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Notes: Sweat Equity in U.S. Private Business*

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1. Introduction

In these notes, we provide additional details for the various versions of the model that we have studied. We start with the dynastic model that extends Aiyagari (1994) to include occupational choice and production by both C corporations and private pass-through entities. We show how to solve the set of static first order conditions as a fixed point in a single variable—which is the problem at the core of each call to the dynamic programming routine. We then extend the model to include life-cycle dynamics, assuming stochastic transitions between youth and old age, and business sales. The final section, is a baseline Lucas span-of-control model typical of the literature on entrepreneurship.

2. Dynastic Model

Individuals start each period with state vector $s = (a, \kappa, \epsilon, z)$ that summarizes their financial asset holdings (a), their sweat capital stock (κ), their productivity if they choose to work as an employee (ϵ), and their productivity if they choose to run a private, pass-through business themselves (z). The value of working is $V_w(s)$ and the value of being a private business owner is $V_p(s)$. Assuming individuals optimize when making their occupational choice, it must be the case that the value of being in state s is

$$V\left(s\right) = \max\{V_w\left(s\right), V_p\left(s\right)\}.$$

2.1. Worker's problem

Let's start by describing the dynamic program solved by workers. Workers choose consumption produced in the two sectors, c_c and c_p, respectively, leisure ℓ, and financial assets next period, a', to to solve the following dynamic program:

$$V_w\left(s\right) = \max_{c_c, c_p, \ell, a'} \{U\left(c, \ell\right) + \tilde{\beta} \sum_{\epsilon', z'} \pi\left(\epsilon', z' | \epsilon, z\right) V\left(s'\right)\} \tag{2.1}$$

subject to

$$a' = \left[\left(1 + r\right) a + w\epsilon n - \left(1 + \tau_c\right)\left(c_c + pc_p\right) - T^n\left(w\epsilon n\right)\right] / \left(1 + \gamma\right)$$

$$\kappa' = (1-\lambda)\,\kappa$$

$$c = c\,(c_c, c_p)$$

$$\ell = 1 - n$$

$$a' \geq \underline{a}$$

$$n \in [0,1]\,.$$

These households take as given the after-tax interest rate r, the before-tax wage rate w, the price of private goods p, the consumption tax rate τ_c, and the net tax function on labor earnings T^n. When comparing the model and data, we also add (exogenous) nonbusiness earnings $\bar{y}_{nb}$ less investments $\bar{x}_{nb}$ to government transfers when specifying T^n.

2.2. Business owner's problem

Next, consider the problem of the small business owner with value function V_p. The dynamic program is:

$$V_p\,(a,\kappa,\epsilon,z) = \max_{\substack{c_c,c_p,h_y,h_\kappa,e,\\ k_p,n_p,a',\kappa'}} \left\{U\,(c,\ell) + \tilde{\beta}\sum_{\epsilon',z'} \pi\,(\epsilon',z'|\epsilon,z)\,V\,(a',\kappa',\epsilon',z')\right\} \tag{2.2}$$

subject to

$$\begin{aligned}
a' &= [(1+r)\,a + py_p - (r_p+\delta_k)\,k_p - e - wn_p - (1+\tau_c)\,(c_c + pc_p) \\
&\qquad - T^b\,(py_p - (r_p+\delta_k)\,k_p - e - wn_p)]/\,(1+\gamma) \\
\kappa' &= \left[(1-\delta_\kappa)\,\kappa + \varsigma h_\kappa^{\vartheta} e^{1-\vartheta}\right]/\,(1+\gamma) \\
y_p &= z k_p^{\alpha}\kappa^{\phi}\left(\omega h_y^{\rho} + (1-\omega)\,n_p^{\rho}\right)^{\frac{\nu}{\rho}}, \quad \alpha+\phi+\nu = 1 \\
c &= \left(\eta c_c^{\varrho} + (1-\eta)\,c_p^{\varrho}\right)^{1/\varrho} \\
\ell &= 1 - h_\kappa - h_y \\
a' &\geq \chi k_p \\
k_p &\geq 0 \\
\ell &\in [0,1]\,,
\end{aligned}$$

taking as given the after-tax interest rate r, the before-tax wage rate w, the price of private goods p, the consumption tax rate τ_c, and the net tax function on business income T^b. Here again, we assume nonbusiness earnings and investments are included with transfers when specifying T^b.

2.3. C corporate problem

There is a competitive C-corporate sector with firms choosing hours n_c and fixed assets k_c to solve the following dynamic program:

$$v_c(k_c) = \max_{n_c, k_c'} \{(1-\tau_d) d_c + \frac{(1+\gamma)}{(1+r)} v_c(k_c')\}$$

subject to

$$k_c' = [(1-\delta_k) k_c + x_c] / (1+\gamma)$$

$$y_c = AF(k_c, n_c)$$

$$d_c = y_c - wn_c - x_c - \tau_p (y_c - wn_c - \delta_k k_c)$$

where d_c are corporate dividends that are taxed at rate τ_d after paying corporate income taxes at rate τ_p, x_c is C-corporate investment, and y_c is C-corporate output from a constant returns to scale technology F with TFP given by A. Employees working for C corporations earn the same hourly wage w as employees in private businesses.

2.4. Financial intermediary

There is a competitive intermediation sector with risk-neutral financial intermediaries that accept deposits and use the funds to invest in C-corporate equities, government bonds, and fixed assets.

At the beginning of each period, the net worth of an intermediary is the value of its equity shares ς, bonds b, and fixed assets k, less the value of deposits owed to households a. During the period, the intermediary receives dividend income from C corporations, interest income from bonds, and rental income on fixed assets and pays interest on deposits. The dynamic program in this case is:

$$v_I(x) = \max_{x'} \{d_I + \left(\frac{1+\gamma}{1+r}\right) v_I(x')\}, \tag{2.3}$$

where the state vector is $x = [\varsigma, b, k, a]'$ and the intermediary dividends are:

$$\begin{aligned} d_I = q\varsigma + \int k_p(s)\,\mu(s)\,ds + b - \int a(s)\,\mu(s)\,ds \\ + (1-\tau_d) d_c \varsigma + rb + (r+\delta_k) \int k_p(s)\,\mu(s)\,ds - r \int a(s)\,\mu(s)\,ds \\ - (1+\gamma) \left[q\varsigma' + b' - \int a'(s)\,\mu(s)\,ds\right] - \int x_p(s)\,\mu(s)\,ds \end{aligned}$$

where $\mu(s)$ is the measure of type s individuals and q is the per-share price of corporate equities.

