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Technical Appendix: Intangible Capital and Measured Productivity*

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* The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1. Introduction

In this appendix, I derive first-order conditions necessary to compute equilibria for the baseline model (without financial frictions) and the extended model (with financial frictions) in Federal Reserve Bank of Minneapolis Staff Report 545. Standard techniques are used to compute log-linear approximations; these techniques rely on log-linearizing the equations derived here around the steady state. All codes and data needed to replicate the results of the paper are available at my website (http://users.econ.umn.edu/∼erm/data/sr545).

2. Baseline Model

I'll start with the household's problem followed by the firm's. I'll also show that the first order conditions are the same for a related problem with the household and firm combined. Here, as in the codes, I allow for stochastic variation in fiscal policy variables. In the main text, I only discuss results for time-varying TFPs.

2.1. Household Problem

Households choose consumption C_t and leisure L_t to maximize expected utility:

$$\max \; E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ \Big[\left(C_t/N_t\right) \left(L_t/N_t\right)^{\psi} \Big]^{1-\alpha} - 1 \Big\} / \left(1-\alpha\right) N_t \tag{2.1}$$

with the population equal to $N_t = N_0(1+g_n)^t$. The maximization is subject to the following per-period budget constraint:

$$(1+\tau_{ct}) \textstyle\sum_j P_{jt} C_{jt} \leq (1-\tau_{ht}) \sum_j W_{jt} H_{jt} + \sum_j \left[\left(\left(1-\tau_{dt}\right) D_{jt} + V_{jt} \right) S_{jt} - S_{jt+1} \right] + \Psi_t \tag{2.2}$$

where the subscript j indexes the sector of production, C_{jt} is consumption of goods made by firms in sector j which are purchased at price P_{jt}, H_{jt} is labor supplied to sector j which is paid W_{jt}, and D_{jt} are dividends paid to the owners of firms in sector j who have S_{jt} outstanding shares that sell at price V_{jt}. Taxes are paid on consumption purchases (τ_{ct}), labor earnings (τ_{ht}) and dividend earnings (τ_{dt}). Any revenues in excess of government purchases of goods and services are lump-sum rebated to the household in the amount Ψ_t.

The composite consumption and leisure that enter the utility function are given by

$$C_t = \Big[\textstyle\sum_j \omega_j C_{jt}^{\frac{\rho-1}{\rho}} \Big]^{\frac{\rho}{\rho-1}} \tag{2.3}$$

$$L_t = N_t - \textstyle\sum_j H_{jt}. \tag{2.4}$$

Notice that here, I assume CES for consumption and linear for hours. This can be extended if I want to assume that labor is not perfectly substitutable across sectors.

The Lagrangian for the household problem is given by

$$\mathcal{L}^H = \max \; E_0 \sum_{t=0}^{\infty} \Big\{ \beta^t \left(\sum_j \omega_j \left(C_{jt}/N_t\right)^{\frac{\rho-1}{\rho}} \right)^{\frac{(1-\alpha)\rho}{\rho-1}} \left(1 - \sum_j H_{jt}/N_t\right)^{\psi(1-\alpha)} / \left(1-\alpha\right) N_t$$
$$+ \mu_t \{ \left(1 - \tau_{ht}\right) \sum_j W_{jt} H_{jt} + \sum_j \left[\left(\left(1 - \tau_{dt}\right) D_{jt} + V_{jt} \right) S_{jt} - S_{jt+1} \right]$$
$$+ \Psi_t - \left(1 + \tau_{ct}\right) \sum_j P_{jt} C_{jt} \} \Big\}.$$

Taking derivatives with respect to C_{jt}, H_{jt}, and S_{jt+1} yields:

$$\begin{aligned} \mu_t \left(1 + \tau_{ct}\right) P_{jt} &= \beta^t \omega_j \left(\sum_j \omega_j \left(C_{jt}/N_t\right)^{\frac{\rho-1}{\rho}} \right)^{(1-\alpha)\rho/(\rho-1)-1} \left(1 - \sum_j H_{jt}/N_t\right)^{\psi(1-\alpha)} \left(C_{jt}/N_t\right)^{-\frac{1}{\rho}} \\ &= \beta^t \left(C_t/N_t\right)^{-\alpha} \left(L_t/N_t\right)^{\psi(1-\alpha)} \omega_j \left(C_{jt}/C_t\right)^{-\frac{1}{\rho}} \end{aligned} \tag{2.5}$$

$$\begin{aligned} \mu_t \left(1 - \tau_{ht}\right) W_{jt} &= \beta^t \psi \left(\sum_j \omega_j \left(C_{jt}/N_t\right)^{\frac{\rho-1}{\rho}} \right)^{\frac{(1-\alpha)\rho}{\rho-1}} \left(1 - \sum_j H_{jt}/N_t\right)^{\psi(1-\alpha)-1} \\ &= \beta^t \psi \left(C_t/N_t\right)^{1-\alpha} \left(L_t/N_t\right)^{\psi(1-\alpha)-1} \end{aligned} \tag{2.6}$$

$$\mu_t V_{jt} = E_t \mu_{t+1} \left(\left(1 - \tau_{dt+1}\right) D_{jt+1} + V_{jt+1} \right) \tag{2.7}$$

where μ is the multiplier for the budget constraint. Taking ratios of prices using the equations in (2.5) yields

$$P_{jt} = P_{kt} C_{kt}^{\frac{1}{\rho}} / \omega_k \left(\omega_j C_{jt}^{\frac{-1}{\rho}} \right). \tag{2.8}$$

I can use this condition and the definition of the aggregate price index P_t, which is given by

$$P_t = \left(\sum_j \omega_j^{\rho} P_{jt}^{1-\rho} \right)^{\frac{1}{1-\rho}} \tag{2.9}$$

to solve for the sectoral consumptions, C_{jt}. More specifically, I raise both sides of (2.8) by $1-\rho$, weight terms by ω_j^{ρ}, and sum:

$$\begin{aligned} \sum_j \omega_j^{\rho} P_{jt}^{1-\rho} &= \left(P_{kt} C_{kt}^{\frac{1}{\rho}} / \omega_k \right)^{1-\rho} \sum_j \omega_j^{\rho} \left(\omega_j C_{jt}^{\frac{-1}{\rho}} \right)^{1-\rho} \\ &= \left(P_{kt} C_{kt}^{\frac{1}{\rho}} / \omega_j \right)^{1-\rho} \sum_j \omega_j C_{jt}^{\frac{\rho-1}{\rho}} \\ &= \left(P_{kt} C_{kt}^{\frac{1}{\rho}} / \omega_k \right)^{1-\rho} C_t^{\frac{\rho-1}{\rho}}. \end{aligned} \tag{2.10}$$

Using the fact that the left-hand side of (2.10) is $P_t^{1-\rho}$, I can easily show that:

$$C_{jt} = C_t \left(\frac{P_{jt}}{\omega_j P_t} \right)^{-\rho}. \tag{2.11}$$

Note that, in equilibrium, $P_t C_t = \sum_j P_{jt} C_{jt}$.

Combining first-order conditions (2.5) and (2.6) for consumption and leisure yields the following intratemporal condition:

$$\frac{\psi C_t}{L_t} = \omega_j \left(C_{jt}/C_t\right)^{\frac{-1}{\rho}} \frac{\left(1-\tau_{ht}\right) W_{jt}}{\left(1+\tau_{ct}\right) P_{jt}} = \frac{\left(1-\tau_{ht}\right) W_{jt}}{\left(1+\tau_{ct}\right) P_t}. \tag{2.12}$$

Since (2.12) holds for all sectors j, the wage must be the same across sectors, and I can replace W_{jt} by W_t. The final set of equations is the household's dynamic Euler equations:

$$V_{jt} \frac{U_{ct}}{P_t\left(1+\tau_{ct}\right)} = \beta E_t \frac{U_{ct+1}}{P_{t+1}\left(1+\tau_{ct+1}\right)} \left(\left(1-\tau_{dt+1}\right) D_{jt+1} + V_{jt+1}\right) \tag{2.13}$$

where U_{ct} is short-hand for the marginal utility function evaluated at C_t/N_t and L_t/N_t.

