (1-\delta)k(i,s^{t-1}) + x(i,s^t) - \phi\left(\frac{x(i,s^t)}{k(i,s^{t-1})}\right)k(i,s^{t-1}) \tag{1.16}$$

and the following constraints on prices:

$$\begin{aligned} P(i,s^{t-1}) &= P(i,s^t) = \ldots P(i,s^{t+N-1}) \\ P(i,s^{t+N}) &= P(i,s^{t+N+1}) = \ldots P(i,s^{t+2N-1}) \\ &\vdots \end{aligned} \tag{1.17}$$

where $\tilde{Q}(s^\tau)$ is the τth period Arrow-Debreu price (that is, a product of the one-period $Q(s^t|s^{t-1})$'s).

The Lagrangian in this case is

$$\begin{aligned} \mathcal{L} = \ldots + \tilde{Q}(s^t)\Bigg\{ \Big[& P(i,s^{t-1})^{\frac{\theta}{\theta-1}} P(s^t)^{\frac{1}{1-\theta}} y(s^t) \\ & - \int W(j,s^{t-1}) l(i,j,s^t)\, dj - P(s^t)x(i,s^t) \\ & + \chi(s^t)\big\{F(k(i,s^{t-1}), A(s^t)L^d(i,s^t)) - P(s^t)^{\frac{1}{1-\theta}} y(s^t) P(i,s^{t-1})^{\frac{1}{\theta-1}}\big\} \\ & + \lambda(s^t)\big\{(1-\delta)k(i,s^{t-1}) + x(i,s^t) \\ & \qquad - \phi(x(i,s^t)/k(i,s^{t-1}))k(i,s^{t-1}) - k(i,s^t)\big\}\Big] \\ & + \kappa(s^t)\Big\{\left[\int l(i,j,s^t)^v \, dj\right]^{\frac{1}{v}} - L^d(i,s^t)\Big\} \\ + \sum_{s_{t+1}} Q(s^{t+1}|s^t)\Big[& P(i,s^{t-1})^{\frac{\theta}{\theta-1}} P(s^{t+1})^{\frac{1}{1-\theta}} y(s^{t+1}) \end{aligned}$$

$$
\begin{aligned}
&- \int W(j,s^t) l(i,j,s^{t+1})\, dj - P(s^{t+1}) x(i,s^{t+1}) \\
+ &\chi(s^{t+1}) \big\{ F(k(i,s^t), L^d(i,s^{t+1})) - P(s^{t+1})^{\frac{1}{1-\theta}} y(s^{t+1}) P(i,s^{t-1})^{\frac{1}{\theta-1}} \big\} \\
+ &\lambda(s^{t+1}) \big\{ (1-\delta) k(i,s^t) + x(i,s^{t+1}) \\
&\qquad - \phi(x(i,s^{t+1})/k(i,s^t)) k(i,s^t) - k(i,s^{t+1}) \big\} \Big] \\
+ &\kappa(s^{t+1}) \Big\{ \left[\int l(i,j,s^{t+1})^v \, dj \right]^{\frac{1}{v}} - L^d(i,s^{t+1}) \Big\} \\
+ &\ldots \Bigg\}
\end{aligned}
\tag{1.18}
$$

The variables χ, λ, and κ are multipliers for constraints (1.14), (1.16), and (1.15), respectively.

Taking the derivative of $\mathcal{L}$ in (1.18) with respect to the monopolist's prices $P(i, s^{t-1})$, I get

$$
\begin{aligned}
\sum_{\tau} \sum_{s^\tau} Q(s^\tau | s^{t-1}) \big\{ &\theta P(i,s^{t-1})^{\frac{1}{\theta-1}} P(s^\tau)^{\frac{1}{1-\theta}} y(s^\tau) \\
&- \chi(s^\tau) P(i,s^{t-1})^{\frac{2-\theta}{\theta-1}} P(s^\tau)^{\frac{1}{1-\theta}} y(s^\tau) \big\} = 0
\end{aligned}
\tag{1.19}
$$

The derivative of $\mathcal{L}$ with respect to $x(i, s^t)$ is:

$$
-P(s^t) + \lambda(s^t) \left[1 - \phi' \left(\frac{x(i,s^t)}{k(i,s^{t-1})} \right) \right] = 0. \tag{1.20}
$$

The derivative of $\mathcal{L}$ with respect to $k(i, s^t)$ is:

$$
\begin{aligned}
-\lambda(s^t) + \sum_{s_{t+1}} Q(s^{t+1}|s^t) \Big\{ &\chi(s^{t+1}) F_k(i,s^{t+1}) + \lambda(s^{t+1}) \Big[1 - \delta \\
&- \phi \left(\frac{x(i,s^{t+1})}{k(i,s^t)} \right) + \phi' \left(\frac{x(i,s^{t+1})}{k(i,s^t)} \right) \frac{x(i,s^{t+1})}{k(i,s^t)} \Big] \Big\} = 0.
\end{aligned}
\tag{1.21}
$$