2.5. Government budget constraint

The government spends g, borrows b and collects taxes on consumption, labor earnings, pass-through earnings, dividends, and profits. In a stationary equilibrium, the government budget constraint is given by:

$$\begin{aligned}
g+(r-\gamma)\,b=\tau_c&\left(\int c_c\,(s)\,\mu\,(s)\;ds+\int pc_p\,(s)\,\mu\,(s)\;ds\right)+\tau_p\,(y_c-wn_c-\delta_k k_c)\\
&+\tau_d\,(y_c-wn_c-(\gamma+\delta_k)\,k_c-\tau_p\,(y_c-wn_c-\delta_k k_c))\\
&+\int T^b\,(py_p\,(s)-(r_p+\delta_k)\,k_p\,(s)-e\,(s)-wn_p\,(s))\,\mu\,(s)\;ds\\
&+\int T^n\,(w\epsilon\,(s)\,n\,(s))\,\mu\,(s)\;ds
\end{aligned}$$

2.6. Market clearing conditions

The market clearing conditions are now:

$$\begin{aligned}
y_c&=\int\left(c_c\,(s)+e\,(s)\right)\mu\,(s)\;ds+(\gamma+\delta_k)\left(k_c+\int k_p\,(s)\,\mu\,(s)\;ds\right)+g+\bar{x}_{nb}-\bar{y}_{nb}\\
n_c&=\int\left(n\,(s)\,\epsilon\,(s)-n_p\,(s)\right)\mu\,(s)\;ds\\
\int a\,(s)\,\mu\,(s)\;ds&=b+(1-\tau_d)\,k_c+\int k_p\,(s)\,\mu\,(s)\;ds\\
\int y_p\,(s)\,\mu\,(s)\;ds&=\int c_p\,(s)\,\mu\,(s)\;ds,
\end{aligned}$$

2.7. Equilibrium

A stationary recursive competitive equilibrium is: value functions V_w, V_p; policy functions a', κ', c_c, c_p, ℓ, n, k_p, n_p, h_y, h_κ, and e; C corporation choices n_c, k_c; prices r, w, p; and a measure over types indexed by the state s such that

- given prices, the policy functions for employees, namely, a', κ', c_c, c_p, ℓ, n solve dynamic programming problem associated with value function V_w;

- given prices, the policy functions for private business owners, namely, a', κ', c_c, c_p, ℓ, k_p, n_p, h_y, h_κ, e solve dynamic programming problems associated with value function V_p;
- given prices, the policy functions for C corporations, namely, n_c and k_c' solve the dynamic programming problem associated with v_c;
- given prices, the policy functions for financial intermediaries namely, $x = [\varsigma, b, k, a]'$ solve the dynamic programming problem associated with v_I;
- the labor market clears: $n_c = \int (n(s)\epsilon(s) - n_p(s))\mu(s)\, ds$;
- the asset market clears: $\int a(s)\mu(s)\, ds = b + (1-\tau_d)k_c + \int k_p(s)\mu(s)\, ds$;
- the private business goods market clears: $\int y_p(s)\mu(s)\, ds = \int c_p(s)\mu(s)\, ds$;
- the C-corporate goods market clears:
$$y_c = \int \left(c_c(s) + \int e(s) \right) \mu(s)\ ds + (\gamma + \delta_k) \left(k_c + \int k_p(s)\, \mu(s)\ ds \right) + g;$$
- the government budget constraint is satisfied;
- the measure of types over states (a, κ, ϵ, z) is invariant.

We should note that in equilibrium, returns on all uses of funds will be equated and the equity price is $q = 1 - \tau_d$. Let the return on equity from $t-1$ to t be denoted by r_{et}. Then, from the intermediary optimization, we have
$$r_{et} = \frac{q_t + (1-\tau_d)\, d_t}{q_{t-1}} - 1,$$
which must be equal to $r_{pt} - \delta_k$ and to r_t, or in the stationary distribution $r_e = r_p - \delta_k = r$. Similarly, from the C-corporate optimization, we have the return on capital
$$r_k = (1-\tau_p)\left(AF_k(k_c, n_c) - \delta_k\right),$$
which must be equal to r and r_e. If we normalize aggregate shares to 1, it follows that the stock value is
$$q = (1-\tau_d)\, k_c.$$
Furthermore, if there is free entry in the intermediary sector, it must be the case that the net worth nw is zero. From that, we get the asset market clearing constraint above.

2.8. Calculations for easier computation

If we use the following functional forms for utility and consumption, namely,

$$U(c,\ell) = \left(c^{1-\psi}\ell^{\psi}\right)^{1-\sigma} / (1-\sigma) \tag{2.4}$$

$$c(c_c,c_p) = \left(\eta c_c^{\varrho} + (1-\eta)\, c_p^{\varrho}\right)^{\frac{1}{\varrho}}, \tag{2.5}$$

then the first-order conditions for c_c, c_p, and n of the worker are given by

$$0 = (1-\psi)\, c^{(1-\psi)(1-\sigma)-1}\ell^{\psi(1-\sigma)}\left[\eta c_c^{\varrho-1}c^{1-\varrho}\right] - (1+\tau_c)\,\varphi \tag{2.6}$$

$$0 = (1-\psi)\, c^{(1-\psi)(1-\sigma)-1}\ell^{\psi(1-\sigma)}\left[(1-\eta)\, c_p^{\varrho-1}c^{1-\varrho}\right] - p\,(1+\tau_c)\,\varphi \tag{2.7}$$

$$0 = -\psi c^{(1-\psi)(1-\sigma)}\ell^{\psi(1-\sigma)-1} + w\epsilon\,(1-\partial T^n/\partial y)\,\varphi \tag{2.8}$$

where $y = w\epsilon n$ and

$$\varphi = \tilde{\beta}\sum_{\epsilon',z'} \pi(\epsilon',z'|\epsilon,z)\, V_a(a',\kappa',\epsilon',z') \,/\, (1+\gamma)\,.$$

In this problem, we assume that variables are stationary so the discount factor $\tilde{\beta}$ has been adjusted by the growth rate of TFP, which is γ. With the utility function in (2.4), the adjusted discount factor is equal to $\tilde{\beta} = \beta(1+\gamma)^{1-\sigma}$, where β is the original discount factor.

Taking the ratio of (2.6) and (2.7) and simplifying, we get

$$\frac{c_p}{c_c} = \left(\frac{\eta p}{1-\eta}\right)^{\frac{1}{\varrho-1}} \equiv \xi_1, \tag{2.9}$$

or $c_p = \xi_1 c_c$, which allows us to write total consumption expenditures in terms of corporate consumption:

$$c_c + pc_p = (1+p\xi_1)\, c_c. \tag{2.10}$$

The consumption bundle can also be written in terms of c_c:

$$c = \left(\eta + (1-\eta)\left(\frac{\eta p}{1-\eta}\right)^{\frac{\varrho}{\varrho-1}}\right)^{1/\varrho} c_c \equiv \xi_2 c_c. \tag{2.11}$$

To derive a value for n, we need to specify the net transfer function T^n. The most flexible and tractable choice is a piecewise linear function:

$$T(y) = \tau_i y - tr_i$$

$$\partial T(y)/\partial y = \tau_i$$

for y in income bracket i, where tr_i includes government transfers and nonbusiness incomes less investment $(\bar{y}_{nb} - \bar{x}_{nb})$.[1]

Taking the ratio of (2.6) and (2.8) and using the definition of ξ_2, we get

$$
\begin{aligned}
c_c &= \left(\frac{(1-\psi)\,\eta}{\psi\xi_2^{\varrho}\,(1+\tau_c)}\right) w\epsilon\,(1-\partial T^n/\partial y)\,(1-n) \\
&\equiv \xi_3 w\epsilon\,(1-\tau_{ni})\,(1-n) \qquad (2.12)
\end{aligned}
$$

assuming the individual is in income bracket i. From the budget constraint, we can write c_c as follows:

$$
\begin{aligned}
c_c &= \frac{(1+r)\,a - (1+\gamma)\,a' + (1-\tau_{ni})\,w\epsilon n + tr_{ni}}{(1+p\xi_1)\,(1+\tau_c)} \\
&\equiv \xi_4 a - \xi_5 a' + \xi_6\left((1-\tau_{ni})\,w\epsilon n + tr_{ni}\right), \qquad (2.13)
\end{aligned}
$$

again assuming that the income $w\epsilon n$ puts the individual in bracket i.

