Notes on Monopoly Pricing and Self-Selection Constraints

Econ 8004 (Spring 2001)
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NOTATION

Let $\theta$ be the type of a consumer. There are $n$ buyer types, $\theta_1 < \theta_2 < \ldots < \theta_n$. Let $f_i$ be the population weight of type $\theta_i$.

Let $V(q, \theta)$ be the reservation price of a buyer of type $\theta$ for a single unit of good of quality $q$.

Let $C(q)$ be the per unit cost to produce one unit of the product with quality $q$.

Define $N(q, \theta) \equiv V(q, \theta) - C(q)$ to be the net surplus.

We use the revelation principle so restrict attention to contracts where the agent reports his or her type $\theta_i$ and the agent is allocated a contract on the basis of this report (any other mechanism is equivalent to some direct revelation mechanism). Let $x_i = (R_i, q_i)$ denote a contract, where $R_i$ is the return to the seller and $q_i$ is the quality (so that the price of the contract is then $P_i = R_i + c(q_i)$).

The utility of a particular contact $x$ to a buyer of type $\theta$ is

$$U(x, \theta) = N(q, \theta) - R.$$

ASSUMPTIONS

1. $N(0, \theta) = 0$ for all $\theta$. 
2. $N(q, \theta)$ is twice continuously differentiable.

3. The net surplus function is concave,

$$\frac{\partial^2 N(q, \theta)}{\partial q^2} < 0,$$

that $\frac{\partial N(q, \theta)}{\partial q} > 0$ and the $\frac{\partial N(q, \theta)}{\partial q} < 0$ for $q$ large enough.

4. The single-crossing condition holds

$$\frac{\partial^2 N(q, \theta)}{\partial q \partial \theta} > 0$$

**THE MONOPOLY PROBLEM WITH PRIVATE INFORMATION**

The monopoly problem with private information can be written as

$$\max_{x_1, x_2, \ldots, x_n} \prod_{i=1}^{n} f_i R_i$$

subject to

Incentive Constraints (IC) : $U(x_i, \theta_i) \geq U(x_j, \theta_i), j \neq i$

Voluntary Participation (VP) : $U(x_1, \theta_1) \geq 0$.

Note that (VP) is only imposed for the lowest type since (IC) and that fact that $U(x_i, \theta_i) \geq U(x, \theta_1)$ for all $i \geq 1$ (from Assumptions 1 and 4) imply that when $U(x_1, \theta_1) \geq 0, U(x_i, \theta_i) \geq 0$ for all higher $i$.

**Lemma 1.** Suppose $x = (R, q)$ and $\tilde{x} = (\tilde{R}, \tilde{q})$ are such that $\tilde{q} > q$. Then

$$\Delta(\theta) = U(\tilde{x}, \theta) - U(x, \theta)$$

is an increasing function of $\theta$.

**Proof.**

$$\Delta'(\theta) = \frac{\partial N(\tilde{q}, \theta)}{\partial \theta} - \frac{\partial N(q, \theta)}{\partial \theta} > 0$$
by A4 and the fact that \( \hat{q} > q \). Q.E.D.

Lemma 1 implies that if \((x_1, x_2, ..., x_n)\) satisfy (IC) then \( q_i \leq q_{i+1} \).

Consider the following relaxed problem

\[
(M') \quad \max_{(x_1, x_2, ..., x_n)} \prod_{i=1}^{n} f_i R_i
\]

subject to

Downward Adjacent Incentive Constraints (DAIC) : \( U(x_i, \theta_i) = U(x_{i-1}, \theta_i), i > 1 \)

Voluntary Participation (VP) : \( U(x_1, \theta_1) \geq 0 \).

Monotonicity : \( q_i \geq q_{i-1}, i > 1 \).

By Lemma 1, it is clear that if \((x_1, x_2, ..., x_n)\) is feasible for \((M)\), then it is feasible for the relaxed problem \((M')\). Below the solution for \((M')\) will be characterized. It will be shown that solutions to \((M')\) all satisfy the incentive constraints (IC) meaning the two problems \((M)\) and \((M')\) have the same set of solutions.

**Proposition.** Let \((x_1, x_2, ..., x_n)\) be a solution to problem \((M')\). Then

(i) \( U(x_i, \theta_i) = U(x_{i-1}, \theta_i), i > 1 \) and \( U(x_1, \theta_1) = 0 \).

(ii) If \( q_i < q_{i+1} \), then \( U(x_i, \theta_i) > U(x_{i+1}, \theta_i) \).

(iii) \((x_1, x_2, ..., x_n)\) satisfies (IC) and is a solution to \((M)\).