Now consider the labor inputs. Taking the derivative of $\mathcal{L}$ with respect to $L^d(i, s^t)$, I get:

$$
\chi(s^t) F_l(i,s^t) - \kappa(s^t) = 0. \tag{1.22}
$$

Taking the derivative with respect to $l(i, j, s^t)$, I get:

$$-W(j, s^{t-1}) + \kappa(s^t) l(i, j, s^t)^{v-1} \left[\int l(i, j, s^t)^v \, dj \right]^{\frac{1}{v}-1} = 0$$

or,

$$W(j, s^{t-1}) = \kappa(s^t) l(i, j, s^t)^{v-1} L^d(i, s^t)^{1-v}. \tag{1.23}$$

If I integrate both sides of (1.23), I get

$$\begin{aligned} \kappa(s^t) &= \left[\int W(j, s^{t-1})^{\frac{v}{v-1}} \, dj \right]^{\frac{v-1}{v}} \\ &\equiv \bar{W}(s^t) \end{aligned} \tag{1.24}$$

which implies that the multipler is equal to the aggregate wage. Substituting that back into (1.23), I have

$$l(i, j, s^t) = \left(\frac{\bar{W}(s^t)}{W(j, s^{t-1})} \right)^{\frac{1}{1-v}} L^d(i, s^t).$$

If I substitute expressions for the multipliers using (1.22) and (1.20) into (1.19) and (1.21), I get

$$P(i, s^{t-1}) = \frac{\sum_\tau \sum_{s^\tau} Q(s^\tau | s^{t-1}) mc(i, s^\tau) P(s^\tau)^{\frac{2-\theta}{1-\theta}} y(s^\tau)}{\theta \sum_\tau \sum_{s^\tau} Q(s^\tau | s^{t-1}) P(s^\tau)^{\frac{1}{1-\theta}} y(s^\tau)} \tag{1.25}$$

$$\begin{aligned} \frac{P(s^t)}{1 - \phi'(i, s^t)} = \beta \sum_{s_{t+1}} Q(s^{t+1} | s^t) P(s^{t+1}) \Bigg\{ & mc(i, s^{t+1}) F_k(i, s^{t+1}) \\ & + \frac{1}{1 - \phi'(i, s^{t+1})} \left[1 - \delta - \phi(i, s^{t+1}) + \phi'(i, s^{t+1}) \frac{x(i, s^{t+1})}{k(i, s^t)} \right] \Bigg\}. \end{aligned} \tag{1.26}$$

Note that I have used the fact that marginal costs of producer i are given by:

$$mc(i, s^t) = w(s^t) / F_l(i, s^t) \tag{1.27}$$

where

$$w(s^t) = \bar{W}(s^t) / P(s^t) \tag{1.28}$$

is the real wage.

1.5. The Government

Monetary policy is modeled as a nominal interest rate rule

$$r(s^t) = a' \begin{bmatrix} r(s^{t-1}) \\ r(s^{t-2}) \\ r(s^{t-3}) \\ E_t \log P(s^{t+1}) - \log P(s^t) \\ \log P(s^t) - \log P(s^{t-1}) \\ \log P(s^{t-1}) - \log P(s^{t-2}) \\ \log P(s^{t-2}) - \log P(s^{t-3}) \\ \log y(s^t) \\ \log y(s^{t-1}) \\ \log y(s^{t-2}) \end{bmatrix} + \text{constant} + \epsilon_{r,t}. \tag{1.29}$$

The government budget constraint is given by:

$$T(s^t) = M(s^t) - M(s^{t-1}). \tag{1.30}$$

where T are transfers to consumers.

1.6. Additional Equilibrium Conditions

I need some additional conditions before computing an equilibrium. The resource constraint is given by

$$y(s^t) = \int_0^1 c(j, s^t)\, dj + \int_0^1 x(i, s^t)\, di + g(s^t). \tag{1.31}$$

Money supply and demand are equated, so that:

$$M(s^t) = \int M^d(j, s^t)\, dj.$$

2. Computing an Equilibrium

I now describe how to compute an equilibrium. First, I normalize variables to make the problem stationary. Second, I derive equations for the steady states of the stationary variables. Third, I linearize the first-order conditions around the steady state. Fourth, I describe in detail the codes used for computing a solution to the linearized system of equations.

To simplify things, I assume from here on (unless noted otherwise) that the ith group of monopolists ($i \in \{1, \ldots N\}$) is the one that set prices i periods ago. Thus, in period t, monopolist 1 is assumed to have set prices conditional on seeing s^{t-1}, monopolist 2 set prices conditional on seeing s^{t-2}, and so on. Similarly I assume that the jth household ($j \in \{1, \ldots \mathcal{N}\}$) is the one that set wages j periods ago.

2.1. Normalization

I assume that prices and wages grow at the rate μ. Thus, I need to normalize them as follows:

$$p(s^t) = P(s^t)/\mu^{t-1}$$

$$p(i, s^{t-1}) = P(i, s^{t-1})/\mu^{t-i}$$

$$\omega(s^{t-1}) = W(s^{t-1})/\mu^{t-1}$$

$$\omega(j, s^{t-1}) = W(j, s^{t-1})/\mu^{t-j}$$

$$\bar{\omega}(s^t) = \bar{W}(s^t)/\mu^{t-1}$$

$$m^d(j, s^t) = M^d(j, s^t)/\mu^t$$

When I normalize the price equation, I get

$$p(1, s^{t-1})\mu^{t-1} = \frac{-\sum_\tau \sum_{s^\tau} \beta^{\tau-1}\pi(s^\tau|s^{t-1})\zeta(\tau - t + 1, s^\tau)mc(\tau - t + 1, s^\tau)(p(s^\tau)\mu^{\tau-1})^{\frac{1}{1-\theta}}y(s^\tau)}{\theta \sum_\tau \sum_{s^\tau} \beta^{\tau-1}\pi(s^\tau|s^{t-1})\zeta(\tau - t + 1, s^\tau)(p(s^\tau)\mu^{\tau-1})^{\frac{\theta}{1-\theta}}y(s^\tau)}$$

