Dynamic Duopoly Assignment for Econ 8601
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With this assignment you will calculate the equilibrium for a dynamic duopoly model as well as for a monopoly model. The technology is the same as in the model in Gowrisankaran and Holmes. To numerically calculate the equilibrium value functions and policy functions, you will use the Chebyshev approximation techniques discussed in Judd (1999). To make your programming easier, I change the notation somewhat to be consistent with Judd’s notation.

Suppose there is a demand function in an industry given by $D(P) = Q$. Let $P(Q) = D^{-1}(Q)$ be the inverse demand. Let $x$ denote an amount of capital (Here use $x$ instead of $K$ to be consistent with Judd). Let $q$ denote an amount of output per unit of capital so total output is $Q = qx$ and next period’s capital stock is $x' = \sigma Q$, where $\sigma = 1 - \delta$ is the amount of capital that survives net of depreciation. Let $h(q)$ be cost per unit of capital from operating at level $q$.

It is useful to begin by solving for the stationary equilibrium price if this industry were perfectly competitive. Then the monopoly problem is discussed followed by the duopoly problem. The last section lays out the specific assignment..

0.1 Perfect Competition

Suppose agents take as given that the price is constant in each period at $p$. Let $v_t$ be the discounted value of owning one unit of capital at the beginning of the period $t$. This must solve

$$v_t = \max_q pq - h(q) + \beta \sigma q v_{t+1}$$

(1)

The FONC of this problem is

$$p - h'(q_t) + \beta \sigma v_{t+1} = 0$$

(2)

In a stationary equilibrium,

$$\sigma q = 1$$

1
or
\[ q^* = \frac{1}{\sigma} \]

And \( v^* \) is
\[ v^* = pq^* - h(q^*) + \beta \sigma q^* v^* \]
\[ = pq^* - h(q^*) + \beta v^* \]

so
\[ v^* = \frac{pq^* - h(q^*)}{1 - \beta} \]

From the FONC
\[ p = h'(q^*) - \beta \sigma v^* \]

Plugging in the formula for \( v^* \) yields
\[ p = h'(q^*) - \beta \sigma pq^* - h(q^*) \frac{1}{1 - \beta} \]

Solving for \( p \) yields the stationary competitive price
\[ p^*_C = (1 - \beta) h'(q^*) + \beta \sigma h(q^*)) \]

Note this has an interpretation as stationary marginal cost. Let \( Q^*_C \) be the stationary competitive output and \( x^*_C = \sigma Q^*_C \) be the stationary competitive capital level.

### 0.2 Pure Monopoly

Consider the monopoly problem. The state variable is \( x \), the beginning of period capital. Let \( w(x) \) be discounted maximized monopoly profit. This solves

\[ w(x) = \max_q P(xq)xq - xh(q) + \beta w(\sigma xq) \]

The FONC is
\[ Px + P'x^2q - xh' + \beta \sigma x \frac{dw}{dx} = 0 \]

or, dividing by \( x \),
\[ P + P'xq - h' + \beta \sigma \frac{dw}{dx} = 0 \]
It is possible to use the envelope theorem to verify that

\[
\frac{dw}{dx} = qh'(q) - h(q)
\]

(Think of \( Q \) has the choice variable. If this is held fixed, a change in \( x \) affects (minus) total cost \( = -xh\left( \frac{Q}{x} \right) \). Differentiating with respect to \( x \) yields the above.) Plugging this into the first-order condition and evaluating at the steady state output level \( q^* = \frac{1}{\sigma} \) yields

\[
p + P'qx - h' + \beta \sigma [qh' - h] = 0
\]

or

\[
p + P'q^*x = (1 - \beta) h' + \beta \sigma h
\]

\[= P^*_C.\]

Thus in the stationary monopoly equilibrium, marginal revenue equals dynamic marginal cost. Let \( x \) solving the above be denoted \( x_M^* \). This is the monopoly stationary capital stock.