Equating (2.12) and (2.13) gives us a linear equation in n (if we know the tax bracket i):

$$
n = \frac{\xi_3 w\epsilon\,(1-\tau_{ni}) - \xi_4 a + \xi_5 a' - \xi_6 tr_{ni}}{(\xi_3+\xi_6)\,(1-\tau_{ni})\,w\epsilon}
$$

To figure out the bracket, we evaluate $T^n(w\epsilon n)$ for each guess of i.

Next, consider writing out equations for the business owner's problem. For now, let's write the equations assuming that $\chi = 0$. Later, we'll rewrite them assuming the constraint is binding. Taking derivatives of the right-hand side of the Bellman equation, with respect to c_c, c_p, h_y, h_κ, e, k_p, and n_p, we get:

$$
0 = (1-\psi)\,c^{(1-\psi)(1-\sigma)-1}\ell^{\psi(1-\sigma)}\left[\eta c_c^{\varrho-1}c^{1-\varrho}\right] - (1+\tau_c)\,\varphi_a \qquad (2.14)
$$

$$
0 = (1-\psi)\,c^{(1-\psi)(1-\sigma)-1}\ell^{\psi(1-\sigma)}\left[(1-\eta)\,c_p^{\varrho-1}c^{1-\varrho}\right] - p\,(1+\tau_c)\,\varphi_a \qquad (2.15)
$$

$$
0 = -\psi c^{(1-\psi)(1-\sigma)}\ell^{\psi(1-\sigma)-1}
$$

[1] We also explored the two-parameter function $T(y) = y - (1-\tau_n)y^{1-p_n}$, where p_n governs the progressivity and τ_n governs the level. This choice has two disadvantages. First, this function is difficult to parameterize across a wide state space. Second, this function gives rise to multiple solutions over some regions of the state space when we compute owner hours.

$$+\varphi_a \nu \omega \frac{p y_p}{h_y} \frac{h_y^\rho}{\left[\omega h_y^\rho + (1-\omega) n_p^\rho\right]} \left(1 - \partial T^b / \partial y\right) \tag{2.16}$$

$$0 = -\psi c^{(1-\psi)(1-\sigma)} \ell^{\psi(1-\sigma)-1} + \varphi_\kappa \varsigma \vartheta \left(e/h_\kappa\right)^{1-\vartheta} \tag{2.17}$$

$$0 = -\varphi_a \left(1 - \partial T^b / \partial y\right) + \varphi_\kappa \varsigma \left(1-\vartheta\right) \left(h_\kappa / e\right)^\vartheta \tag{2.18}$$

$$0 = p \alpha y_p / k_p - r_p - \delta_k \tag{2.19}$$

$$0 = \nu \left(1-\omega\right) \frac{p y_p}{n_p} \frac{n_p^\rho}{\left[\omega h_y^\rho + (1-\omega) n_p^\rho\right]} - w \tag{2.20}$$

where

$$\varphi_a = \tilde{\beta} \sum_{\epsilon', z'} \pi\left(\epsilon', z' | \epsilon, z\right) V_a\left(a', \kappa', \epsilon', z'\right) / \left(1+\gamma\right)$$

$$\varphi_\kappa = \tilde{\beta} \sum_{\epsilon', z'} \pi\left(\epsilon', z' | \epsilon, z\right) V_\kappa\left(a', \kappa', \epsilon', z'\right) / \left(1+\gamma\right).$$

Let's first consider the case with no sweat capital and $\kappa = h_y = k_p = n_p = 0$. In this case, we only need to solve the following equations for c_c, h_κ, and e assuming income falls in bracket i:

$$c_c = \frac{(1-\psi)\,\eta}{\psi \xi_2^\varrho \left(1+\tau_c\right)} \frac{\vartheta e}{(1-\vartheta)\, h_\kappa} \left(1-\tau_{bi}\right)\left(1-h_\kappa\right) \tag{2.21}$$

$$c_c = \left(\frac{(1+r)\,a - (1+\gamma)\,a' + tr_{bi}}{(1+p\xi_1)(1+\tau_c)}\right) - \left(\frac{1-\tau_{bi}}{(1+p\xi_1)(1+\tau_c)}\right) e \tag{2.22}$$

$$e = \left(\frac{(1+\gamma)\,\kappa'}{\varsigma}\right)^{\frac{1}{1-\vartheta}} h_\kappa^{-\frac{\vartheta}{1-\vartheta}} \tag{2.23}$$

where $c_p = \xi_1 c_c$ as before. We can equate the c_c equations and subtitute in for e. That leaves us with one equation in one unknown, namely, h_κ:

$$\alpha_1 \left(1-h_\kappa\right) = \alpha_2 h_\kappa^{\frac{1}{1-\vartheta}} - \alpha_3 h_\kappa \tag{2.24}$$

where

$$\alpha_1 = \frac{(1-\psi)\,\eta}{\psi \xi_2^\varrho \left(1+\tau_c\right)}$$

$$\alpha_2 = \frac{(1-\vartheta)\left(\xi_4 a - \xi_5 a' + \xi_6 tr_{bi}\right)}{\vartheta \left((1+\gamma)\,\kappa'/\zeta\right)^{\frac{1}{1-\vartheta}}} \frac{1}{1-\tau_{bi}}$$

$$\alpha_3 = \frac{1-\vartheta}{\vartheta\left(1+p\xi_1\right)\left(1+\tau_c\right)}.$$

At each step of the bisection, we need to figure out the tax bracket i in order to evaluate the α's. The term $\alpha_1(1-h_\kappa)$ on the left-hand side is positive for $h_\kappa = 0$ and zero for $h_\kappa = 1$. The right-hand side is zero at $h_\kappa = 0$ and crosses the left-hand side once if $\alpha_2 > \alpha_3$. If $\vartheta \in (0,1)$, we can ensure that a Newton iteration will converge for any point between h_κ^* and 1, where

$$h_\kappa^* = \left(\frac{\alpha_3\left(1-\vartheta\right)}{\alpha_2}\right)^{\frac{1-\vartheta}{\vartheta}}.$$

This point is the minimum of the right-hand side of (2.24). Alternatively, we could pick the point where the right-hand side is zero because the curve has to cross zero before crossing the downward sloping line. In this case:

$$h_\kappa^* = \left(\frac{\alpha_3}{\alpha_2}\right)^{\frac{1-\vartheta}{\vartheta}}.$$

Notice that in either case, α_2 has to be greater than zero. Otherwise, there is no fixed point in the interval [0,1].