2.2. Firm Problem

Firms maximize the present value of after-tax dividends on behalf of their owners, that is, the households:

$$\max \; E_0 \sum_{t=0}^{\infty} \beta^t U_{ct} \left(1-\tau_{dt}\right) D_{jt} / \left[P_t\left(1+\tau_{ct}\right)\right] \tag{2.14}$$

where

$$\begin{aligned} D_{jt} = {} & P_{jt} Y_{jt} + Q_{jt} X_{Ijt} - W_{jt} H_{jt} - \left(1+\tau_{xt}\right) \textstyle\sum_l P_{lt} X_{Tljt} - \sum_l P_{lt} M_{ljt} - \sum_l Q_{lt} X_{Iljt} \\ & - \tau_{pt} \{ P_{jt} Y_{jt} + Q_{jt} X_{Ijt} - W_{jt} H_{jt} - \left(\delta_T + \tau_{kt}\right) P_{jt} K_{Tjt} \\ & - \textstyle\sum_l P_{lt} M_{ljt} - \sum_l Q_{lt} X_{Iljt} \} - \tau_{kt} P_{jt} K_{Tjt} \end{aligned} \tag{2.15}$$

$$Y_{jt} = \left(K_{Tjt}^1\right)^{\theta_j} \left(K_{Ijt}\right)^{\phi_j} \left(\textstyle\prod_l \left(M_{ljt}^1\right)^{\gamma_{lj}}\right) \left(Z_{jt}^1 H_{jt}^1\right)^{1-\theta_j-\phi_j-\gamma_j} \tag{2.16}$$

$$X_{Ijt} = \left(K_{Tjt}^2\right)^{\theta_j} \left(K_{Ijt}\right)^{\phi_j} \left(\textstyle\prod_l \left(M_{ljt}^2\right)^{\gamma_{lj}}\right) \left(Z_{jt}^2 H_{jt}^2\right)^{1-\theta_j-\phi_j-\gamma_j} \tag{2.17}$$

$$K_{Tjt+1} = \left(1-\delta_T\right) K_{Tjt} + \textstyle\prod_l X_{Tljt}^{\zeta_{lj}} \tag{2.18}$$

$$K_{Ijt+1} = \left(1-\delta_I\right) K_{Ijt} + \textstyle\prod_l X_{Iljt}^{\nu_{lj}} \tag{2.19}$$

$$M_{ljt} = M_{ljt}^1 + M_{ljt}^2 \tag{2.20}$$

The Lagrangian in this case is:

$$\begin{aligned} \mathcal{L}^F = E_0 \sum_{t=0}^{\infty} \lambda_t \Big\{ & \left(1-\tau_{pt}\right) P_{jt} \left(K_{Tjt}^1\right)^{\theta_j} \left(K_{Ijt}\right)^{\phi_j} \left(\prod \left(M_{ljt}^1\right)^{\gamma_{lj}}\right) \left(Z_{jt}^1 H_{jt}^1\right)^{1-\theta_j-\phi_j-\gamma_j} \\ & + \left(1-\tau_{pt}\right) Q_{jt} \left(K_{Tjt} - K_{Tjt}^1\right)^{\theta_j} \left(K_{Ijt}\right)^{\phi_j} \left(\prod \left(M_{ljt} - M_{ljt}^1\right)^{\gamma_{lj}}\right) \left(Z_{jt}^2 \left(H_{jt} - H_{jt}^1\right)\right)^{1-\theta_j-\phi_j-\gamma_j} \end{aligned}$$

$$- (1 - \tau_{pt}) W_{jt} H_{jt}$$

$$- (1 + \tau_{xt}) \textstyle\sum_l P_{lt} X_{Tljt}$$

$$- (1 - \tau_{pt}) \textstyle\sum_l P_{lt} M_{ljt}$$

$$- (1 - \tau_{pt}) \textstyle\sum_l Q_{lt} X_{Iljt}$$

$$+ [\tau_{pt} (\delta_T + \tau_{kt}) - \tau_{kt}] P_{jt} K_{Tjt} \Big\}$$

$$+ E_0 \sum_{t=0}^{\infty} \chi_{jt} \{(1 - \delta_T) K_{Tjt} + \textstyle\prod_l X_{Tljt}^{\zeta_{lj}} - K_{Tjt+1}\}$$

$$+ E_0 \sum_{t=0}^{\infty} \mu_{jt} \{(1 - \delta_I) K_{Ijt} + \textstyle\prod_l X_{Iljt}^{\nu_{lj}} - K_{Ijt+1}\}$$

where $\lambda_t = \beta^t U_{ct} (1 - \tau_{dt}) / [P_t (1 + \tau_{ct})]$.

Taking derivatives with respect to K_{Tjt+1}, K_{Ijt+1}, K_{Tjt}^1, H_{jt}, H_{jt}^1, M_{ljt}, M_{ljt}^1, X_{Tljt}, and X_{Iljt}, I get:

$$\chi_{jt} = E_t \chi_{jt+1} (1 - \delta_T) + E_t \lambda_{t+1} [(\tau_{pt+1} (\delta_T + \tau_{kt+1}) - \tau_{kt+1}) P_{jt+1}] \\ + E_t \lambda_{t+1} (1 - \tau_{pt+1}) \theta_j Q_{jt+1} X_{Ijt+1} / K_{Tjt+1}^2 \tag{2.21}$$

$$\mu_{jt} = E_t \mu_{jt+1} (1 - \delta_I) + E_t \lambda_{t+1} (1 - \tau_{pt+1}) \phi_j Q_{jt+1} X_{Ijt+1} / K_{Ijt+1} \} \\ + E_t \lambda_{t+1} (1 - \tau_{pt+1}) \phi_j P_{jt+1} Y_{jt+1} / K_{Ijt+1} \tag{2.22}$$

$$\lambda_t (1 - \tau_{pt}) P_{jt} Y_{jt} / K_{Tjt}^1 = \lambda_t (1 - \tau_{pt}) Q_{jt} X_{Ijt} / K_{Tjt}^2 \tag{2.23}$$

$$\lambda_t (1 - \tau_{pt}) W_{jt} = \lambda_t (1 - \tau_{pt}) (1 - \theta_j - \phi_j - \gamma_j) Q_{jt} X_{Ijt} / H_{jt}^2 \tag{2.24}$$

$$\lambda_t (1 - \tau_{pt}) P_{jt} Y_{jt} / H_{jt}^1 = \lambda_t (1 - \tau_{pt}) Q_{jt} X_{Ijt} / H_{jt}^2 \tag{2.25}$$

$$\lambda_t (1 - \tau_{pt}) \gamma_{lj} Q_{jt} X_{Ijt} / M_{ljt}^2 = \lambda_t (1 - \tau_{pt}) P_{lt} \tag{2.26}$$

$$\lambda_t (1 - \tau_{pt}) \gamma_{lj} P_{jt} Y_{jt} / M_{ljt}^1 = \lambda_t (1 - \tau_{pt}) \gamma_{lj} Q_{jt} X_{Ijt} / M_{jt}^2 \tag{2.27}$$

$$\lambda_t (1 + \tau_{xt}) P_{lt} = \chi_{jt} \zeta_{lj} \mathcal{X}_{Tjt} / X_{Tljt} \tag{2.28}$$

$$\lambda_t (1 - \tau_{pt}) Q_{lt} = \mu_{jt} \nu_{lj} \mathcal{X}_{Ijt} / X_{Iljt} \tag{2.29}$$

where $\mathcal{X}_{Tjt} = \prod_l X_{Tljt}^{\zeta_{lj}}$ and $\mathcal{X}_{Ijt} = \prod_l X_{Iljt}^{\nu_{lj}}$.

Simplifying the equations in (2.21)-(2.22), I get:

$$\chi_{jt} = E_t \chi_{jt+1} (1 - \delta_T) + E_t \lambda_{t+1} [(\tau_{pt+1} (\delta_T + \tau_{kt+1}) - \tau_{kt+1}) P_{jt+1}] \\ + E_t \lambda_{t+1} (1 - \tau_{pt+1}) \theta_j Q_{jt+1} X_{Ijt+1} / K_{Tjt+1}^2 \tag{2.30}$$

$$\mu_{jt} = E_t \mu_{jt+1} (1 - \delta_I) + E_t \lambda_{t+1} (1 - \tau_{pt+1}) \phi_j Q_{jt+1} X_{Ijt+1} / K_{Ijt+1} \tag{2.31}$$

$$+ E_t \lambda_{t+1} \left(1 - \tau_{pt+1}\right) \phi_j P_{jt+1} Y_{jt+1} / K_{Ijt+1}$$

$$P_{jt} Y_{jt} / K^1_{Tjt} = Q_{jt} X_{Ijt} / K^2_{Tjt} \tag{2.32}$$

$$W_{jt} = \left(1 - \theta_j - \phi_j - \gamma_j\right) Q_{jt} X_{Ijt} / H^2_{jt} \tag{2.33}$$

$$P_{jt} Y_{jt} / H^1_{jt} = Q_{jt} X_{Ijt} / H^2_{jt} \tag{2.34}$$

$$\gamma_{lj} Q_{jt} X_{Ijt} / M^2_{ljt} = P_{lt} \tag{2.35}$$

$$P_{jt} Y_{jt} / M^1_{ljt} = Q_{jt} X_{Ijt} / M^2_{ljt} \tag{2.36}$$

$$\lambda_t \left(1 + \tau_{xt}\right) P_{lt} = \chi_{jt} \zeta_{lj} \mathcal{X}_{Tjt} / X_{Tljt} \tag{2.37}$$

$$\lambda_t \left(1 - \tau_{pt}\right) Q_{lt} = \mu_{jt} \nu_{lj} \mathcal{X}_{Ijt} / X_{Iljt} \tag{2.38}$$