or

$$p(s^{t-1}) = \frac{-\sum_\tau \sum_{s^\tau} (\beta\mu^{\frac{1}{1-\theta}})^{\tau-1}\pi(s^\tau|s^{t-1})\zeta(\tau-t+1,s^\tau)mc(\tau-t+1,s^\tau)p(s^\tau)^{\frac{1}{1-\theta}}y(s^\tau)}{\theta\sum_\tau \sum_{s^\tau} (\beta\mu^{\frac{\theta}{1-\theta}})^{\tau-1}\pi(s^\tau|s^{t-1})\zeta(\tau-t+1,s^\tau)p(s^\tau)^{\frac{\theta}{1-\theta}}y(s^\tau)} \tag{2.1}$$

Notice that the indices for ζ and marginal cost are $\tau - t + 1$ which is $1, 2, \ldots, \mathcal{N}$ when we write out the sums.

When I normalize the wage equation, I get

$$\omega(1, s^{t-1})\mu^{t-1} = \frac{-\sum_\tau \sum_{s^\tau} \beta^{\tau-1}\pi(s^\tau|s^{t-1})(\bar\omega(s^\tau)\mu^{\tau-1})^{\frac{1}{1-v}}L^d(s^\tau)U_l(\tau-t+1,s^\tau)}{v\sum_\tau \sum_{s^\tau} \beta^{\tau-1}\pi(s^\tau|s^{t-1})(\bar\omega(s^\tau)\mu^{\tau-1})^{\frac{1}{1-v}}L^d(s^\tau)\zeta(\tau-t+1,s^\tau)/[p(s^\tau)\mu^{\tau-1}]}$$

or

$$\omega(s^{t-1}) = \frac{-\sum_\tau \sum_{s^\tau} (\beta\mu^{\frac{1}{1-v}})^{\tau-1}\pi(s^\tau|s^{t-1})\bar\omega(s^\tau)^{\frac{1}{1-v}}L^d(s^\tau)U_l(\tau-t+1,s^\tau)}{v\sum_\tau \sum_{s^\tau} (\beta\mu^{\frac{1}{1-v}})^{\tau-1}\pi(s^\tau|s^{t-1})\bar\omega(s^\tau)^{\frac{v}{1-v}}L^d(s^\tau)\zeta(\tau-t+1,s^\tau)/p(s^\tau)} \tag{2.2}$$

Again, notice that the indices for ζ and the marginal utility are $\tau-t+1$ which is $1, 2, \ldots, \mathcal{N}$ when we write out the sums.

The relationship between the aggregate and individual wages is normalized as follows:

$$\begin{aligned}\bar\omega(s^t) &= \left[\frac{1}{\mathcal{N}}\sum_{j=1}^{\mathcal{N}}\left(\frac{W(s^{t-j})}{\mu^{t-1}}\right)^{\frac{v}{v-1}}\right]^{\frac{v-1}{v}} \\ &= \left[\frac{1}{\mathcal{N}}\omega(s^{t-1})^{\frac{v}{v-1}} + \frac{1}{\mathcal{N}}\left(\frac{\omega(s^{t-2})}{\mu}\right)^{\frac{v}{v-1}} + \cdots + \frac{1}{\mathcal{N}}\left(\frac{\omega(s^{t-\mathcal{N}})}{\mu^{\mathcal{N}-1}}\right)^{\frac{v}{v-1}}\right]^{\frac{v-1}{v}}\end{aligned}$$

When I normalize the wage equation, I get

$$\omega(1, s^{t-1})\mu^{t-1} = \frac{-\sum_\tau \sum_{s^\tau} \beta^{\tau-1}\pi(s^\tau|s^{t-1})(\bar\omega(s^\tau)\mu^{\tau-1})^{\frac{1}{1-v}}L^d(s^\tau)U_l(\tau-t+1,s^\tau)}{v\sum_\tau \sum_{s^\tau} \beta^{\tau-1}\pi(s^\tau|s^{t-1})(\bar\omega(s^\tau)\mu^{\tau-1})^{\frac{1}{1-v}}L^d(s^\tau)\zeta(\tau-t+1,s^\tau)}$$

or

$$\omega(s^{t-1}) = \frac{-\sum_\tau \sum_{s^\tau} (\beta\mu^{\frac{1}{1-v}})^{\tau-1}\pi(s^\tau|s^{t-1})\bar\omega(s^\tau)^{\frac{1}{1-v}}L^d(s^\tau)U_l(\tau-t+1,s^\tau)}{v\sum_\tau \sum_{s^\tau} (\beta\mu^{\frac{1}{1-v}})^{\tau-1}\pi(s^\tau|s^{t-1})\bar\omega(s^\tau)^{\frac{1}{1-v}}L^d(s^\tau)\zeta(\tau-t+1,s^\tau)} \tag{2.3}$$

Notice that the indices for the marginal utilities are $\tau - t + 1$ which is $1, 2, \ldots, \mathcal{N}$ when we write out the sums.