Next consider the case with positive sweat capital ($\kappa > 0$). In this case, it will turn out to be relatively simple to solve the problem if we we introduce a new variable $\tilde{h}_y$:

$$\tilde{h}_y = \left(\omega h_y^\rho + \left(1-\omega\right) n_p^\rho\right)^{\frac{1}{\rho}}.$$

Specifically, we can show that solving the static first-order conditions boil down to solving a fixed point problem in $\tilde{h}_y$. To do this, we first write n_p as a function of $\tilde{h}_y$ and k_p:

$$n_p = \left(\frac{\nu\left(1-\omega\right)\left(r_p+\delta_k\right)}{\alpha w}\frac{k_p}{\tilde{h}_y^\rho}\right)^{\frac{1}{1-\rho}} \equiv \xi_7\left(\frac{k_p}{\tilde{h}_y^\rho}\right)^{\frac{1}{1-\rho}}$$

assuming $\rho < 1$. Given n_p, we can write k_p and y_p as follows:

$$k_p = \left(\frac{\alpha p z \kappa^\phi}{r_p+\delta_k}\right)^{\frac{1}{1-\alpha}} \tilde{h}_y^{\frac{\nu}{1-\alpha}} = \xi_8 z^{\frac{1}{1-\alpha}} \kappa^{\frac{\phi}{1-\alpha}} \tilde{h}_y^{\frac{\nu}{1-\alpha}}$$

$$y_p = z k_p^\alpha \kappa^\phi \tilde{h}_y^\nu = \xi_8^\alpha z^{\frac{1}{1-\alpha}} \kappa^{\frac{\phi}{1-\alpha}} \tilde{h}_y^{\frac{\nu}{1-\alpha}}$$

Notice that we now have both $\tilde{h}_y$ and h_y in the equations. If we know $\tilde{h}_y$, then we can compute k_p and n_p and therefore, h_y:

$$h_y = \left(\frac{\tilde{h}_y^\rho - \left(1-\omega\right) n_p^\rho}{\omega}\right)^{\frac{1}{\rho}}$$

$$= \left(\frac{\tilde{h}_y^{\rho} - (1-\omega)\, \xi_7^{\rho} k_p^{\frac{\rho}{1-\rho}} \tilde{h}_y^{-\frac{\rho^2}{1-\rho}}}{\omega} \right)^{\frac{1}{\rho}}$$

$$= \left(\frac{1 - (1-\omega)\, \xi_7^{\rho} \left(\xi_8 \left(z\kappa^{\phi} \right)^{\frac{1}{1-\alpha}} \right)^{\frac{\rho}{1-\rho}} \tilde{h}_y^{\left(\frac{\nu}{1-\alpha}-\rho\right)\left(\frac{\rho}{1-\rho}\right)-\rho}}{\omega} \right)^{\frac{1}{\rho}} \tilde{h}_y \tag{2.25}$$

This is a big mess but we can just write $h_y(\tilde{h}_y)$ below.

Given the substitutions we have already made, the first-order conditions can be written intermediately as four equations in c_c, $\tilde{h}_y$, h_κ, and e:

$$c_c = \left(\frac{(1-\psi)\, \eta\nu\omega p \xi_8^{\alpha} z^{\frac{1}{1-\alpha}} \kappa^{\frac{\phi}{1-\alpha}} (1-\tau_{bi})}{\psi \xi_2^{\varrho} (1+\tau_c)} \right) \tilde{h}_y^{\frac{\nu}{1-\alpha}-\rho} h_y\left(\tilde{h}_y\right)^{\rho-1} \left(1 - h_y\left(\tilde{h}_y\right) - h_\kappa\right)$$

$$= \xi_9 (1-\tau_{bi}) \left(z\kappa^{\phi}\right)^{\frac{1}{1-\alpha}} \tilde{h}_y^{\frac{\nu}{1-\alpha}-\rho} h_y\left(\tilde{h}_y\right)^{\rho-1} \left(1 - h_y\left(\tilde{h}_y\right) - h_\kappa\right) \tag{2.26}$$

$$\begin{aligned} c_c = {} & \left(\frac{(1+r)\, a - (1+\gamma)\, a' + tr_{bi}}{(1+p\xi_1)(1+\tau_c)} \right) \\ & + \left(\frac{p\xi_8^{\alpha} - (r_p + \delta_k)\, \xi_8}{(1+p\xi_1)(1+\tau_c)} \right) (1-\tau_{bi}) \left(z\kappa^{\phi}\right)^{\frac{1}{1-\alpha}} \tilde{h}_y^{\frac{\nu}{1-\alpha}} \\ & - \left(\frac{1}{(1+p\xi_1)(1+\tau_c)} \right) (1-\tau_{bi})\, e \\ & - \left(\frac{w\xi_7 \xi_8^{\frac{1}{1-\rho}}}{(1+p\xi_1)(1+\tau_c)} \right) (1-\tau_{bi}) \left(z\kappa^{\phi}\right)^{\frac{1}{(1-\alpha)(1-\rho)}} \tilde{h}_y^{\left(\frac{\nu}{1-\alpha}-\rho\right)\left(\frac{1}{1-\rho}\right)} \\ \equiv {} & \xi_4 a - \xi_5 a' + \xi_6 tr_{bi} + \xi_{10} (1-\tau_{bi}) \left(z\kappa^{\phi}\right)^{\frac{1}{1-\alpha}} \tilde{h}_y^{\frac{\nu}{1-\alpha}} - \xi_{11} (1-\tau_{bi})\, e \\ & - \xi_{12} (1-\tau_{bi}) \left(z\kappa^{\phi}\right)^{\frac{1}{(1-\alpha)(1-\rho)}} \tilde{h}_y^{\left(\frac{\nu}{1-\alpha}-\rho\right)\left(\frac{1}{1-\rho}\right)} \end{aligned} \tag{2.27}$$

$$\varsigma h_\kappa^{\vartheta} e^{1-\vartheta} = (1+\gamma)\, \kappa' - (1-\delta_\kappa)\, \kappa \tag{2.28}$$

$$\frac{\varepsilon e}{\vartheta h_\kappa} = \left(\nu\omega p \xi_8^{\alpha}\right) \left(z^{\frac{1}{1-\alpha}} \kappa^{\frac{\phi}{1-\alpha}} \right) \tilde{h}_y^{\frac{\nu}{1-\alpha}-\rho} h_y\left(\tilde{h}_y\right)^{\rho-1}. \tag{2.29}$$

Equations (2.28) and (2.29) can be combined to get h_κ:

$$e = \left(\frac{1-\vartheta}{\vartheta} \nu\omega p \xi_8^{\alpha} \right) \left(z\kappa^{\phi}\right)^{\frac{1}{1-\alpha}} \tilde{h}_y^{\frac{\nu}{1-\alpha}-\rho} h_y\left(\tilde{h}_y\right)^{\rho-1} h_\kappa$$

$$= \xi_{13}\left(z\kappa^{\phi}\right)^{\frac{1}{1-\alpha}}\tilde{h}_y^{\frac{\nu}{1-\alpha}-\rho}h_y\left(\tilde{h}_y\right)^{\rho-1}h_\kappa \tag{2.30}$$

$$h_\kappa = \left(\frac{\left(\frac{1}{\zeta}\left(\left(1+\gamma\right)\kappa' - \left(1-\delta_\kappa\right)\kappa\right)\right)^{\frac{1}{1-\vartheta}}}{\xi_{13}\left(z\kappa^{\phi}\right)^{\frac{1}{1-\alpha}}}\right)^{1-\vartheta}\left(\tilde{h}_y^{\frac{\nu}{1-\alpha}-\rho}h_y\left(\tilde{h}_y\right)^{\rho-1}\right)^{\vartheta-1}. \tag{2.31}$$