Using the first-order conditions for the investments, I can rewrite the multipliers χ_{jt} and μ_{jt} in terms of λ_t as follows:

$$\chi_{jt} = \lambda_t \left(1 + \tau_{xt}\right) \prod_l \left[P_{lt}/\zeta_{lj}\right]^{\zeta_{lj}} \equiv \lambda_t \left(1 + \tau_{xt}\right) \pi^T_{jt} \tag{2.39}$$

$$\mu_{jt} = \lambda_t \left(1 - \tau_{pt}\right) \prod_l \left[Q_{lt}/\nu_{lj}\right]^{\nu_{lj}} \equiv \lambda_t \left(1 - \tau_{pt}\right) \pi^I_{jt}. \tag{2.40}$$

Using the first-order conditions for the intermediate goods, I can derive an intermediate goods price:

$$\begin{aligned} \pi^M_{jt} &= \gamma_j \prod_l \left[P_{lt}/\gamma_{lj}\right]^{\gamma_{lj}/\gamma_j} \\ &= \gamma_j \prod_l \left[P_{jt} Y_{jt} / M^1_{ljt}\right]^{\gamma_{lj}/\gamma_j} \\ &= \gamma_j P_{jt} Y_{jt} \prod_l \left[1/M^1_{ljt}\right]^{\gamma_{lj}/\gamma_j} \\ &= \gamma_j P_{jt} Y_{jt} / M^1_{jt} \end{aligned} \tag{2.41}$$

and similarly for $\pi^M_{jt} = \gamma_j Q_{jt} X_{Ijt} / M^2_{jt}$.

Substituting the multipliers and prices into the original equations, I have a simplified set of first-order conditions for the firms in sector j:

$$\begin{aligned} &\left(1 + \tau_{xt}\right) \left(1 - \tau_{dt}\right) U_{ct} \pi^T_{jt} / \left[P_t \left(1 + \tau_{ct}\right)\right] \\ &\quad = \beta E_t \left(1 - \tau_{dt+1}\right) U_{ct+1} P_{jt+1} / \left[P_{t+1} \left(1 + \tau_{ct+1}\right)\right] \end{aligned}$$

$$\cdot \left\{1 + (1-\tau_{pt+1})\left(\theta_j Y_{jt+1}/K^1_{Tts+1} - \delta_T - \tau_{kt+1}\right) + \left(\pi^T_{jt+1}/P_{jt+1}\left(1+\tau_{xt+1}\right) - 1\right)(1-\delta_T)\right\} \tag{2.42}$$

$$(1-\tau_{pt})(1-\tau_{dt})\,U_{ct}\pi^I_{jt}/\left[P_t\left(1+\tau_{ct}\right)\right] = \beta E_t\left(1-\tau_{pt+1}\right)\left(1-\tau_{dt+1}\right)U_{ct+1}Q_{jt+1}/\left[P_{t+1}\left(1+\tau_{ct+1}\right)\right] \cdot \left\{\phi_j(Q_{jt+1}X_{Ijt+1} + P_{jt+1}Y_{jt+1})/\left(Q_{jt+1}K_{Ijt+1}\right) + \pi^I_{jt+1}/Q_{jt+1}\left(1-\delta_I\right)\right\} \tag{2.43}$$

$$W_{jt} = (1-\theta_j-\phi_j-\gamma_j)\,Q_{jt}X_{Ijt}/H^2_{jt} \tag{2.44}$$

$$\frac{P_{jt}Y_{jt}}{Q_{jt}X_{Ijt}} = \frac{K^1_{Tjt}}{K^2_{Tjt}} = \frac{H^1_{jt}}{H^2_{jt}} = \frac{M^1_{ljt}}{M^2_{ljt}} \tag{2.45}$$

Using the first-order conditions, I can derive simple formulas for the income and cost shares in the model's input-output table:

$$\gamma_{lj} = \frac{P_{lt}M^1_{ljt}}{P_{jt}Y_{jt}} = \frac{P_{lt}M^2_{ljt}}{Q_{jt}X_{Ijt}} \tag{2.46}$$

$$\gamma_j = \frac{\pi^M_{jt}M^1_{jt}}{P_{jt}Y_{jt}} = \frac{\pi^M_{jt}M^2_{jt}}{Q_{jt}X_{Ijt}} \tag{2.47}$$

$$\zeta_{lj} = \frac{P_{lt}X_{Tljt}}{\pi^T_{jt}\mathcal{X}_{Tjt}} \tag{2.48}$$

$$\nu_{lj} = \frac{Q_{lt}X_{Iljt}}{\pi^I_{jt}\mathcal{X}_{Ijt}}, \tag{2.49}$$

where recall that $\pi^T_{jt} = \prod_l [P_{lt}/\zeta_{lj}]^{\zeta_{lj}}$ (ptvec in the codes), $\mathcal{X}_{Tjt} = \prod_l X^{\zeta_{lj}}_{Tljt}$ (prodxt in the codes), $\pi^I_{jt} = \prod_l [P_{lt}/\nu_{lj}]^{\nu_{lj}}$ (pivec in the codes), and $\mathcal{X}_{Ijt} = \prod_l X^{\nu_{lj}}_{Iljt}$ (prodxi in the codes). These formulas will be useful when matching up the model parameters with the benchmark BEA input-output table.

2.3. Steady State

I can use the first-order conditions to compute the steady state variables of the detrended system (with lower case letters indicating that the variable is stationary):

$$(1+\tau_x)\,\pi^T_j/p_j = \tilde{\beta}\left\{1 + (1-\tau_p)\left(\theta_j y_j/k^1_{Tj} - \delta_T - \tau_k\right) + \left(\pi^T_j/p_j\left(1+\tau_x\right) - 1\right)(1-\delta_T)\right\}$$

$$\pi^I_j/q_j = \tilde{\beta}\left\{\phi_j\left(q_j x_{Ij} + p_j y_j\right)/\left(q_j k_{Ij}\right) + \pi^I_j/q_j\left(1-\delta_I\right)\right\}$$

$$w_j = (1-\theta_j-\phi_j-\gamma_j)\, q_j x_{Ij}/h_j^2$$

$$\frac{p_j y_j}{q_j x_{Ij}} = \frac{k_{Tj}^1}{k_{Tj}^2} = \frac{h_j^1}{h_j^2} = \frac{m_{lj}^1}{m_{lj}^2}$$

$$c_j + \sum_l x_{Tjl} + \sum_l \left(m_{jl}^1 + m_{jl}^2\right) = y_j = \left(k_{Tj}^1\right)^{\theta_j} (k_{Ij})^{\phi_j} \prod_l \left(m_{lj}^1\right)^{\gamma_{lj}} \left(z_j^1 h_j^1\right)^{1-\theta_j-\phi_j-\gamma_j}$$

$$\sum_l x_{Ijl} = x_{Ij} = \left(k_{Tj}^2\right)^{\theta_j} (k_{Ij})^{\phi_j} \prod_l \left(m_{lj}^2\right)^{\gamma_{lj}} \left(z_j^2 h_j^2\right)^{1-\theta_j-\phi_j-\gamma_j}$$

$$[(1+g_n)(1+g_z)-1+\delta_T]\, k_{Tj} = \prod_l x_{Tlj}^{\zeta_{lj}}$$

$$[(1+g_n)(1+g_z)-1+\delta_I]\, k_{Ij} = \prod_l x_{Ilj}^{\nu_{lj}}$$

$$\frac{\psi c}{l} = \omega_j (c_j/c)^{-1/\rho} \frac{(1-\tau_h)\, w_j}{(1+\tau_c)\, p_j} = \frac{(1-\tau_h)\, w_j}{(1+\tau_c)\, p}.$$

To solve these equations, I make an initial guess of $3J$ variables: pc, $\{p_j\}$ (with p_1 normalized), $\{q_j\}$, and $\{h_j\}$. Given these variables, I can use definitions to construct $\{c_j\}$, $\{\pi_j^T\}$, $\{\pi_j^I\}$, $\{\pi_j^M\}$.

