Equations for BQUAH.M

The Blanchard-Quah structural VAR is found as follows. We run regressions of X on lags using OLS:

$$X_t = B_1 X_{t-1} + B_2 X_{t-2} + \ldots B_p X_{t-p} + v_t \tag{1}$$

with $Ev_t v_t' = \Omega$ found from the residuals of the equations. Invert (1) to get the Wold:

$$X_t = v_t + C_1 v_{t-1} + C_2 v_{t-2} + \ldots \tag{2}$$

where the C's satisfy

$$I = (I - B_1 L - B_2 L^2 \ldots - B_p L^p)(I + C_1 L + C_2 L^2 + \ldots)$$

for all values of L.

The Blanchard-Quah (BQ) structural MA is

$$X_t = A_0 \epsilon_t + A_1 \epsilon_{t-1} + A_2 v_{t-2} + \ldots$$

with $A_0 \epsilon_t = v_t$ and $A_j = C_j A_0$, $j \geq 1$.

The identifying restrictions for BQ are $E\epsilon_t \epsilon_t' = I$ and the (1,1) element of $\sum_{j=0}^{\infty} A_j$, or equivalently $[\sum_{j=0}^{\infty} C_j] A_0$, is equal to 0. This implies that $A_0 A_0' = \Omega$ and the (1,1) element of SA_0 equals 0, where $S = (I - B_1 - \ldots - B_p)^{-1} I$. This in turn implies a system of 4 equations and 4 unknowns:

$$\omega_{11} = a_{11}^2 + a_{12}^2 \tag{3}$$

$$\omega_{12} = a_{11} a_{21} + a_{12} a_{22} \tag{4}$$

$$\omega_{22} = a_{21}^2 + a_{22}^2 \tag{5}$$

$$0 = s_{11} a_{11} + s_{12} a_{21} \tag{6}$$

where a_{ij}, ω_{ij}, and s_{ij} are the elements of A_0, Ω, and S, respectively.

We will need to impose a sign convention since impulse responses could be positive or negative. Assume $a_{12} > 0$ so that productivity rises with technology and $a_{21} < 0$ so that hours fall with a demand shock (taxes in our case).

Eliminate a_{11} using the fact that $a_{11} = -s_{12} a_{21} / s_{11}$:

$$\omega_{11} = f^2 a_{21}^2 + a_{12}^2$$

$$\omega_{12} = f a_{21}^2 + a_{12} a_{22}$$

$$\omega_{22} = a_{21}^2 + a_{22}^2$$

where $f = -s_{12}/s_{11}$. Solve for a_{12} and a_{22}:

$$a_{12} = [\omega_{11} - f^2 a_{21}^2]^{1/2} \tag{7}$$
$$a_{22} = [\omega_{22} - a_{21}^2]^{1/2} \tag{8}$$

and substitute to get 2 equations, 2 unknowns:

$$\omega_{12} = f a_{21}^2 + [\omega_{11} - f^2 a_{21}^2]^{1/2}[\omega_{22} - a_{21}^2]^{1/2}.$$

Let $\lambda = a_{21}^2$ and we have a quadratic in λ:

$$(\omega_{12} - f\lambda)^2 = (\omega_{11} - f^2\lambda)(\omega_{22} - \lambda)$$

which can be writte out:

$$\omega_{12}^2 - 2f\lambda\omega_{12} + f^2\lambda^2 = \omega_{11}\omega_{22} - f^2\lambda\omega_{22} - \omega_{11}\lambda + f^2\lambda^2$$

and simplified as follows:

$$\lambda = \frac{\omega_{11}\omega_{22} - \omega_{12}^2}{\omega_{11} + f^2\omega_{22} - 2f\omega_{12}}.$$

Since we want $a_{21} < 0$, we set $a_{21} = -\sqrt{\lambda}$. Note the numerator of λ has to be positive because Ω is a variance-covariance matrix. The denominator is postive if $f\omega_{12} < 0$ or not too big.

With a_{21}, we have a_{12} and a_{22} by (7) and (8). We also have $a_{11} = f a_{21}$. To keep the sign convention correct, we do the following sequence of operations in our computer code given inputs Ω and S (and hence λ and f as defined above):

$$a_{21} = -\sqrt{\lambda}$$
$$a_{11} = f a_{21}$$
$$a_{12} = \sqrt{\omega_{11} - f^2\lambda}$$
$$a_{22} = \begin{cases} \sqrt{\omega_{22} - \lambda} & \text{if } \omega_{12} - f\lambda > 0 \\ -\sqrt{\omega_{22} - \lambda} & \text{otherwise} \end{cases}$$

where the conditional statement on the sign of $\omega_{12} - f\lambda$ uses (4) and the solutions for a_{11} and a_{21}.