For ease of notation below, define the function $g(x)$ to be:

$$g\left(x\right) = \left(x^{\frac{\nu}{1-\alpha}-\rho}h_y\left(x\right)^{\rho-1}\right)^{\vartheta-1}.$$

If we multiply (2.26) and (2.27) by $\tilde{h}_y^{\rho-\frac{\nu}{1-\alpha}}h_y(\tilde{h}_y)^{1-\rho}$, equate them, use the equations (2.30) and (2.31) for e and h_κ, and divide all terms by $(z\kappa^{\phi})^{\frac{1}{1-\alpha}}$, then we have:

$$\begin{aligned}&\alpha_1\left(1-h_y\left(\tilde{h}_y\right)-\alpha_5 g\left(\tilde{h}_y\right)\right)\\&\quad=\left(\alpha_2\tilde{h}_y^{-\frac{\nu}{1-\alpha}}-\alpha_6\tilde{h}_y^{\left(\frac{\nu}{1-\alpha}-\rho\right)\left(\frac{\rho}{1-\rho}\right)-\rho}+\alpha_3\right)\tilde{h}_y^{\rho}h_y\left(\tilde{h}_y\right)^{1-\rho}-\alpha_4 g\left(\tilde{h}_y\right)\end{aligned} \tag{2.32}$$

where

$$\alpha_1 = \xi_9$$

$$\alpha_2 = \frac{\xi_4 a - \xi_5 a' + \xi_6 tr_{bi}}{\left(1-\tau_{bi}\right)\left(z\kappa^{\phi}\right)^{\frac{1}{1-\alpha}}}$$

$$\alpha_3 = \xi_{10}$$

$$\alpha_4 = \xi_{11}\xi_{13}\alpha_5$$

$$\alpha_5 = \left(\frac{\left(\frac{1}{\zeta}\left(\left(1+\gamma\right)\kappa' - \left(1-\delta_\kappa\right)\kappa\right)\right)^{\frac{1}{1-\vartheta}}}{\xi_{13}\left(z\kappa^{\phi}\right)^{\frac{1}{1-\alpha}}}\right)^{1-\vartheta}.$$

$$\alpha_6 = \xi_{12}\left(z\kappa^{\phi}\right)^{\left(\frac{1}{1-\alpha}\right)\left(\frac{\rho}{1-\rho}\right)}$$

In the case of $\kappa = 0$, we can again use (2.24) to find h_κ.

Next, we consider making a good guess for the bounds on $\tilde{h}_y$. We'll bound this using the values of $\tilde{h}_y$ associated with $h_y = 0$ and $h_y = 1$. To ease notation, let

$$\xi_{14} = \left(1-\omega\right)\xi_7^{\rho}\left(\xi_8\left(z\kappa^{\phi}\right)^{\frac{1}{1-\alpha}}\right)^{\frac{\rho}{1-\rho}}.$$

Let's start with the lower end. From (2.25), we find the lower bound by setting:

$$\tilde{h}_y^{lb} = \xi_{14}^{\overline{-\left(\frac{\nu}{1-\alpha}-\rho\right)\left(\frac{\rho}{1-\rho}\right)+\rho}^{\,1}}.$$

At this value, $h_y = 0$ and the residual equation is equal to $\alpha_1 > 0$.

For the upper bound, we could solve

$$1 = \xi_{14}\tilde{h}_y^{\left(\frac{\nu}{1-\alpha}-\rho\right)\left(\frac{\rho}{1-\rho}\right)-\rho} + \omega\tilde{h}_y^{-\rho}$$

or we could take the maximum of the two terms on the right hand side, which puts a cap on $\tilde{h}_y$. Then, the upper bound is:

$$\tilde{h}_y^{ub} = \max\left((\xi_{14}+\omega)^{\frac{1}{\rho}}, (\xi_{14}+\omega)^{\overline{-\left(\frac{\nu}{1-\alpha}-\rho\right)\left(\frac{\rho}{1-\rho}\right)+\rho}^{\,1}}\right).$$

We can compute $\tilde{h}_y$ assuming the constraint on a' does not bind and then check if $a' < \chi k_p$. Suppose it is. In this case, the equations change as follows. We replace (2.19) with:

$$k_p = a'/\chi.$$

In this case, we need to change ξ_7:

$$\begin{aligned} n_p &= \left(\frac{\nu(1-\omega)pk_p^\alpha}{w}\right)^{\frac{1}{1-\rho}}\tilde{h}_y^{\frac{\nu-\rho}{1-\rho}} \\ &= \xi_7\left(z\kappa^\phi\right)^{\frac{1}{1-\rho}}\tilde{h}_y^{\frac{\nu-\rho}{1-\rho}} \end{aligned}$$

and drop ξ_8 since y_p can be written directly in terms of the unknown $\tilde{h}_y$:

$$y_p = k_p^\alpha z\kappa^\phi\tilde{h}_y^\nu.$$

The new expression for $h_y(\tilde{h}_y)$ is:

$$h_y = \left(\frac{\tilde{h}_y^\rho - (1-\omega)\left[\xi_7\left(z\kappa^\phi\right)^{\frac{1}{1-\rho}}\tilde{h}_y^{\frac{\nu-\rho}{1-\rho}}\right]^\rho}{\omega}\right)^{\frac{1}{\rho}}$$

The equations for c_c, $\tilde{h}_y$, h_κ and e are now:

$$c_c = \left(\frac{(1-\psi)\,\eta\nu\omega p z\kappa^\phi k_p^\alpha\,(1-\tau_{bi})}{\psi\xi_2^\varrho\,(1+\tau_c)}\right)\tilde{h}_y^{\nu-\rho} h_y\left(\tilde{h}_y\right)^{\rho-1}\left(1-h_y\left(\tilde{h}_y\right)-h_\kappa\right)$$

$$= \xi_9\,(1-\tau_{bi})\,z\kappa^\phi\tilde{h}_y^{\nu-\rho}h_y\left(\tilde{h}_y\right)^{\rho-1}\left(1-h_y\left(\tilde{h}_y\right)-h_\kappa\right) \tag{2.33}$$

$$\begin{aligned} c_c = {} & \left(\frac{(1+r)\,a-(1+\gamma)\,a'+tr_{bi}}{(1+p\xi_1)\,(1+\tau_c)}\right) \\ & -\left(\frac{1}{(1+p\xi_1)\,(1+\tau_c)}\right)(1-\tau_{bi})\,e \\ & +\left(\frac{pk_p^\alpha}{(1+p\xi_1)\,(1+\tau_c)}\right)(1-\tau_{bi})\,z\kappa^\phi\tilde{h}_y^\nu \\ & -\left(\frac{(r_p+\delta_k)\,k_p}{(1+p\xi_1)\,(1+\tau_c)}\right)(1-\tau_{bi}) \\ & -\left(\frac{w\xi_7}{(1+p\xi_1)\,(1+\tau_c)}\right)(1-\tau_{bi})\left(z\kappa^\phi\right)^{\frac{1}{1-\rho}}\tilde{h}_y^{\frac{\nu-\rho}{1-\rho}} \\ \equiv {} & \xi_4 a-\xi_5 a'+\xi_6 tr_{bi}-\xi_{11}\,(1-\tau_{bi})\,e+\xi_{10}\,(1-\tau_{bi})\,z\kappa^\phi\tilde{h}_y^\nu \\ & -\xi_{12}\,(1-\tau_{bi})\left(z\kappa^\phi\right)^{\frac{1}{1-\rho}}\tilde{h}_y^{\frac{\nu-\rho}{1-\rho}} \end{aligned} \tag{2.34}$$

$$\varsigma h_\kappa^\vartheta e^{1-\vartheta} = (1+\gamma)\,\kappa'-(1-\delta_\kappa)\,\kappa \tag{2.35}$$

$$\frac{\vartheta e}{(1-\vartheta)\,h_\kappa} = \nu\omega pk_p^\alpha z\kappa^\phi\tilde{h}_y^{\nu-\rho}h_y\left(\tilde{h}_y\right)^{\rho-1}. \tag{2.36}$$