A more efficient algorithm starts with the assumption that I know aggregate c, the price vector with element p_j (and $p_1 = 1$), the ratio of hours by sector with element h_j^2/h_j^1, and the vector of hours by sector with element h_j. Then, in the order listed, I make the following computations:

$$\begin{aligned}
\pi_j^M &= \gamma_j \prod_l (p_l/\gamma_{lj})^{\gamma_{lj}/\gamma_j} \\
\pi_j^T &= \prod_l (p_l/\zeta_{lj})^{\zeta_{lj}} \\
p &= \left(\sum_j \omega_j^\rho p_j^{1-\rho}\right)^{1/(1-\rho)} \\
c_j &= (\omega_j p/p_j)^\rho\, c \\
l &= 1 - \sum_j h_j \\
m_j^1/y_j &= \gamma_j p_j/\pi_j^M \\
h_j^1 &= h_j/\left(1 + h_j^2/h_j^1\right) \\
h_j^2 &= h_j - h_j^1 \\
q_j x_{Ij}/(p_j y_j) &= h_j^2/h_j^1 \\
x_{Ij}/y_j &= (q_j x_{Ij}/(p_j y_j))^{1-\phi_j} \left(z_j^2/z_j^1\right)^{1-\theta_j-\phi_j-\gamma_j} \\
q_j/p_j &= (q_j x_{Ij}/(p_j y_j))\,/\,(x_{Ij}/y_j) \\
q_j &= (q_j/p_j)\, p_j
\end{aligned}$$

$$
\begin{aligned}
\pi_j^I &= \prod_l \left(q_l/\nu_{lj}\right)^{\nu_{lj}} \\
k_{Ij}/y_j &= \tilde{\beta}\phi\left(q_j x_{Ij}/y_j + p_j\right)/\left(\pi_j^I\left(1-\tilde{\beta}\left(1-\delta_I\right)\right)\right) \\
ret &= \left(\left(1+\tau_x\right)\pi_j^T/p_j - \tilde{\beta}\left(1-\left(1-\tau_p\right)\left(\delta_T+\tau_k\right)+\left(1-\delta_T\right)\left(\left(1+\tau_x\right)\pi_j^T/p_j-1\right)\right)\right) \\
&\quad /\left(\tilde{\beta}\left(1-\tau_p\right)\right) \\
k_{Tj}^1/y_j &= \theta/ret \\
y_j &= \left(\left(k_{Tj}^1/y_j\right)^{\theta_j}\left(k_{Ij}/y_j\right)^{\phi_j}\left(m_j^1/y_j\right)^{\gamma_j}\right)^{1/(1-\theta_j-\phi_j-\gamma_j)} z_j^1 h_j^1 \\
k_{Tj}^1 &= \left(k_{Tj}^1/y_j\right)y_j \\
k_{Tj}^2 &= k_{Tj}^1 h_j^2/h_j^1 \\
k_{Tj} &= k_{Tj}^1 + k_{Tj}^2 \\
m_j^1 &= \left(m_j^1/y_j\right)y_j \\
x_{Ij} &= \left(x_{Ij}/y_j\right)y_j \\
k_{Ij} &= \left(k_{Ij}/y_j\right)y_j \\
q_j x_{Ij}/p_j &= \left(q_j x_{Ij}/\left(p_j y_j\right)\right)y_j \\
\mathcal{X}_{Tj} &= \left(\left(1+g_z\right)\left(1+g_n\right)-1+\delta_T\right)k_{Tj} \\
\mathcal{X}_{Ij} &= \left(\left(1+g_z\right)\left(1+g_n\right)-1+\delta_I\right)k_{Ij} \\
m_{lj}^1 &= \gamma_{lj}p_j y_j/p_l \\
m_{lj}^2 &= \gamma_{lj}q_j x_{Ij}/p_l \\
x_{Tlj} &= \zeta_{lj}\pi_j^T\mathcal{X}_{Tj}/p_l \\
x_{Ilj} &= \nu_{lj}\pi_j^I\mathcal{X}_{Ij}/q_l
\end{aligned}
$$

and then construct the 3×1 residual vector, R_j for each sector j:

$$
\begin{aligned}
&R_j(1) = y_j - \textstyle\sum_l x_{Tjl} - \sum_l m_{jl}^1 - \sum_l m_{jl}^2 - g_j - c_j \\
&R_j(2) = x_{Ij} - \textstyle\sum_l x_{Ijl} \\
&R_j(3) = \left(1-\tau_h\right)l\omega_j\left(1-\theta_j-\phi_j-\gamma_j\right)y_j c_j^{-1/\rho}/h_j^1 - \left(1+\tau_c\right)\psi c^{(\rho-1)/\rho}.
\end{aligned}
$$

To find the steady state, I update using Newton's method.

Given the vector of steady state values, the system of first-order conditions is log-linearized around it. Then a version of Vaughan's method is applied to compute an equilibrium.

2.4. Combining the Household and Firm

To check the equations derived separately for the households and the firms, consider solving the household's and firm's problem as one, namely:

$$\max \ E_t \sum_{t=0}^{\infty} \beta^t \left\{ \left[(C_t/N_t)\,(L_t/N_t)^{\psi} \right]^{1-\alpha} - 1 \right\} / \,(1-\alpha)\, N_t \tag{2.50}$$

subject to:

$$\begin{aligned}
&\textstyle\sum_j P_{jt}\{C_{jt} + \sum_l (X_{Tjl} + M_{jlt})\} + \sum_j Q_{jt} \sum_l X_{Ijlt} \\
&\quad = \textstyle\sum_j \{r_{Tjt}K_{Tjt} + r_{Ij}K_{Ijt} + W_{jt}H_{jt} + \sum_l P_{Mljt}M_{ljt}\} + Tr_t \\
&\qquad - \tau_{ct} \textstyle\sum_j P_{jt}C_{jt} \\
&\qquad - \tau_{xt} \textstyle\sum_j \sum_l P_{jt}X_{Tjl} \\
&\qquad - \tau_{ht} \textstyle\sum_j W_{jt}H_{jt} \\
&\qquad - \tau_{kt} \textstyle\sum_j P_{jt}K_{Tjt} \\
&\qquad - \tau_{pt} \textstyle\sum_j \{r_{Tjt}K_{Tjt} + r_{Ijt}K_{Ijt} - (\delta_T + \tau_{kt})\, P_{jt}K_{Tjt} - \sum_l Q_{lt}X_{Iljt}\} \\
&\qquad - \tau_{dt} \textstyle\sum_j \{r_{Tjt}K_{Tjt} + r_{Ijt}K_{Ijt} - (1+\tau_{xt}) \sum_l P_{lt}X_{Tljt} - \sum_l Q_{lt}X_{Iljt} - \tau_{kt}P_{jt}K_{Tjt} \\
&\qquad\qquad - \tau_{pt}\{r_{Tjt}K_{Tjt} + r_{Ijt}K_{Ijt} - (\delta_T + \tau_{kt})\, P_{jt}K_{Tjt} - \textstyle\sum_l Q_{lt}X_{Iljt}\}
\end{aligned} \tag{2.51}$$

$$C_t = \left[\textstyle\sum_j \omega_j C_{jt}^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}} \tag{2.52}$$

$$L_t = N_t - \textstyle\sum_j H_{jt} \tag{2.53}$$

$$K_{Tjt+1} = (1-\delta_T)\, K_{Tjt} + \textstyle\prod_l X_{Tljt}^{\zeta_{lj}} \tag{2.54}$$

$$K_{Ijt+1} = (1-\delta_I)\, K_{Ijt} + \textstyle\prod_l X_{Iljt}^{\nu_{lj}} \tag{2.55}$$

As before, the technologies are given by:

$$Y_{jt} = \left(K_{Tjt}^1\right)^{\theta_j} (K_{Ijt})^{\phi_j} \left(\textstyle\prod_l \left(M_{ljt}^1\right)^{\gamma_{lj}} \right) \left(Z_{jt}^1 H_{jt}^1\right)^{1-\theta_j-\phi_j-\gamma_j} \tag{2.56}$$

$$X_{Ijt} = \left(K_{Tjt}^2\right)^{\theta_j} (K_{Ijt})^{\phi_j} \left(\textstyle\prod_l \left(M_{ljt}^2\right)^{\gamma_{lj}} \right) \left(Z_{jt}^2 H_{jt}^2\right)^{1-\theta_j-\phi_j-\gamma_j} . \tag{2.57}$$