The relationship between the aggregate and individual wages is normalized as follows:

$$
\begin{aligned}
\bar{\omega}(s^t) &= \left[\frac{1}{\mathcal{N}}\sum_{j=1}^{\mathcal{N}}\left(\frac{W(s^{t-j})}{\mu^{t-1})}\right)^{\frac{v}{v-1}}\right]^{\frac{v-1}{v}} \\
&= \left[\frac{1}{\mathcal{N}}\omega(s^{t-1})^{\frac{v}{v-1}} + \frac{1}{\mathcal{N}}\left(\frac{\omega(s^{t-2})}{\mu}\right)^{\frac{v}{v-1}} + \cdots + \frac{1}{\mathcal{N}}\left(\frac{\omega(s^{t-\mathcal{N}})}{\mu^{\mathcal{N}-1}}\right)^{\frac{v}{v-1}}\right]^{\frac{v-1}{v}}
\end{aligned}
$$

2.2. Steady State

To compute the steady state, I drop s^t arguments in the first-order conditions and solve for a fixed point. Consider doing this iteratively. Start with a guess for the capital stocks, $k(i)$, $i = 1, \ldots N$, output y, the consumption levels $c(j)$, $j = 2, \ldots \mathcal{N}$, and the money demands $m^d(j)$, $j = 1, \ldots \mathcal{N}$. Because I will assume that there is habit persistence, I can either assume some random $c(j)$'s or set them in a particular way. Below I will assume that the steady state $c(j)$'s equate $U_1(j)$'s across households.

With the $k(i)'s$, we can back out the investments from the law of motion for capital

$$
k(i) = (1-\delta)k(i-1) + x(i) - \phi\left(\frac{x(i)}{k(i-1)}\right)k(i-1), \quad i = 1, \ldots, N.
$$

With y, we can get the steady state input demands:

$$
y(i) = \mu^{\frac{i-1}{1-\theta}}.
$$

Using $y(i)$'s and $k(i)$'s, we can back out the labor demands for each firm i, i.e., $L^d(i)$, using the production technology.

Having the $L^d(i)$'s we can determine the $F_k(i)$'s and then back out marginal costs via the capital Euler equations:

$$
\begin{aligned}
\frac{1}{1-\phi'(i)} &= \beta\big(mc(i+1)F_k(i+1) \\
&\quad + \frac{1}{1-\phi'(i+1)}\left[1-\delta-\phi(i+1)+\phi'(i+1)x(i+1)/k(i)\right]\big), \quad i = 1, \ldots N
\end{aligned}
$$

where $\phi(i) = \phi(x(i)/k(i-1))$.

The first consumption level is derived with the resource constraint

$$c(1) = \mathcal{N}(y - x - g) - \sum_{j=2}^{\mathcal{N}} c(j).$$

We can back out $L^s(j)$'s from the labor demand functions:

$$L^s(j) = \left(\frac{\mu^{j-1}\bar{\omega}}{\omega}\right)^{\frac{1}{1-v}} \ \frac{1}{N}\sum_{i=1}^{N} L^d(i) = \left(\frac{\mu^{j-1}\bar{\omega}}{\omega}\right)^{\frac{1}{1-v}} \ L^d$$

Note that $L^s(j)$ is a function of the total labor demand and μ's because

$$\bar{\omega}/\omega = \left[\frac{1}{\mathcal{N}}\left(1 + \mu^{\frac{v}{1-v}} + \dots \mu^{\frac{(\mathcal{N}-1)v}{1-v}}\right)\right]^{\frac{v-1}{v}}.$$

With consumptions, money demands and labor supplies, I can compute all derivatives of utility. Using U_m's and U_l's, I have

$$\zeta(j) = U_m(j)/(1 - \beta/\mu)$$

and the steady state real wage:

$$\frac{\omega}{p} = -\frac{1}{v}\left(\frac{U_l(1) + U_l(2)\beta\mu^{\frac{1}{1-v}} + U_l(3)\beta^2\mu^{\frac{2}{1-v}} + \ldots + U_l(\mathcal{N})\beta^{\mathcal{N}-1}\mu^{\frac{\mathcal{N}-1}{1-v}}}{\zeta(1) + \zeta(2)\beta\mu^{\frac{v}{1-v}} + \zeta(3)\beta^2\mu^{\frac{2v}{1-v}} + \ldots + \zeta(\mathcal{N})\beta^{\mathcal{N}-1}\mu^{\frac{(\mathcal{N}-1)v}{1-v}}}\right)$$

We can use the following equations to check that we have a fixed point:

$$mc(i) = \bar{\omega}/(pF_l(i)), \quad i = 1, \ldots, N$$

$$1 = \frac{1}{\theta}\left(\frac{mc(1) + mc(2)\beta\mu^{\frac{1}{1-\theta}} + mc(3)\beta^2\mu^{\frac{2}{1-\theta}} + \ldots + mc(N)\beta^{N-1}\mu^{\frac{N-1}{1-\theta}}}{1 + \beta\mu^{\frac{\theta}{1-\theta}} + \beta^2\mu^{\frac{2\theta}{1-\theta}} + \ldots + \beta^{N-1}\mu^{\frac{(N-1)\theta}{1-\theta}}}\right)$$

$$U_1(j) = \zeta(j) - \beta U_2(j) \quad j = 1, \ldots, \mathcal{N}$$

$$U_1(j) = U_1(1) \quad j = 2, \ldots \mathcal{N}.$$

2.3. Solving the Linearized System

The system of equations that we solve has $N + 2\mathcal{N} + 3$ **dynamic** equations:

- 1 pricing equation, (2.1)
- 1 wage-setting equations (2.3);
- N Euler equations for capital (1.26);
- $\mathcal{N}$ dynamic consumption equations (1.9);
- $\mathcal{N}$ money demand equations (1.10);
- 1 resource equation (1.31)
- 1 interest rate equation (1.29)
- and **static** equations and definitions that determine:
  - $\hat{y}_i$ from (1.3);
  - $\hat{p}$ from (1.4);
  - $\hat{L}_i^d$ from (1.14)
  - $\hat{x}_i$ from (1.16)
  - $\hat{mc}_i$ from (1.27)
  - $\hat{L}_j^s$ from (1.6)
  - $\hat{w}$ from (1.28)

We can write the system of equations in terms of a subset of our variables and back out all variables via the static conditions listed above. We turn to this next.