Combining (2.35) and (2.36) yields:

$$\begin{aligned} e &= \left(\frac{1-\vartheta}{\vartheta}\nu\omega pk_p^\alpha\right)z\kappa^\phi\tilde{h}_y^{\nu-\rho}h_y\left(\tilde{h}_y\right)^{\rho-1}h_\kappa \\ &= \xi_{13}z\kappa^\phi\tilde{h}_y^{\nu-\rho}h_y\left(\tilde{h}_y\right)^{\rho-1}h_\kappa \end{aligned} \tag{2.37}$$

$$h_\kappa = \left(\frac{\left(\frac{1}{\varsigma}\left((1+\gamma)\,\kappa'-(1-\delta_\kappa)\,\kappa\right)\right)^{\frac{1}{1-\vartheta}}}{\xi_{13}z\kappa^\phi}\right)^{1-\vartheta}\left(\tilde{h}_y^{\nu-\rho}h_y\left(\tilde{h}_y\right)^{\rho-1}\right)^{\vartheta-1}. \tag{2.38}$$

The function $g(x)$ has to be replaced by:

$$g\,(x) = \left(x^{\nu-\rho}h_y\,(x)^{\rho-1}\right)^{-\frac{\vartheta}{\vartheta+\varepsilon}}.$$

If we multiply (2.33) and (2.34) by $\tilde{h}_y^{\rho-\nu} h_y(\tilde{h}_y)^{1-\rho}$, equate them, use the equations (2.37) and (2.38) for e and h_κ, and divide all terms by $z\kappa^\phi$, then we have:

$$\begin{aligned} &\alpha_1 \left(1 - h_y\left(\tilde{h}_y\right) - \alpha_5 g\left(\tilde{h}_y\right)\right) \\ &\quad = \left(\alpha_2 \tilde{h}_y^{-\nu} - \alpha_6 \tilde{h}_y^{(\nu-\rho)\left(\frac{\rho}{1-\rho}\right)-\rho} + \alpha_3\right) \tilde{h}_y^{\rho} h_y\left(\tilde{h}_y\right)^{1-\rho} - \alpha_4 g\left(\tilde{h}_y\right) \end{aligned} \tag{2.39}$$

where

$$\begin{aligned} \alpha_1 &= \xi_9 \\ \alpha_2 &= \frac{\xi_4 a - \xi_5 a' + \xi_6 tr_{bi}}{\left(1-\tau_{bi}\right) z\kappa^\phi} \\ \alpha_3 &= \xi_{10} \\ \alpha_4 &= \xi_{11}\xi_{13}\alpha_5 \\ \alpha_5 &= \left(\frac{\left(\frac{1}{\zeta}\left(\left(1+\gamma\right)\kappa' - \left(1-\delta_\kappa\right)\kappa\right)\right)^{\frac{1}{1-\vartheta}}}{\xi_{13} z\kappa^\phi}\right)^{1-\vartheta}. \\ \alpha_6 &= \xi_{12}\left(z\kappa^\phi\right)^{\frac{\rho}{1-\rho}} \end{aligned}$$

3. Add Life-Cycle Dynamics

Now we allow for young and old agents that face the probability of aging (that is, transiting from young to old) and the probability of death (that is, replacing an old by a young offspring). For both young and old, there is a choice of running a business or working for others, so the value functions must satisfy:

$$\begin{aligned} V_y\left(a,\kappa,\epsilon,z\right) &= \max\{V_{y,w}\left(a,\kappa,\epsilon,z\right), V_{y,p}\left(a,\kappa,\epsilon,z\right)\} \\ V_o\left(a,\kappa,\epsilon,z\right) &= \max\{V_{o,w}\left(a,\kappa,\epsilon,z\right), V_{o,p}\left(a,\kappa,\epsilon,z\right)\}. \end{aligned}$$

In this version, young businesses solve:

$$\begin{aligned} V_{y,p}\left(a,\kappa,\epsilon,z\right) = \max_{c_c,c_p,k_p,h_y,h_\kappa,a',\kappa'} &\{U\left(c,\ell\right) \\ &+ \tilde{\beta}\pi_y \sum_{\epsilon',z'} \pi\left(\epsilon',z'|\epsilon,z\right) V_y\left(a',\kappa',\epsilon',z'\right) \\ &+ \tilde{\beta}\left(1-\pi_y\right) \sum_{\epsilon',z'} \pi\left(\epsilon',z'|\epsilon,z\right) V_o\left(a',\kappa',\epsilon',z'\right)\} \end{aligned}$$

subject to

$$
\begin{aligned}
a' &= [(1+r)\,a + py_p - (r_p+\delta_k)\,k_p - wn_p - e - (1+\tau_c)\,(c_c + pc_p) \\
&\qquad - T^b\,(py_p - (r_p+\delta_k)\,k_p - e) + \bar{y}_{nb} - \bar{x}_{nb}]/\,(1+\gamma) \\
\kappa' &= \left[(1-\delta_\kappa)\,\kappa + \varsigma h_\kappa^{\vartheta} e^{1-\vartheta}\right]/\,(1+\gamma) \\
y_p &= z k_p^{\alpha} \kappa^{\phi} \left(\omega h_y^{\rho} + (1-\omega)\,n_p^{\rho}\right)^{\frac{\nu}{\rho}}, \quad \alpha+\phi+\nu = 1 \\
c &= \left(\eta c_c^{\varrho} + (1-\eta)\,c_p^{\varrho}\right)^{1/\varrho} \\
\ell &= 1 - h_\kappa - h_y \\
a' &\geq \chi k_p \\
k_p &\geq 0 \\
\ell &\in [0,1]\,.
\end{aligned}
$$