I assume that markets are competitive and, therefore, the the factor prices are equal to marginal products:

$$r_{Tjt} = \theta_j P_{jt} Y_{jt} / K_{Tjt}^1$$

$$= \theta_j Q_{jt} X_{Ijt} / K_{Tjt}^2 \tag{2.58}$$

$$r_{Ijt} = \phi_j \left(P_{jt} Y_{jt} + Q_{jt} X_{Ijt} \right) / K_{Ijt} \tag{2.59}$$

$$P_{Mljt} = \gamma_{lj} P_{jt} Y_{jt} / M_{ljt}^1$$

$$= \gamma_{lj} Q_{jt} X_{Ijt} / M_{ljt}^2 \tag{2.60}$$

$$W_{jt} = \left(1 - \theta_j - \phi_j - \gamma_j\right) P_{jt} Y_{jt} / H_{jt}^1$$

$$= \left(1 - \theta_j - \phi_j - \gamma_j\right) Q_{jt} X_{Ijt} / H_{jt}^2. \tag{2.61}$$

The resource constraints and the government budget constraint are as follows:

$$C_{jt} + G_{jt} + \textstyle\sum_l \left(X_{Tjlt} + M_{jlt}^1 + M_{jlt}^2 \right) = Y_{jt} \tag{2.62}$$

$$\textstyle\sum_l X_{Ijlt} = X_{Ijt} \tag{2.63}$$

$$\textstyle\sum_j P_{jt} G_{jt} + \Psi_t = \text{all taxes}, \tag{2.64}$$

where "all taxes" is short-hand for tax revenues from consumption taxes, labor taxes, dividend taxes, property taxes, and profits taxes.

The Lagrangian for the combined household problem is:

$$\begin{aligned}
\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ & \left[\left(C_t / N_t\right) \left(1 - \textstyle\sum_j H_{jt} / N_t\right)^{\psi} \right]^{1-\alpha} - 1 \Big\} / \left(1 - \alpha\right) N_t \\
& + \lambda_t \Big[r_{Tjt} K_{Tjt} + r_{Ijt} K_{Ijt} + W_{jt} H_{jt} + \textstyle\sum_l P_{Mljt} M_{ljt} + Tr_t \\
& \quad - \tau_{ct} \textstyle\sum_j P_{jt} C_{jt} \\
& \quad - \tau_{xt} \textstyle\sum_j \sum_l P_{jt} X_{Tjl} \\
& \quad - \tau_{ht} \textstyle\sum_j W_{jt} H_{jt} \\
& \quad - \tau_{kt} \textstyle\sum_j r_{Tjt} K_{Tjt} \\
& \quad - \tau_{pt} \textstyle\sum_j \{ r_{Tjt} K_{Tjt} + r_{Ijt} K_{Ijt} - \left(\delta_T + \tau_{kt}\right) P_{jt} K_{Tjt} - \sum_l Q_{lt} X_{Iljt} \} \\
& \quad - \tau_{dt} \textstyle\sum_j \{ r_{Tjt} K_{Tjt} + r_{Ijt} K_{Ijt} - \left(1 + \tau_{xt}\right) \sum_l P_{lt} X_{Tljt} - \sum_l Q_{lt} X_{Iljt} - \tau_{kt} P_{jt} K_{Tjt} \\
& \qquad\qquad - \tau_{pt} \{ r_{Tjt} K_{Tjt} + r_{Ijt} K_{Ijt} - \left(\delta_T + \tau_{kt}\right) P_{jt} K_{Tjt} - \textstyle\sum_l Q_{lt} X_{Iljt} \} \\
& \quad - \textstyle\sum_j P_{jt} \{ C_{jt} + \sum_l \left(X_{Tjl} + M_{jlt} \right) \} + \sum_j Q_{jt} \sum_l X_{Ijlt} \\
& + \mu_{Ct} \Big\{ \left[\textstyle\sum_j \omega_j C_{jt}^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}} - C_t \Big\} \\
& + \mu_{Tjt} \{ \left(1 - \delta_T\right) K_{Tjt} + \textstyle\prod_l X_{Tljt}^{\zeta_{lj}} - K_{Tjt+1} \} \\
& + \mu_{Ijt} \{ \left(1 - \delta_I\right) K_{Ijt} + \textstyle\prod_l X_{Iljt}^{\nu_{lj}} - K_{Ijt+1} \}
\end{aligned}$$

The first-order conditions with respect to C_t, C_{jt}, H_{jt}, K_{Tjt}, K_{Ijt}, X_{Tljt}, X_{Iljt}, and M_{ljt}:

$$0 = \beta^t U_{ct} - \mu_{Ct} \tag{2.65}$$

$$0 = \mu_{Ct}\omega_j \left[\sum_j \omega_j C_{jt}^{\frac{\rho-1}{\rho}}\right]^{\frac{1}{\rho-1}} C_{jt}^{-1/\rho} - \lambda_t P_{jt}\left(1+\tau_{ct}\right) \tag{2.66}$$

$$0 = -\beta^t U_{lt} + \lambda_t W_{jt}\left(1-\tau_{ht}\right) \tag{2.67}$$

$$\begin{aligned} 0 = &-\mu_{Tjt} + \mu_{Tjt+1}\left(1-\delta_T\right) + \lambda_{t+1}\big[r_{Tjt+1} - \tau_{kt+1}P_{jt+1} \\ &- \tau_{pt+1}\left(r_{Tjt+1} - \left(\delta_T + \tau_{kt+1}\right)P_{jt+1}\right) \\ &- \tau_{dt+1}\left(r_{Tjt+1} - \tau_{kt+1}P_{jt+1} - \tau_{pt+1}\left(r_{Tjt+1} - \left(\delta_T+\tau_{kt+1}\right)P_{jt+1}\right)\right)\big] \end{aligned} \tag{2.68}$$

$$0 = -\mu_{Ijt} + \mu_{Ijt+1}\left(1-\delta_I\right) + \lambda_{t+1}\left[r_{Ijt+1} - \tau_{pt+1}r_{Ijt+1} - \tau_{dt+1}\left(r_{Ijt+1} - \tau_{pt+1}r_{Ijt+1}\right)\right] \tag{2.69}$$

$$0 = -\lambda_t\left(1+\tau_{xt}\right)\left(1-\tau_{dt}\right)P_{lt} + \mu_{Tjt}\zeta_{lj}\left[\prod_l X_{Tljt}^{\zeta_{lj}}\right]/X_{Tljt} \tag{2.70}$$

$$0 = -\lambda_t\left(1-\tau_{pt}\right)\left(1-\tau_{dt}\right)Q_{lt} + \mu_{Ijt}\nu_{lj}\left[\prod_l X_{Iljt}^{\nu_{lj}}\right]/X_{Iljt} \tag{2.71}$$

$$0 = \lambda_t\left(P_{Mljt} - P_{lt}\right) \tag{2.72}$$

I can solve for λ_t using the first two first order conditions, namely, (2.65) and (2.66), and I get:

$$\lambda_t = \frac{\beta^t U_{ct}}{1+\tau_{ct}}\left[\omega_j\left(\frac{C_t}{C_{jt}}\right)^{1/\rho}\frac{1}{P_{jt}}\right] \equiv \frac{\beta^t U_{ct}}{1+\tau_{ct}}\frac{1}{P_t}$$

where recall that $P_t = (\sum_j \omega_j^\rho P_{jt}^{1-\rho})^{\frac{1}{1-\rho}}$ and $P_t C_t = \sum_j P_{jt}C_{jt}$. I can solve for μ_{Tjt} and μ_{Ijt} by noting:

$$\begin{aligned} \frac{\mu_{Tjt}}{\lambda_t\left(1+\tau_{xt}\right)\left(1-\tau_{dt}\right)} &= \prod_l \left(\tfrac{\mu_{Tjt}}{\lambda_t(1+\tau_{xt})(1-\tau_{dt})}\right)^{\zeta_{lj}} \\ &= \prod_l \left(\frac{P_{lt}X_{Tljt}}{\zeta_{lj}\prod_l X_{Tljt}^{\zeta_{lj}}}\right)^{\zeta_{lj}} \\ &= \prod_l \left(\tfrac{P_{lt}}{\zeta_{lj}}\right)^{\zeta_{lj}} \\ &\equiv \pi_{jt}^T. \end{aligned} \tag{2.73}$$

and I can do a similar thing with intangibles using the relation:

$$\mu_{Ijt} = \lambda_t\left(1-\tau_{pt}\right)\left(1-\tau_{dt}\right)\pi_{jt}^I \tag{2.74}$$