We introduce a new index $\aleph = \max(N, \mathcal{N})$ because we will need to record sufficient lags and leads of the variables. We will use the following vectors in our computation:

$$z_t = [\hat{p}_{t-1}, \hat{\omega}_{t-1}, \hat{k}_{1,t}, \ldots \hat{k}_{N,t}, \hat{c}_{1,t}, \ldots \hat{c}_{\mathcal{N},t}, \hat{m}^d_{1,t}, \ldots \hat{m}^d_{\mathcal{N},t}, y_t, r_t]' \qquad (n_z \times 1)$$

$$\begin{aligned} X_t = [&\hat{p}_{t-2}, \ldots, \hat{p}_{t-N}, \hat{p}_{t-(N+1)}, \hat{p}_{t-(N+2)}, \hat{p}_{t-(N+3)}, \hat{\omega}_{t-2}, \ldots, \hat{\omega}_{t-\mathcal{N}}, \\ &\hat{k}_{1,t-1}, \ldots, \hat{k}_{N,t-1}, \hat{c}_{1,t-1}, \ldots, \hat{c}_{\mathcal{N},t-1}, \hat{y}_{t-1}, \hat{y}_{t-2}, r_{t-1}, r_{t-2}, r_{t-3}] \qquad (n_X \times 1) \end{aligned}$$

$$\mathcal{Z}_t = [z_{t+\aleph-1}, z_{t+\aleph-2}, \ldots, z_t, X_t, \epsilon_{r,t},$$

$$\hat{g}_{t+\aleph-1},\ldots,\hat{g}_t,\hat{a}_{t+\aleph-1},\ldots,\hat{a}_t,\hat{\varphi}_{t+\aleph-1},\ldots,\hat{\varphi}_t]'$$

$$Z_t = [z_t, z_{t-1},\ldots,z_{t-\aleph-1}]' \qquad (n_Z \times 1)$$

$$S_t = [\epsilon_{r,t},\epsilon_{r,t-1},\hat{g}_t,\hat{g}_{t-1},\hat{a}_t,\hat{a}_{t-1},\hat{\varphi}_t,\hat{\varphi}_{t-1}]' \qquad (n_S \times 1)$$

The vector z_t contains the choice variables at time t. It has $n_z = N + 2\mathcal{N} + 4$ elements. The vector X_t are the state variables at time t. There are $n_X = 2N + 2\mathcal{N} + 6$ state variables. The vector $\mathcal{Z}_t$ contains all variables that appear in the residual equations. The vectors Z_t and S_t are used when we characterize the solution,

$$Z_t = AZ_{t-1} + BS_t \tag{2.4}$$

with $S_{t+1} = \mathcal{P}S_t + \epsilon_{t+1}$. The vector Z has $n_Z = (\aleph + 2)n_z$ elements and S has $n_S = 8$ elements.

The residual equations can be written succinctly as follows:

$$\mathcal{E}\left[A_1\begin{bmatrix} X_{t+1} \\ z_{t+\aleph-1} \\ \vdots \\ z_{t+1}\end{bmatrix} + A_2\begin{bmatrix} X_t \\ z_{t+\aleph-2} \\ \vdots \\ z_t\end{bmatrix} + \text{shock terms}|\Omega_t\right] = 0$$

where $\mathcal{E}$ implies that expectations are taken – but we will assume that different information sets for the different residual equations. For our example, the residuals are denoted $R(\mathcal{Z})$ and the matrix A_1 is given by

$$A_1 = \begin{bmatrix} I_{n_X,n_X} & 0_{n_X,n_z} & 0_{n_X,n_z} & \ldots & 0_{n_X,n_z} \\ 0_{n_z,n_X} & \frac{dR}{d\mathcal{Z}}(:,1:n_z) & \frac{dR}{d\mathcal{Z}}(:,n_z{+}1:2n_z) & \ldots & \frac{dR}{d\mathcal{Z}}(:,(\aleph{-}2)n_z{+}1:(\aleph{-}1)n_z) \\ 0_{n_z,n_X} & 0_{n_z,n_z} & I_{n_z,n_z} & \ldots & 0_{n_z,n_z} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n_z,n_X} & 0_{n_z,n_z} & 0_{n_z,n_z} & \ldots & I_{n_z,n_z} \end{bmatrix} \tag{2.5}$$

and matrix A_2 is given by:

$$A_2 = \begin{bmatrix} -\mathcal{I}_1 & 0_{n_X,n_z} & \ldots & 0_{n_X,n_z} & -\mathcal{I}_2 \\ \frac{dR}{d\mathcal{Z}}(:,\aleph n_z{+}1:\aleph n_z{+}n_X) & 0_{nz,nz} & \ldots & 0_{n_z,n_z} & \frac{dR}{d\mathcal{Z}}(:,(\aleph{-}1)n_z{+}1:\aleph n_z) \\ 0_{n_z,n_X} & I_{n_z,n_z} & 0_{n_z,n_z} & \ldots & 0_{n_z,n_z} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n_z,n_X} & 0_{n_z,n_z} & \ldots & I_{n_z,n_z} & 0_{n_z,n_z} \end{bmatrix}. \tag{2.6}$$