1. **Duopoly**

Now consider duopoly. Let \( x \) be the initial capital stock of firm 1 and \( y \) the initial stock of firm 2. Let \( w_1(x, y) \) value function for firm 1 and let \( q_1(x, y) \) be the value function and the policy function. Restrict attention to symmetric Markov-perfect equilibrium so

\[
q_2(y, x) = q_1(x, y).
\]

In a symmetric MPE, \( q_1(x, y) \) solves

\[
\max_q q x P(qx + q_2y) - x h(q) + \beta w_1(\sigma qx, \sigma q_2y)
\]

For \( q_2 = q_2(x, y) \). The FONC is

\[
xP + qxP'x - xh'(q) + \beta \sigma x \frac{\partial w_1}{\partial x} = 0
\]

\[
P + qxP' - h'(q) + \beta \sigma \frac{\partial w_1}{\partial x} = 0
\]
2. Assignment

Take the parameterization

\[ h(q) = \frac{q^2}{2} \]
\[ p(Q) = Q^{-\frac{1}{2}} \]
\[ Q = p^{-2} \]

Assume \( \beta = .5 \) and \( \sigma = .5 \). Calculate \( K_c^* \) and \( K_M^* \).

2.1 Part 1: Monopoly Problem

Use value function iteration and Chebyshev approximation (page 223 in Judd) to calculate the equilibrium value function for the monopoly problem.

Use \( n = 5 \) (the order of the polynomials) and \( m = 10 \) (the number of grid points). Let \( a = .5x_M^* \) and \( b = 1.5x_M^* \); these are the endpoints of the grid using Judd’s notation.

Iterate on the vector \((a_0, a_1, ..., a_n)\) which is the vector determining the approximation of \( w(x) \) (Sorry for the awkward notation where \( a \) denotes two things; this is Judd’s fault). Start with \( a_i = 0 \) for all \( i \) and stop when

\[ \max_{i \in \{0, n\}} |a_i^{t+1} - a_i^t| < .000001 \]

where \( t \) denotes a particular iteration.

After the value function converges approximate the policy function \( q(x) \). Let the initial capital level be \( x_0 = a = .5x_M^* \) and calculate for periods 1-25 the following variables: \( x_t, q_t, P_t \) and \( w_t(x_t) \). Compare with \( x_M^*, q^*, P_M^* \) and \( w_M^* \), the stationary monopoly levels.

2.2 Part 2: Duopoly Problem

Let \((a_0, ..., a_n)\) be the coefficient vector for the value function \( v_1(x,y) \) approximation and \((b_0, ..., b_n)\) the coefficient vector for the policy function \( q_1(x,y) \) approximation. Use Judd’s techniques for approximation in \( R^2 \) (page 238) to approximate the Markov perfect equilibrium. Note you need to iterate on \( q_1 \) as well as \( v_1 \) since firm 1 takes firm 2’s action as given in the problem (and \( q_2(x,y) = q_1(y,x) \)).
Let \( a = 0.25x_M^* \) and \( b = x_C^* \) be the end points of the grid.

Solve for the equilibrium path for the first 25 periods starting at \( x_0 = b \) and \( y_0 = a \).

Print out \( q_{1,t}, q_{2,t}, x_t, y_t, \) and \( P_t \). What happens to market share over time?

Let \( (x_D^*, y_D^*) \) be the stationary duopoly capital stocks. Let \( MR_1^* \) be the marginal revenue of firm 1 at the stationary level,

\[
MR_1^* = P_D^* + P_D^*q^*x_D^*
\]

Verify that

\[
MR_1^* < p_C^*
\]

(Note the marginal revenue at \( t = 25 \) is a sufficiently close approximation). Thus marginal revenue is less than stationary marginal cost. What is the intuition for why this is the case?

To help with the intuition for why this is the case, consider the following alternative duopoly problem. Suppose that at time 0 the initial state is \((x_0, y_0)\) where \( x_0 = y_0 \). Suppose at time 0 each firm selects an output choice \( q_1 \) and \( q_2 \) that is constant over time. The environment is therefore one of a one-shot game. Suppose that firm 1 takes as given the \( q_2 = \frac{1}{\sigma} = q^* \). Solve for the initial capital levels \( x_0 = y_0 \) such that it is optimal for firm 1 to choose \( q_1 = q^* \) taking as given that \( q_2 = q^* \). In what way does the analysis of this alternative environment help with the intuition mentioned above?