The old businesses solve:

$$
\begin{aligned}
V_{o,p}\,(a,\kappa,\epsilon,z) = \max_{c_c,c_p,k_p,h_y,h_\kappa,a',\kappa'} \{&U\,(c,\ell) \\
&+ \tilde{\beta}\pi_o \sum_{\epsilon',z'} \pi\,(\epsilon',z'|\epsilon,z)\,V_o\,(a',\kappa',\epsilon',z') \\
&+ \tilde{\beta}\iota\,(1-\pi_o) \sum_{\epsilon',z'} \pi\,(\epsilon',z')\,V_y\,(a',\kappa',\epsilon',z')\}
\end{aligned}
$$

$$
\begin{aligned}
a' &= [(1+r)\,a + py_p - (r_p+\delta_k)\,k_p - wn_p - e - (1+\tau_c)\,(c_c + pc_p) \\
&\qquad - T^b\,(py_p - (r_p+\delta_k)\,k_p - e) + T^r]/\,(1+\gamma) \\
\kappa_0' &= \left[(1-\delta_\kappa)\,\kappa + \varsigma h_\kappa^{\vartheta} e^{1-\vartheta}\right]/\,(1+\gamma) \\
\kappa' &= \begin{cases} \kappa_0' & \text{if old to old} \\ (1-\lambda_d)\,\kappa_0' & \text{if old to young} \end{cases} \\
y_p &= \zeta_o z k_p^{\alpha} \kappa^{\phi} \left(\omega h_y^{\rho} + (1-\omega)\,n_p^{\rho}\right)^{\frac{\nu}{\rho}} \\
c &= \left(\eta c_c^{\varrho} + (1-\eta)\,c_p^{\varrho}\right)^{1/\varrho} \\
\ell &= 1 - h_\kappa - h_y \\
a' &\geq \chi k_p \\
k_p &\geq 0 \\
\ell &\in [0,1]\,.
\end{aligned}
$$

where $\iota \in [0,1]$ is a measure of altruism, ζ_o is a measure of lowered productivity in old age, and λ_d is a measure of business capital that is lost when the owner dies.

The young workers solve:

$$\begin{aligned} V_{y,w}\left(a,\kappa,\epsilon,z\right) = \max_{c_c,c_p,\ell,a'} &\{U\left(c,\ell\right) \\ &+ \tilde{\beta}\pi_y \sum_{\epsilon',z'} \pi\left(\epsilon',z'|\epsilon,z\right) V_y\left(a',\kappa',\epsilon',z'\right) \\ &+ \tilde{\beta}\left(1-\pi_y\right) \sum_{\epsilon',z'} \pi\left(\epsilon',z'|\epsilon,z\right) V_o\left(a',\kappa',\epsilon',z'\right)\} \end{aligned}$$

subject to

$$\begin{aligned} a' &= \left[\left(1+r\right)a + w\epsilon n - \left(1+\tau_c\right)\left(c_c + pc_p\right) - T^n\left(w\epsilon n\right) + \bar{y}_{nb} - \bar{x}_{nb}\right]/\left(1+\gamma\right) \\ \kappa' &= \lambda\kappa \\ c &= \left(\eta c_c^{\varrho} + \left(1-\eta\right)c_p^{\varrho}\right)^{1/\varrho} \\ \ell &= 1-n \\ a &\geq \underline{a} \\ n &\in [0,1] \end{aligned}$$

The old workers solve:

$$\begin{aligned} V_{o,w}\left(a,\kappa,\epsilon,z\right) = \max_{c_c,c_p,\ell,a'} &\{U\left(c,\ell\right) \\ &+ \tilde{\beta}\pi_o \sum_{\epsilon',z'} \pi\left(\epsilon',z'|\epsilon,z\right) V_o\left(a',\kappa',\epsilon',z'\right) \\ &+ \tilde{\beta}\iota\left(1-\pi_o\right) \sum_{\epsilon',z'} \pi\left(\epsilon',z'\right) V_y\left(a',\kappa',\epsilon',z'\right)\} \end{aligned}$$

subject to

$$\begin{aligned} a' &= [\left(1+r\right)a + w\epsilon_o n - \left(1+\tau_c\right)\left(c_c + pc_p\right) \\ &\qquad - T^n\left(w\epsilon_o n\right) + T^r + \bar{y}_{nb} - \bar{x}_{nb}]/\left(1+\gamma\right) \\ \kappa' &= \lambda\kappa \\ \epsilon_o &= \zeta_o\epsilon \\ c &= \left(\eta c_c^{\varrho} + \left(1-\eta\right)c_p^{\varrho}\right)^{1/\varrho} \end{aligned}$$

$$\ell = 1 - n$$

$$a \geq \underline{a}$$

$$n \in [0,1]$$

where again ζ_o is a measure of lowered productivity in old age.

The market clearing conditions are now:

$$\begin{aligned}
y_c &= \int \left(c_c\left(s\right) + e\left(s\right)\right)\mu\left(s\right)\, ds + \left(\gamma + \delta_k\right)\left(k_c + \int k_p\left(s\right)\mu\left(s\right)\right) ds + g + \bar{x}_{nb} - \bar{y}_{nb} \\
n_c &= \int n\left(s\right)\epsilon\left(s\right)\mu\left(s\right)\, ds \\
\int a\left(s\right)\mu\left(s\right)\, ds &= b + \left(1 - \tau_d\right)k_c + \int k_p\left(s\right)\mu\left(s\right)\, ds \\
\int y_p\left(s\right)\mu\left(s\right)\, ds &= \int c_p\left(s\right)\mu\left(s\right)\, ds,
\end{aligned}$$

and the government budget constraint is:

$$\begin{aligned}
g + \left(r - \gamma\right)b = \tau_c &\int \left(c_c\left(s\right) + pc_p\left(s\right)\right)\mu\left(s\right)\, ds + \tau_p\left(y_c - wn_c - \delta_k k_c\right) \\
&+ \tau_d\left(y_c - wn_c - \left(\gamma + \delta_k\right)k_c - \tau_p\left(y_c - wn_c - \delta_k k_c\right)\right) \\
&+ \int T^b\left(py_p\left(s\right) - \left(r_p + \delta_k\right)k_p\left(s\right) - wn_p\left(s\right) - e\left(s\right)\right)\mu\left(s\right)\, ds \\
&+ \int T^n\left(w\epsilon\left(s\right)n\left(s\right)\right)\mu\left(s\right)\, ds - T^r\int \chi_{s,old}\mu\left(s\right)\, ds,
\end{aligned}$$

where $\epsilon(s)$ includes the ζ_o factor.

The equations used for computing $\tilde{h}_y$ are the same in this model as in the dynastic model. The only difference here is some additional parameters in the case of older workers.

4. Add Business Sales

In this section, we introduce business sales, setting us up to eventually match model predictions to IRS data on Form 8594.

Let's start with the market arrangement for sale of private businesses. There is an intermediary who buys and sells sweat capital. All agents have an option to sell their sweat capital. Selling entails a capital gains tax τ_{cg}. Let $v_\kappa(s)$ be the price offered by the intermediary to the buy the entire stock of $\kappa(s)$ from agent with state s. The intermediary raises funds by offering a *homogeneous* price quantity bundle $(\bar{p_\kappa}, \bar{\kappa})$ to all potential buyers.