The matrices $\mathcal{I}_1$ and $\mathcal{I}_2$ in A_2 are given by

$$\mathcal{I}_1 = \begin{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} & 0_{N+2,\mathcal{N}-1} & 0_{N+2,N+\mathcal{N}} & 0_{N+2,5} \\ 0_{\mathcal{N}-1,N+2} & \begin{bmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} & 0_{\mathcal{N}-1,N+\mathcal{N}} & 0_{\mathcal{N}-1,5} \\ 0_{N+\mathcal{N},N+2} & 0_{N+\mathcal{N},\mathcal{N}-1} & 0_{N+\mathcal{N},N+\mathcal{N}} & 0_{N+\mathcal{N},5} \\ 0_{5,N+2} & 0_{5,\mathcal{N}-1} & 0_{5,N+\mathcal{N}} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{bmatrix}$$

$$\mathcal{I}_2 = \begin{bmatrix} \begin{bmatrix} 1 \\ 0_{N+1,1} \end{bmatrix} & 0_{N+2,1} & 0_{N+2,N+\mathcal{N}} & 0_{N+2,\mathcal{N}} & 0_{N+2,2} \\ 0_{\mathcal{N}-1,1} & \begin{bmatrix} 1 \\ 0_{\mathcal{N}-2,1} \end{bmatrix} & 0_{\mathcal{N}-1,N+\mathcal{N}} & 0_{\mathcal{N}-1,\mathcal{N}} & 0_{\mathcal{N}-1,2} \\ 0_{N+\mathcal{N},1} & 0_{N+\mathcal{N},1} & I_{N+\mathcal{N},N+\mathcal{N}} & 0_{N+\mathcal{N},\mathcal{N}} & 0_{N+\mathcal{N},2} \\ 0_{5,1} & 0_{5,1} & 0_{5,N+\mathcal{N}} & 0_{5,\mathcal{N}} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

Using the method laid out in Blanchard and Kahn (1980), we construct eigenvalues of $-A_1^{-1}A_2$ if A_1 is invertible and generalized eigenvalues otherwise. Then, ignoring shock terms, I have

$$\begin{bmatrix} X_{t+1} \\ z_{t+\aleph-1} \\ \vdots \\ z_{t+1} \end{bmatrix} = V\Lambda V^{-1} \begin{bmatrix} X_t \\ z_{t+\aleph-2} \\ \vdots \\ z_t \end{bmatrix}.$$

We can sort eigenvalues inside and outside the unit circle. If there are n_X stable eigenvalues (which is the number of state variables in X), then I have a locally determinate system.

Suppose that the eigenvectors in V and eigenvalues in Λ are sorted so that the upper left partition of Λ contains the stable eigenvalues. Then,

$$X_{t+1} = V_{11}\Lambda_1 V_{11}^{-1} X_t$$

$$\begin{bmatrix} z_{t+\aleph-2} \\ \vdots \\ z_t \end{bmatrix} = V_{21}V_{11}^{-1} X_t.$$

The last n_z elements imply a relationship between the decision variables z and the state variables X. If I want to write the system as (2.4), then I can use this relationship between z and X to fill in the elements of A. In particular, we set

$$A(1:n_z, 1:n_z:(N+2)n_z) \;= A_{zX}(:, 1:N+2)$$

$$A(1:n_z, 2:n_z:(\mathcal{N}-1)n_z) \;= A_{zX}(:, N+3:N+\mathcal{N}+1)$$

$$A(1:n_z, 3:N+2) \;= A_{zX}(:, N+\mathcal{N}+2:2N+\mathcal{N}+1)$$

$$A(1:n_z, N+3:N+\mathcal{N}+2) \;= A_{zX}(:, 2N+\mathcal{N}+2:2N+2\mathcal{N}+1)$$

$$A(1:n_z, n_z-1) \;= A_{zX}(:, 2N+2\mathcal{N}+2)$$

$$A(1:n_z, 2n_z-1) \;= A_{zX}(:, 2N+2\mathcal{N}+3)$$

$$A(1:n_z, n_z) \;= A_{zX}(:, 2N+2\mathcal{N}+4)$$

$$A(1:n_z, 2n_z) \;= A_{zX}(:, 2N+2\mathcal{N}+5)$$

$$A(1:n_z, 3n_z) \;= A_{zX}(:, 2N+2\mathcal{N}+6)$$

$$A(n_z+1:n_Z, 1:n_Z-n_z) \;= I_{n_Z-n_z, n_Z-n_z}$$

where A_{zX} comes from $z_t = A_{zX}X_t$.

The next step is to compute B:

$$B = \begin{bmatrix} B_1 \\ 0_{n_z,n_S} \\ \vdots \\ 0_{n_z,n_S} \end{bmatrix} = \begin{bmatrix} I_{n_z,n_z} \\ 0_{n_z,n_z} \\ \vdots \\ 0_{n_z,n_z} \end{bmatrix} B_1 \equiv \mathcal{S}B_1. \tag{2.7}$$

Note that the dimension of B_1 is $n_z \times n_S$. We will use $\mathcal{S}$ below in order to reduce the problem of computing B to one of computing B_1.