(iv) \( q_n = q_n^*, q_i < q_i^*, \) for \( i < n \), if \( q_i^* > 0 \), where \( q_i^* \) is the first-best quality level satisfying

\[
\frac{\partial N(q_i^*, \theta_i)}{\partial q} = 0.
\]

(v) \( q_i \neq q_{i+1} \) implies \( q_i < q_{i+1} \) and \( R_i < R_{i+1} \).

\( q_i = q_{i+1} \) implies that \( R_i = R_{i+1} \).

**Proof of (i)**

Suppose \( U(x_i, \theta_i) > U(x_{i-1}, \theta_i) \) for some \( i > 1 \). Increasing the markup on contract \( x_i \) to \( R_i + \varepsilon \) for \( \varepsilon > 0 \) small enough does not violate (DAIC) but increases profits, a contradiction. Similarly \( U(x_1, \theta_1) > 0 \) results in a contradiction.
Proof of (ii)
This follows from (i) above and Lemma 1.

Proof of (iii).
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Proof of (iv)
The Lagrangian can be written
\[ \mathcal{L} = \sum_{i=1}^{n} f_i R_i + \sum_{i=2}^{n} \lambda_i [U(x_i, \theta_i) - U(x_{i-1}, \theta_i)] + \lambda_1 U(x_1, \theta_1) + \sum_{i=2}^{n} \mu_i [q_i - q_{i-1}] . \]

The first-order conditions are
\[
\begin{align*}
\partial_{R_i} &= f_i + \lambda_{i+1} - \lambda_i = 0, \quad i < n \\
\partial_{R_n} &= f_n - \lambda_n = 0 \\
\partial_{q_1} &= -\lambda_2 \frac{\partial N(q_1, \theta_2)}{\partial q} + \lambda_1 \frac{\partial N(q_1, \theta_1)}{\partial q} - \mu_2 = 0 \\
\partial_{q_i} &= -\lambda_{i+1} \frac{\partial N(q_i, \theta_{i+1})}{\partial q} + \lambda_i \frac{\partial N(q_i, \theta_i)}{\partial q} - \mu_{i+1} + \mu_i = 0, \quad 1 < i < n \\
\partial_{q_n} &= \lambda_n \frac{\partial N(q_n, \theta_n)}{\partial q} + \mu_n = 0
\end{align*}
\]

From $\partial_{R_i}$,
\[ \lambda_i = \frac{X_i}{f_i} > 0. \]

It is shown first that $q_1 < q_1^*$. If $q_1 \geq q_1^*$, then $\frac{\partial N(q_1, \theta_1)}{\partial q} \leq 0$. By A4,
\[ \frac{\partial N(q_1, \theta_2)}{\partial q} > \frac{\partial N(q_1, \theta_1)}{\partial q} . \]

Since $\lambda_1 = \frac{P}{1} f_i < \frac{P}{2} f_i = \lambda_2$, these facts imply that
\[ -\lambda_2 \frac{\partial N(q_1, \theta_2)}{\partial q} + \lambda_1 \frac{\partial N(q_1, \theta_1)}{\partial q} < 0, \]

which contradicts $\partial_{q_1} = 0$. 
Suppose now that \( q_{i-1} < q_i^\ast - 1 \). It will be shown that if \( i < n \), then \( q_i < q_i^\ast \), which by induction will show that \( q_i < q_j^\ast \) for all \( j < n \). If \( q_{i-1} = q_i \), then this follows from the fact that \( q_i = q_{i-1} < q_{i-1}^\ast < q_i^\ast \). If \( q_{i-1} < q_i \), then the multiplier \( \mu_i = 0 \). It can then be shown that \( q_i < q_i^\ast \) in the same manner used to show that \( q_i < q_i^\ast \).

It remains to show that \( q_n = q_n^\ast \). If \( q_n = q_{n-1} \), then \( \frac{\partial N(q_n, \theta_n)}{\partial q} > 0 \) since \( q_n = q_{n-1} < q_{n-1}^\ast < q_n^\ast \), which contradicts \( q_n = 0 \). Therefore \( \mu_n = 0 \) and \( q_n = q_n^\ast \) follows from \( q_n = 0 \).

Proof of (v).

From (i) above,

\[
R_{i+1} - R_i = N(q_{i+1}, \theta_{i+1}) - N(q_i, \theta_{i+1}).
\]

If \( q_{i+1} = q_i \), then \( R_{i+1} = R_i \). If \( q_{i+1} > q_i \), then \( R_{i+1} > R_i \) follows from the fact that \( q_{i+1}^\ast \geq q_{i+1} > q_i \).

Q.E.D.