Substituting the expressions for the multipliers into (2.65)-(2.72) and simplifying, I get:

$$\left(1+\tau_{xt}\right)\left(1-\tau_{dt}\right)\pi_{jt}^T U_{ct}/\left[P_t\left(1+\tau_{ct}\right)\right]$$

$$
\begin{aligned}
&= \beta E_t \left(1 - \tau_{dt+1}\right) U_{ct+1} P_{jt+1} / \left[P_{t+1} \left(1 + \tau_{ct+1}\right)\right] \\
&\qquad \cdot \left\{1 + \left(1 - \tau_{pt+1}\right) \left(r_{Tjt+1}/P_{jt+1} - \delta_T - \tau_{kt+1}\right)\right. \\
&\qquad\quad \left. + \left(\left(1 + \tau_{xt+1}\right) \pi^T_{jt+1}/P_{jt+1} - 1\right) \left(1 - \delta_T\right)\right\}
\end{aligned}
\tag{2.75}
$$

$$
\begin{aligned}
&\left(1 - \tau_{pt}\right) \left(1 - \tau_{dt}\right) \pi^I_{jt} U_{ct} / \left[P_t \left(1 + \tau_{ct}\right)\right] \\
&\quad = \beta E_t \left(1 - \tau_{pt+1}\right) \left(1 - \tau_{dt+1}\right) U_{ct+1} Q_{jt+1} / \left[P_{t+1} \left(1 + \tau_{ct+1}\right)\right] \\
&\qquad \cdot \left\{r_{Ijt+1}/Q_{jt+1} + \pi^I_{jt+1}/Q_{jt+1} \left(1 - \delta_I\right)\right\}
\end{aligned}
\tag{2.76}
$$

$$
\frac{U_{lt}}{U_{ct}} = \frac{\left(1 - \tau_{ht}\right) W_{jt}}{\left(1 + \tau_{ct}\right) P_t} \tag{2.77}
$$

as before. (See (2.12), (2.42), and (2.43).)

3. Adding Financial Frictions

Here, I update the model to include frictions analyzed in Jermann and Quadrini (AER 2012). Specifically, I assume that there are tax advantages of debt financing and adjustment costs incurred when firms pay dividends to the households.

3.1. Household Problem

The household problem now includes a choice of holding debt:

$$
\max\ E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left[\left(C_t/N_t\right) \left(L_t/N_t\right)^{\psi} \right]^{1-\alpha} - 1 \right\} / \left(1 - \alpha\right) N_t \tag{3.1}
$$

subject to the per-period budget constraint:

$$
\begin{aligned}
&\left(1 + \tau_{ct}\right) \textstyle\sum_j P_{jt} C_{jt} + \dfrac{B_{t+1}}{1 + r_t} \\
&\qquad \leq \left(1 - \tau_{ht}\right) \textstyle\sum_j W_{jt} H_{jt} + \sum_j \left[\left(\left(1 - \tau_{dt}\right) D_{jt} + V_{jt}\right) S_{jt} - S_{jt+1}\right] + B_t + \Psi_t
\end{aligned}
\tag{3.2}
$$

This adds one additional first order condition, namely:

$$
\mu_t = \left(1 + r_t\right) E_t \mu_{t+1}
$$

or,

$$
\frac{U_{ct}}{P_t \left(1 + \tau_{ct}\right)} = \beta \left(1 + r_t\right) E_t \frac{U_{ct+1}}{P_{t+1} \left(1 + \tau_{ct+1}\right)}. \tag{3.3}
$$

3.2. Firm Problem

The firm has the same objective function as before:

$$\max \; E_0 \sum_{t=0}^{\infty} \beta^t U_{ct} \left(1-\tau_{dt}\right) D_{jt} / \left[P_t \left(1+\tau_{ct}\right)\right]$$

but the equation for dividends has several more terms:

$$\begin{aligned} D_{jt} = {} & P_{jt}Y_{jt} + Q_{jt}X_{Ijt} - W_{jt}H_{jt} - \left(1+\tau_{xt}\right)\textstyle\sum_l P_{lt}X_{Tljt} - \sum_l P_{lt}M_{ljt} - \sum_l Q_{lt}X_{Iljt} - \tau_{kt}P_{jt}K_{Tjt} \\ & - \tau_{pt}\{P_{jt}Y_{jt} + Q_{jt}X_{Ijt} - W_{jt}H_{jt} - \left(\delta_T + \tau_{kt}\right) P_{jt}K_{Tjt} - \textstyle\sum_l P_{lt}M_{ljt} - \sum_l Q_{lt}X_{Iljt}\} \\ & + \frac{B_{jt+1}}{1+r_t\left(1-\tau_{bj}\right)} - B_{jt} - \kappa_j \left(D_{jt} - \bar{D}_j\right)^2 \end{aligned} \tag{3.4}$$

There is also an enforcement constraint:

$$\xi_{jt}\left(P_{jt}K_{Tjt+1} - \frac{B_{jt+1}}{1+r_t}\right) \geq P_{jt}Y_{jt} + Q_{jt}X_{Ijt} \tag{3.5}$$

which will be binding if τ_{bj} is sufficiently large.

The Lagrangian in this case is:

$$\begin{aligned} \mathcal{L}^F = {} & E_0 \sum_{t=0}^{\infty} \beta^t U_{ct} \left(1-\tau_{dt}\right) D_{jt} / \left[P_t \left(1+\tau_{ct}\right)\right] \\ & + E_0 \sum_{t=0}^{\infty} \lambda_{jt} \Big\{ -D_{jt} - \kappa_j \left(D_{jt} - \bar{D}_j\right)^2 - B_{jt} + \frac{B_{jt+1}}{1+r_t\left(1-\tau_{bj}\right)} \\ & \quad \left(1-\tau_{pt}\right) P_{jt} \left(K^1_{Tjt}\right)^{\theta_j} \left(K_{Ijt}\right)^{\phi_j} \left(\prod \left(M^1_{ljt}\right)^{\gamma_{lj}}\right) \left(Z^1_{jt}H^1_{jt}\right)^{1-\theta_j-\phi_j-\gamma_j} \\ & \quad + \left(1-\tau_{pt}\right) Q_{jt} \left(K_{Tjt} - K^1_{Tjt}\right)^{\theta_j} \left(K_{Ijt}\right)^{\phi_j} \left(\prod \left(M_{ljt} - M^1_{ljt}\right)^{\gamma_{lj}}\right) \left(Z^2_{jt}\left(H_{jt} - H^1_{jt}\right)\right)^{1-\theta_j-\phi_j-\gamma_j} \\ & \quad - \left(1-\tau_{pt}\right) W_{jt}H_{jt} \\ & \quad - \left(1+\tau_{xt}\right) \textstyle\sum_l P_{lt}X_{Tljt} \\ & \quad - \left(1-\tau_{pt}\right) \textstyle\sum_l P_{lt}M_{ljt} \\ & \quad - \left(1-\tau_{pt}\right) \textstyle\sum_l Q_{lt}X_{Iljt} \\ & \quad + \left[\tau_{pt}\left(\delta_T + \tau_{kt}\right) - \tau_{kt}\right] P_{jt}K_{Tjt} \Big\} \\ & \quad + E_0 \sum_{t=0}^{\infty} \chi_{jt} \{ \left(1-\delta_T\right) K_{Tjt} + \textstyle\prod_l X_{Tljt}^{\zeta_{lj}} - K_{Tjt+1} \} \\ & \quad + E_0 \sum_{t=0}^{\infty} \mu_{jt} \{ \left(1-\delta_I\right) K_{Ijt} + \textstyle\prod_l X_{Iljt}^{\nu_{lj}} - K_{Ijt+1} \} \\ & \quad + E_0 \sum_{t=0}^{\infty} \beta^t \varphi_{jt} \{ \xi_{jt} \left(P_{jt}K_{Tjt+1} - \frac{B_{jt+1}}{1+r_t}\right) \\ & \qquad - P_{jt} \left(K^1_{Tjt}\right)^{\theta_j} \left(K_{Ijt}\right)^{\phi_j} \left(\prod \left(M^1_{ljt}\right)^{\gamma_{lj}}\right) \left(Z^1_{jt}H^1_{jt}\right)^{1-\theta_j-\phi_j-\gamma_j} \\ & \qquad - Q_{jt} \left(K_{Tjt} - K^1_{Tjt}\right)^{\theta_j} \left(K_{Ijt}\right)^{\phi_j} \left(\prod \left(M_{ljt} - M^1_{ljt}\right)^{\gamma_{lj}}\right) \left(Z^2_{jt}\left(H_{jt} - H^1_{jt}\right)\right)^{1-\theta_j-\phi_j-\gamma_j} \} \end{aligned}$$