Next, consider the timing. The timing is as follows: Agents enter the period with a, κ. After shocks (z, ϵ) are realized, they decide between the following four options: (1) sell and be a worker; (2) not sell and be a worker; (3) run a pass-through business without buying new sweat capital; or (4) buy new sweat capital and then run pass-through business. Notice that the second and third are what we had before, while the first and fourth options deal with buying and selling. In this case, the occupational choice is summarized by maximization over the four possible options:

$$V(s) = \max\left\{V_w(\tilde{s}), V_w(s), V_p(s), V_p(\tilde{s})\right\} \tag{4.1}$$

where the elements of state $\tilde{s}$ are defined as follows:

$$\begin{aligned} a(\tilde{s}) &= a(s)\, \mathit{II}_{\{\text{option 2 or 3}\}}(s) \\ &\quad + (a(s) + (1-\tau_{cg})\, \upsilon_\kappa(s))\left(\mathit{II}_{\{\text{option 1}\}}(s)\right) \\ &\quad + (a(s) - \bar{p}_\kappa)\left(\mathit{II}_{\{\text{option 4}\}}(s)\right) \\ \kappa(\tilde{s}) &= \kappa(s)\, \mathit{II}_{\{\text{option 2 or 3}\}}(s) + (\kappa(s) + \bar{\kappa})\left(\mathit{II}_{\{\text{option 4}\}}(s)\right) \end{aligned}$$

The last two elements of $\tilde{s}$, are given by $z(\tilde{s})$ and $\epsilon(\tilde{s})$ and are equal to $z(s)$ and $\epsilon(s)$, respectively.

Once the occupational choices are made, agents choose: consumption, savings, leisure, and conditional on running a business production choices. The formulations of V_w and V_p are unchanged.

With business sales, we need to modify the defintion of a competitive equilibrium. The equilibrium definition now includes $(\upsilon_\kappa(s), \bar{p}_\kappa, \bar{\kappa})$ such that

$$V_w(s) = V_w\left(a + (1-\tau_{cg})\, \upsilon_\kappa(s), 0, z(s), \epsilon(s)\right) \tag{4.2}$$

This equation means that (i) the seller has no bargaining power and (ii) and that the intermediary offers a price that equals the outside option of the seller which I take to be as entering the workforce and allowing his sweat capital to depreciate at λ.

The exiting entrepreneurs are indifferent between selling their sweat to the intermediary or holding on to it. We break the tie by requiring π_f of the individuals to sell. This is an exogenously set parameter and we can pin it down using the SBO data on business acquisitions. Owners are asked whether they founded, purchased, inherited or received the business as a gift or transfer. This allows us to determine how many owners built their businesses on their own and how many purchased bought or inherited someone else's sweat capital.

The chunk size $\bar{\kappa}$ is given by

$$\bar{\kappa} = \int \kappa\left(s\right)\mu\left(s\right)\pi_f I\!I_{\{\text{option 1 or 2}\}}\left(s\right)ds \tag{4.3}$$

The zero profit condition for the intermediary is

$$\bar{p_\kappa}\int I\!I_{\{\text{option 4}\}}\left(s\right)\mu\left(s\right)ds = \left(1-\psi_1\right)\int v_\kappa\left(s\right)\mu\left(s\right)\pi_f I\!I_{\{\text{option 1 or 2}\}}\left(s\right)ds \tag{4.4}$$

Given the sale is a taxable transaction, there will be an additional term in the governments' budget constraint, namely,

$$\tau_{cg}\int v_\kappa\left(s\right)\mu\left(s\right)\pi_f I\!I_{\{\text{option 1 or 2}\}}\left(s\right)ds.$$

In order to compute equilibria, we need to make modifications to the computational algorithm. Specifically, we'll need to take the following steps:

- Inner loop: Fix $(p, r, w, \bar{p}_\kappa, \bar{\kappa})$
  - Guess V_w, V_p
  - Compute $v_\kappa(s)$ using (4.2). This step is a bit costly and would require a bisection. Treat $v_\kappa(s)$ in the same way as other value functions and interpolate when simulating.
  - Compute $V(s)$ using (4.1).
  - Solve the Bellman optimization problems to update V_w and V_p
- Outer loop: Use market clearing conditions as before and additionally (4.3) to pin down the additional equilibrium objects $(\bar{p}_\kappa, \bar{\kappa})$.

5. Lucas Span of Control

The reference model for our tax experiments is the model of Lucas (1978). This model is a nested problem with inelastic hours for business owners and no sweat capital κ or expensing e. In this case, the young pass-through business owner solves:

$$V_{y,p}(a,\epsilon,z) = \max_{c_c,c_p,k_p,a'} \{U\left(c,\bar{\ell}\right) + \tilde{\beta} \sum_{\epsilon',z'} \pi\left(\epsilon',z'|\epsilon,z\right) V\left(a',\epsilon',z'\right)\} \tag{5.1}$$

subject to

$$\begin{aligned}
a' &= [(1+r)\,a + py_p - (r_p+\delta_k)\,k_p - wn_p - (1+\tau_c)\,(c_c+pc_p) \\
&\qquad - T^b\left(py_p - (r_p+\delta_k)\,k_p - wn_p\right)]/\left(1+\gamma\right) \\
y_p &= zk_p^{\alpha} n_p^{\nu} \\
c &= \left(\eta c_c^{\varrho} + (1-\eta)\,c_p^{\varrho}\right)^{1/\varrho} \\
a' &\geq \chi k_p \\
k_p &\geq 0.
\end{aligned}$$

The problem for employees is the same as before except the value function does not depend on κ.

Let's start by assuming that $\chi = 0$. The first-order conditions for business owners with respect to c_c, c_p, and k_p are as follows:

$$0 = (1-\psi)\,c^{(1-\psi)(1-\sigma)-1}\bar{\ell}^{\psi(1-\sigma)}\left[\eta c_c^{\varrho-1}c^{1-\varrho}\right] - (1+\tau_c)\,\varphi_a \tag{5.2}$$

$$0 = (1-\psi)\,c^{(1-\psi)(1-\sigma)-1}\bar{\ell}^{\psi(1-\sigma)}\left[(1-\eta)\,c_p^{\varrho-1}c^{1-\varrho}\right] - p\,(1+\tau_c)\,\varphi_a \tag{5.3}$$

$$0 = p\alpha y_p/k_p - (r_p+\delta_k) \tag{5.4}$$

$$0 = p\nu y_p/n_p - w, \tag{5.5}$$

where

$$\varphi_a = \tilde{\beta} \sum_{\epsilon',z'} \pi\left(\epsilon',z'|\epsilon,z\right) V_a\left(a',\epsilon',z'\right)/\left(1+\gamma\right).$$

The ratio of (5.2) and (5.5) yields

$$\frac{c_p}{c_c} = \left(\frac{\eta p}{1-\eta}\right)^{\frac{1}{\varrho-1}}$$

or $c_p = \xi_1 c_c$ as before. Again, we can write out the consumption bundle as

$$c = \left(\eta + (1-\eta)\left(\frac{\eta p}{1-\eta}\right)^{\frac{\varrho}{\varrho-1}}\right)^{1/\varrho} c_c \equiv \xi_2 c_c.$$

In this case, assuming $\alpha + \nu < 1$, we have

$$k_p = \left(\frac{pz\alpha^{1-\nu}\nu^{\nu}}{(r_p + \delta_k)^{1-\nu} w^{\nu}} \right)^{\frac{1}{1-\alpha-\nu}}$$

$$n_p = \frac{\nu (r_p + \delta_k)}{\alpha w} k_p.$$

Relative to the baseline model, the static problem is trivial since we do not need to compute hours in the business.

Now consider the case with $\chi > 0$. We can first solve the static problems assuming $\chi = 0$ and check if $a' \geq \chi k_p$. If it is not, we replace (5.4) with $k_p = a'/\chi$ and

$$n_p = \left(\frac{p\nu z k_p^{\alpha}}{w} \right)^{\frac{1}{1-\nu}}.$$