To derive expressions for the elements of B, first note that the residuals can be written as follows:

$$\mathcal{E}\Big[a_0 Z_{t+\aleph-1} + a_1 Z_{t+\aleph-2} + \ldots + a_{\aleph-1} Z_t + a_\aleph Z_{t-1}$$
$$+ b_0 S_{t+\aleph-1} + b_1 S_{t+\aleph-2} + \ldots + b_{\aleph-1} S_t | \Omega_t\Big] = 0$$

Using the definitions of Z and $\mathcal{Z}$, we can write:

$$a_0 = [dR/d\mathcal{Z}(:, 1 : \aleph n_z), 0_{n_z, 2n_z}]$$

$$a_\aleph(:, 1 : n_z : (N+2)n_z) = dR/d\mathcal{Z}(:, \aleph n_z + 1 : \aleph n_z + N + 2)$$

$$a_\aleph(:, 2 : n_z : (\mathcal{N} - 1)n_z) = dR/d\mathcal{Z}(:, \aleph n_z + N + 3 : \aleph n_z + N + \mathcal{N} + 1)$$

$$a_\aleph(:, 3 : N + 2) = dR/d\mathcal{Z}(:, \aleph n_z + N + \mathcal{N} + 2 : \aleph n_z + 2N + \mathcal{N} + 1)$$

$$a_\aleph(:, N + 3 : N + \mathcal{N} + 2) = dR/d\mathcal{Z}(:, \aleph n_z + 2N + \mathcal{N} + 2 : \aleph n_z + 2N + 2\mathcal{N} + 1)$$

$$a_\aleph(:, n_z - 1 : n_z : 2n_z - 1) = dR/d\mathcal{Z}(:, \aleph n_z + 2N + 2\mathcal{N} + 2 : \aleph n_z + 2N + 2\mathcal{N} + 3)$$

$$a_\aleph(:, n_z : n_z : 3n_z) = dR/d\mathcal{Z}(:, \aleph n_z + 2N + 2\mathcal{N} + 4 : \aleph n_z + 2N + 2\mathcal{N} + 6)$$

$$b_k(:, 3) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 2 + k), \quad k = 0, \ldots \aleph - 1$$

$$b_k(:, 5) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 2 + \aleph + k), \quad k = 0, \ldots \aleph - 1$$

$$b_k(:, 7) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 2 + 2\aleph + k), \quad k = 0, \ldots \aleph - 1.$$

$$b_{\aleph-1}(:, 1) = dR/d\mathcal{Z}(:, \aleph n_z + n_X + 1)$$

Using the solution in (2.4) I get:

$$\mathcal{E}\Big[a_0 \left(A^\aleph Z_{t-1} + BS_{t+\aleph-1} + ABS_{t+\aleph-2} + \ldots + A^{\aleph-1} BS_t\right)$$
$$+ a_1 \left(A^{\aleph-1} Z_{t-1} + BS_{t+\aleph-2} + ABS_{t+\aleph-3} + \ldots + A^{\aleph-2} BS_t\right) + \ldots$$
$$+ a_{\aleph-1} \left(AZ_{t-1} + BS_t\right) + a_\aleph Z_{t-1}$$
$$+ b_0 S_{t+\aleph-1} + b_1 S_{t+\aleph-2} + \ldots + b_{\aleph-1} S_t | \Omega_t\Big] = 0 \qquad (2.8)$$

I need to derive expressions for $\mathcal{E}[\mathcal{M}S_{t+j}|\Omega_t]$ as a function of S_t, where $\mathcal{M}$ is assumed to be one of the coefficients in (2.8). First, using the fact that $S_{t+1} = \mathcal{P}S_t + \epsilon_{t+1}$ we have

$$\mathcal{E}[\mathcal{M}S_{t+j}|\Omega_t] = \mathcal{M}\mathcal{P}^j\mathcal{E}[S_t|\Omega_t].$$

The matrix $\mathcal{P}$ is assumed to be:

$$\mathcal{P} = \begin{bmatrix} \rho_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_\varphi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.9}$$

If $\Omega_t = \{\epsilon_s, \hat{g}_s, \hat{a}_s, \hat{\varphi}_s\}_{s=0}^{t-1}$ (as is the case for the pricing equations) and $\mathcal{P}$ is given by (2.9) then

$$\begin{aligned} &\mathcal{E}[\mathcal{M}_1\epsilon_t + \mathcal{M}_2\epsilon_{t-1} + \mathcal{M}_3\hat{g}_t + \mathcal{M}_4\hat{g}_{t-1} + \mathcal{M}_5\hat{a}_t + \mathcal{M}_6\hat{a}_{t-1} + \mathcal{M}_7\hat{\varphi}_t + \mathcal{M}_8\hat{\varphi}_{t-1}|\Omega_t] \\ &\quad = [0,\, \mathcal{M}_1\rho_r + \mathcal{M}_2,\, 0,\, \mathcal{M}_3\rho_g + \mathcal{M}_4,\, 0,\, \mathcal{M}_5\rho_a + \mathcal{M}_6,\, 0,\, \mathcal{M}_7\rho_\varphi + \mathcal{M}_8]\, S_t. \end{aligned}$$

If $\Omega_t = \{\epsilon_s, \hat{g}_s, \hat{a}_s, \hat{\varphi}_s\}_{s=0}^{t}$ (as is the case for the capital Euler equations, consumption equations, and the money demand equations), then $\mathcal{E}[\mathcal{M}S_t|\Omega_t] = \mathcal{M}S_t$.