Most first-order conditions now change because of the enforcement condition, which has total revenues included. Taking derivatives of the Lagrangian with respect to K_{Tjt+1}, K_{Ijt+1}, K^1_{Tjt}, H_{jt}, H^1_{jt}, M_{ljt}, M^1_{ljt}, X_{Tljt}, X_{Iljt}, D_{jt}, and B_{jt+1}, I get:

$$\begin{aligned}\chi_{jt} - \beta^t \varphi_{jt}\xi_{jt}P_{jt} = {} & E_t\chi_{jt+1}(1-\delta_T) + E_t\lambda_{jt+1}\left[(\tau_{pt+1}(\delta_T+\tau_{kt+1}) - \tau_{kt+1})P_{jt+1}\right] \\ & + E_t\lambda_{jt+1}(1-\tau_{pt+1})\theta_j Q_{jt+1}X_{Ijt+1}/K^2_{Tjt+1} \\ & - E_t\beta^{t+1}\varphi_{jt+1}\theta_j Q_{jt+1}X_{Ijt+1}/K^2_{Tjt+1}\end{aligned} \quad (3.6)$$

$$\begin{aligned}\mu_{jt} = {} & E_t\mu_{jt+1}(1-\delta_I) + E_t\lambda_{jt+1}(1-\tau_{pt+1})\phi_j Q_{jt+1}X_{Ijt+1}/K_{Ijt+1} \\ & + E_t\lambda_{jt+1}(1-\tau_{pt+1})\phi_j P_{jt+1}Y_{jt+1}/K_{Ijt+1} \\ & - E_t\beta^{t+1}\varphi_{jt+1}\phi_j\left(P_{jt+1}Y_{jt+1} + Q_{jt+1}X_{Ijt+1}\right)/K_{Ijt+1}\end{aligned} \quad (3.7)$$

$$\left(\lambda_{jt}(1-\tau_{pt}) - \beta^t\varphi_{jt}\right)P_{jt}Y_{jt}/K^1_{Tjt} = \left(\lambda_{jt}(1-\tau_{pt}) - \beta^t\varphi_{jt}\right)Q_{jt}X_{Ijt}/K^2_{Tjt} \quad (3.8)$$

$$\lambda_{jt}(1-\tau_{pt})W_{jt} = \left(\lambda_{jt}(1-\tau_{pt}) - \beta^t\varphi_{jt}\right)(1-\theta_j-\phi_j-\gamma_j)Q_{jt}X_{Ijt}/H^2_{jt} \quad (3.9)$$

$$\left(\lambda_{jt}(1-\tau_{pt}) - \beta^t\varphi_{jt}\right)P_{jt}Y_{jt}/H^1_{jt} = \left(\lambda_{jt}(1-\tau_{pt}) - \beta^t\varphi_{jt}\right)Q_{jt}X_{Ijt}/H^2_{jt} \quad (3.10)$$

$$\left(\lambda_{jt}(1-\tau_{pt}) - \beta^t\varphi_{jt}\right)\gamma_{lj}Q_{jt}X_{Ijt}/M^2_{ljt} = \lambda_{jt}(1-\tau_{pt})P_{lt} \quad (3.11)$$

$$\left(\lambda_{jt}(1-\tau_{pt}) - \beta^t\varphi_{jt}\right)\gamma_{lj}P_{jt}Y_{jt}/M^1_{ljt} = \left(\lambda_{jt}(1-\tau_{pt}) - \beta^t\varphi_{jt}\right)\gamma_{lj}Q_{jt}X_{Ijt}/M^2_{jt} \quad (3.12)$$

$$\lambda_{jt}(1+\tau_{xt})P_{lt} = \chi_{jt}\zeta_{lj}\mathcal{X}_{Tjt}/X_{Tljt} \quad (3.13)$$

$$\lambda_{jt}(1-\tau_{pt})Q_{lt} = \mu_{jt}\nu_{lj}\mathcal{X}_{Ijt}/X_{Iljt} \quad (3.14)$$

$$\beta^t U_{ct}(1-\tau_{dt})/\left[P_t(1+\tau_{ct})\right] - \lambda_{jt}\left(1+2\kappa_j\left(D_{jt}-\bar{D}_j\right)\right) \quad (3.15)$$

$$\lambda_{jt} = (1+r_t(1-\tau_{bj}))\left(\lambda_{jt+1} + \beta^t\varphi_{jt}\xi_{jt}/(1+r_t)\right) \quad (3.16)$$

where $\mathcal{X}_{Tjt} = \prod_l X^{\zeta_{lj}}_{Tljt}$ and $\mathcal{X}_{Ijt} = \prod_l X^{\nu_{lj}}_{Iljt}$.

After simplifying, the equations in (3.6)-(3.16) can be written as follows:

$$\begin{aligned}\chi_{jt} - \beta^t \varphi_{jt}\xi_{jt}P_{jt} = {} & E_t\chi_{jt+1}(1-\delta_T) + E_t\lambda_{jt+1}\left[(\tau_{pt+1}(\delta_T+\tau_{kt+1}) - \tau_{kt+1})P_{jt+1}\right] \\ & + E_t\lambda_{jt+1}(1-\tau_{pt+1})\theta_j Q_{jt+1}X_{Ijt+1}/K^2_{Tjt+1} \\ & - E_t\beta^{t+1}\varphi_{jt+1}\theta_j Q_{jt+1}X_{Ijt+1}/K^2_{Tjt+1}\end{aligned} \quad (3.17)$$

$$\begin{aligned}\mu_{jt} = {} & E_t\mu_{jt+1}(1-\delta_I) + E_t\lambda_{jt+1}(1-\tau_{pt+1})\phi_j Q_{jt+1}X_{Ijt+1}/K_{Ijt+1} \\ & + E_t\lambda_{jt+1}(1-\tau_{pt+1})\phi_j P_{jt+1}Y_{jt+1}/K_{Ijt+1} \\ & - E_t\beta^{t+1}\varphi_{jt+1}\phi_j\left(P_{jt+1}Y_{jt+1} + Q_{jt+1}X_{Ijt+1}\right)/K_{Ijt+1}\end{aligned} \quad (3.18)$$

$$\lambda_{jt} = (1+r_t(1-\tau_{bj}))\left(E_t\lambda_{jt+1} + \beta^t\varphi_{jt}\xi_{jt}/(1+r_t)\right) \quad (3.19)$$

$$P_{jt}Y_{jt}/K^1_{Tjt} = Q_{jt}X_{Ijt}/K^2_{Tjt} \tag{3.20}$$

$$P_{jt}Y_{jt}/H^1_{jt} = Q_{jt}X_{Ijt}/H^2_{jt} \tag{3.21}$$

$$P_{jt}Y_{jt}/M^1_{ljt} = Q_{jt}X_{Ijt}/M^2_{jt} \tag{3.22}$$

$$W_{jt} = \left(1 - \frac{\beta^t \varphi_{jt}}{\lambda_{jt}\left(1-\tau_{pt}\right)}\right)\left(1-\theta_j-\phi_j-\gamma_j\right) Q_{jt}X_{Ijt}/H^2_{jt} \tag{3.23}$$

$$P_{lt} = \left(1 - \frac{\beta^t \varphi_{jt}}{\lambda_{jt}\left(1-\tau_{pt}\right)}\right)\gamma_{lj} Q_{jt}X_{Ijt}/M^2_{ljt} \tag{3.24}$$

$$\chi_{jt} = \lambda_{jt}\left(1+\tau_{xt}\right)P_{lt}/\{\zeta_{lj}\mathcal{X}_{Tjt}/X_{Tljt}\} = \lambda_{jt}\left(1+\tau_{xt}\right)\pi^T_{jt} \tag{3.25}$$

$$\mu_{jt} = \lambda_{jt}\left(1-\tau_{pt}\right)Q_{lt}/\{\nu_{lj}\mathcal{X}_{Ijt}/X_{Iljt}\} = \lambda_{jt}\left(1-\tau_{pt}\right)\pi^I_{jt} \tag{3.26}$$

$$\lambda_{jt} = \{\beta^t U_{ct}\left(1-\tau_{dt}\right)/\left[P_t\left(1+\tau_{ct}\right)\right]\}/\{1+2\kappa_j\left(D_{jt}-\bar{D}_j\right)\} \tag{3.27}$$

The first three equations are the dynamic first order conditions for the three state variables: K_{Tjt}, K_{Ijt}, and B_{jt}. The last three equations can be used to eliminate the multipliers: χ_{jt}, μ_{jt}, and λ_{jt}. That still leaves φ_{jt}, which appears everywhere in the ratio $\beta^t\varphi_{jt}/\lambda_{jt}$. To ease notation, I'll denote this ratio as ς_{jt}.