For the model above,

$$\begin{aligned} &\mathcal{E}\Bigg[\bigg((a_0B + b_0)\mathcal{P}^{\aleph-1} + (a_0AB + a_1B + b_1)\mathcal{P}^{\aleph-2} + \ldots. + \\ &\qquad (a_0A^{\aleph-1}B + a_1A^{\aleph-2}B + \ldots a_{-1}B + b_{N-1})\mathcal{P}^0\bigg)S_t|\Omega_t\Bigg] = \mathcal{E}[\mathcal{M}S_t|\Omega_t] \equiv \hat{\mathcal{M}}S_t \end{aligned}$$

where $\mathcal{M}$ and $\hat{\mathcal{M}}$ both have dimension $n_z \times n_S$. Applying the method of undetermined coefficients, we want to find the matrix B_1 of (2.7) such that every element of $\hat{\mathcal{M}}$ is equal to 0. Because of the timing of the pricing decisions, this will imply $n_z \times n_S - 8$ equations in $n_z \times n_S - 8$ unknowns. In other words, the coefficients on ϵ_t, $\hat{g}_t$, $\hat{a}_t$, and $\hat{\varphi}_t$ in the first two rows of B_1 will be set equal to 0 because prices cannot respond immediately to these shocks.

The following steps are taken to set up the system of equations. First, I stack the nonzero elements of $\hat{\mathcal{M}}$ in a vector. Second, I construct a matrix $\mathcal{D}$ that relates this vector to vec$(\mathcal{M}')$. In my case, this relation is:

$$\underbrace{\begin{bmatrix} \mathcal{M}_{1,1}\rho_r + \mathcal{M}_{1,2} \\ \mathcal{M}_{1,3}\rho_g + \mathcal{M}_{1,4} \\ \mathcal{M}_{1,5}\rho_a + \mathcal{M}_{1,6} \\ \mathcal{M}_{1,7}\rho_\varphi + \mathcal{M}_{1,8} \\ \mathcal{M}_{2,1}\rho_r + \mathcal{M}_{2,2} \\ \mathcal{M}_{2,3}\rho_g + \mathcal{M}_{2,4} \\ \mathcal{M}_{2,5}\rho_a + \mathcal{M}_{2,6} \\ \mathcal{M}_{2,7}\rho_\varphi + \mathcal{M}_{2,8} \\ \mathcal{M}_{3,1} \\ \vdots \\ \mathcal{M}_{n_z,n_S} \end{bmatrix}}_{\text{nonzeros of vec}(\hat{\mathcal{M}}')} = \underbrace{\begin{bmatrix} \Psi & 0_{8,n_z n_S-16} \\ 0_{n_z n_S-16,16} & I_{n_z n_S-16} \end{bmatrix}}_{\mathcal{D}} \underbrace{\begin{bmatrix} \mathcal{M}_{1,1} \\ \mathcal{M}_{1,2} \\ \vdots \\ \mathcal{M}_{1,n_S} \\ \mathcal{M}_{2,1} \\ \mathcal{M}_{2,2} \\ \vdots \\ \mathcal{M}_{n_z,n_S} \end{bmatrix}}_{\text{vec}(\mathcal{M}')} \tag{2.10}$$

where

$$\Psi = I_{2,2} \otimes \begin{bmatrix} \rho_r & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_g & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_a & 1 \end{bmatrix} \qquad (8 \times 16).$$

Third, I set $\mathcal{D}$vec$(\mathcal{M}')$ equal to zero (which ensures that $\hat{\mathcal{M}} = 0$),

$$\begin{aligned} \mathcal{D}\text{vec}(\mathcal{M}') = \mathcal{D}\text{vec}\big([a_0\mathcal{S}B_1\mathcal{P}^{\aleph-1}]' + [a_0A\mathcal{S}B_1\mathcal{P}^{\aleph-2} + a_1\mathcal{S}B_1\mathcal{P}^{\aleph-2}]' + \ldots + \\ [a_0A^{\aleph-1}\mathcal{S}B_1\mathcal{P}^0 + a_1A^{\aleph-2}\mathcal{S}B_1\mathcal{P}^0 + \ldots + a_{\aleph-1}\mathcal{S}B_1\mathcal{P}^0]'\big) \\ + \mathcal{D}\text{vec}\left([b_0\mathcal{P}^{\aleph-1} + b_1\mathcal{P}^{\aleph-2} + \ldots + b_{\aleph-1}\mathcal{P}^0]'\right) \\ \\ \equiv \mathcal{Q}\text{vec}(B_1') + \mathcal{R}. \end{aligned}$$

To construct $\mathcal{Q}$ we need to use the fact that vec(ABC) is equal to $[C' \otimes A]$vec(B). At this point, we can write the equation explicitly in terms of B_1 – or more precisely, the nonzero elements of B_1:

$$\text{vec}(B1')(\text{nonzero elements}) = -\left[\mathcal{Q}(:, \text{nonzero elements})\right]^{-1}\mathcal{R}.$$

For the model above, the nonzero elements of B_1 are all (i,j) except $(i,j) \in \{(1,1)$, (1,3), (1,5), (1,7), (2,1), (2,3), (2,5), $(2,7)\}$. These are the coefficients on contemporaneous shocks in the pricing decision rules.