Substituting the multipliers χ_{jt} and μ_{jt} gives us the following dynamic equations:

$$\begin{aligned}\lambda_{jt}&\left[\left(1+\tau_{xt}\right)\pi^T_{jt} - \varsigma_{jt}\xi_{jt}P_{jt}\right] \\ &= E_t\lambda_{jt+1}P_{jt+1}\{1+\left(1-\tau_{pt+1}\right)\left(\theta_j Y_{jt+1}/K^1_{Tjt+1}-\delta_T-\tau_{kt+1}\right) \\ &\qquad - \varsigma_{jt+1}\theta_j Y_{jt+1}/K^1_{Tjt+1} + \left(\pi^T_{jt+1}/P_{jt+1}\left(1+\tau_{xt+1}\right)-1\right)\left(1-\delta_T\right)\}\end{aligned} \tag{3.28}$$

$$\begin{aligned}\lambda_{jt}&\left(1-\tau_{pt}\right)\pi^I_{jt} \\ &= E_t\lambda_{jt+1}\{\left(1-\tau_{pt+1}-\varsigma_{jt+1}\right)\phi_j\left(Q_{jt+1}X_{Ijt+1}+P_{jt+1}Y_{jt+1}\right)/K_{Ijt+1} \\ &\qquad + \left(1-\tau_{pt+1}\right)\pi^I_{jt+1}\left(1-\delta_I\right)\}\end{aligned} \tag{3.29}$$

$$\lambda_{jt}\left(1-\varsigma_{jt}\xi_{jt}\left(1+r_t\left(1-\tau_{bj}\right)\right)/\left(1+r_t\right)\right) = \left(1+r_t\left(1-\tau_{bj}\right)\right)E_t\lambda_{jt+1} \tag{3.30}$$

where λ_j and ς_j are defined above.

To compute the steady state, I apply a Newton's method as follows. I start by assuming I know aggregate c, the price vector with element p_j (and $p_1 = 1$), the ratio of hours by sector with element h^2_j/h^1_j, and the vector of hours by sector with element h_j. Then, in order, make the following computations:

$$\begin{aligned} r &= 1/\tilde{\beta} - 1 \\ \varsigma_j &= \left(1+r\right)\left(1-\tilde{\beta}\left(1+r\left(1-\tau_{bj}\right)\right)\right)/\left(\xi_j\tilde{\beta}\left(1+r\left(1-\tau_{bj}\right)\right)\right)\end{aligned}$$

$$
\begin{aligned}
\pi_j^M &= \gamma_j \prod_l \left(p_l/\gamma_{lj}\right)^{\gamma_{lj}/\gamma_j} \\
\pi_j^T &= \prod_l \left(p_l/\zeta_{lj}\right)^{\zeta_{lj}} \\
p &= \left(\sum_j \omega_j^\rho p_j^{1-\rho}\right)^{1/(1-\rho)} \\
c_j &= \left(\omega_j p/p_j\right)^\rho c \\
l &= 1 - \sum_j h_j \\
m_j^1/y_j &= \left(1 - \varsigma_j/\left(1-\tau_p\right)\right)\gamma_j p_j/\pi_j^M \\
h_j^1 &= h_j/\left(1 + h_j^2/h_j^1\right) \\
h_j^2 &= h_j - h_j^1 \\
q_j x_{Ij}/\left(p_j y_j\right) &= h_j^2/h_j^1 \\
x_{Ij}/y_j &= \left(q_j x_{Ij}/\left(p_j y_j\right)\right)^{1-\phi_j}\left(z_j^2/z_j^1\right)^{1-\theta_j-\phi_j-\gamma_j} \\
q_j/p_j &= \left(q_j x_{Ij}/\left(p_j y_j\right)\right)/\left(x_{Ij}/y_j\right) \\
q_j &= \left(q_j/p_j\right)p_j \\
\pi_j^I &= \prod_l \left(q_l/\nu_{lj}\right)^{\nu_{lj}} \\
k_{Ij}/y_j &= \left(1 - \varsigma_j/\left(1-\tau_p\right)\right)\tilde{\beta}\phi\left(q_j x_{Ij}/y_j + p_j\right)/\left(\pi_j^I\left(1-\tilde{\beta}\left(1-\delta_I\right)\right)\right) \\
ret &= \Big(\left(1+\tau_x\right)\pi_j^T/p_j - \varsigma_j\xi_j - \tilde{\beta}\big(1-\left(1-\tau_p\right)\left(\delta_T+\tau_k\right) \\
&\qquad + \left(1-\delta_T\right)\left(\left(1+\tau_x\right)\pi_j^T/p_j - 1\right)\big)\Big)/\left(\tilde{\beta}\left(1-\tau_p-\varsigma_j\right)\right) \\
k_{Tj}^1/y_j &= \theta/ret \\
y_j &= \left(\left(k_{Tj}^1/y_j\right)^{\theta_j}\left(k_{Ij}/y_j\right)^{\phi_j}\left(m_j^1/y_j\right)^{\gamma_j}\right)^{1/(1-\theta_j-\phi_j-\gamma_j)} z_j^1 h_j^1 \\
k_{Tj}^1 &= \left(k_{Tj}^1/y_j\right)y_j \\
k_{Tj}^2 &= k_{Tj}^1 h_j^2/h_j^1 \\
k_{Tj} &= k_{Tj}^1 + k_{Tj}^2 \\
m_j^1 &= \left(m_j^1/y_j\right)y_j \\
x_{Ij} &= \left(x_{Ij}/y_j\right)y_j \\
k_{Ij} &= \left(k_{Ij}/y_j\right)y_j \\
q_j x_{Ij}/p_j &= \left(q_j x_{Ij}/\left(p_j y_j\right)\right)y_j \\
\mathcal{X}_{Tj} &= \left(\left(1+g_z\right)\left(1+g_n\right) - 1 + \delta_T\right)k_{Tj} \\
\mathcal{X}_{Ij} &= \left(\left(1+g_z\right)\left(1+g_n\right) - 1 + \delta_I\right)k_{Ij} \\
m_{lj}^1 &= \left(1 - \varsigma_j/\left(1-\tau_p\right)\right)\gamma_{lj} p_j y_j/p_l
\end{aligned}
$$

$$
\begin{aligned}
m^2_{lj} &= (1-\varsigma_j/(1-\tau_p))\,\gamma_{lj} q_j x_{Ij}/p_l \\
x_{Tlj} &= \zeta_{lj}\pi^T_j \mathcal{X}_{Tj}/p_l \\
x_{Ilj} &= \nu_{lj}\pi^I_j \mathcal{X}_{Ij}/q_l \\
w_j &= (1-\varsigma_j/(1-\tau_p))\,(1-\theta_j-\phi_j-\gamma_j)\,p_j y_j/h^1_j \\
b_j &= \left[p_j k_{Tj} - (p_j y_j + q_j x_{Ij})/\xi_j\right]/\tilde{\beta} \\
d_j &= (1-\tau_p)\,(p_j y_j + q_j x_{Ij} - w_j h_j - \textstyle\sum_l p_l M_{lj} - \sum_l q_l X_{Ilj}) - (1+\tau_x)\sum_l p_l x_{Tlj} \\
&\quad + \left[\tau_p(\delta_T+\tau_k) - \tau_k\right] p_j k_{Tj} + b_j\,(1/(1+r\,(1-\tau_{bj})) - 1) \\
\lambda_j &= U_c\,(1-\tau_d)/\left[p\,(1+\tau_c)\right] \\
\varphi_j &= \lambda_j \varsigma_j
\end{aligned}
$$

and then construct the 3×1 residual vector, R_j for each sector j:

$$
\begin{aligned}
R_j(1) &= y_j - \textstyle\sum_l x_{Tjl} - \sum_l m^1_{jl} - \sum_l m^2_{jl} - g_j - c_j \\
R_j(2) &= x_{Ij} - \textstyle\sum_l x_{Ijl} \\
R_j(3) &= \psi\,(1+\tau_c)\,pc - (1-\tau_h)\,w_j l.
\end{aligned}
$$

The remaining step, as in the baseline model, is to apply a version of Vaughan's method to the system of first-order conditions after log-linearizing them around the steady state.