Consider the following duopoly model. There are two firms, 1 and 2, and time is discrete, $t \in \{0, 1, 2, \ldots\}$. Let $q_{1,t}$ and $q_{2,t}$ be the output of each firm in period $t$ and let $Q_t = q_{1,t} + q_{2,t}$ be total output. Suppose the inverse demand is constant over time and linear, $P = A - Q$.

The average production cost in period $t$ is constant in current production but depends upon industry production in the previous period. Let $X_t = Q_{t-1}$ denote total industry output from the previous period. Specifically, the marginal cost (and average cost) in period $t$ of each firm is $c(X_t)$, where $c(0) \equiv \bar{c} < A$, $c' < 0$, $c'' > 0$, and $\lim_{X \to \infty} c(X) \equiv \underline{c} > 0$. So given last period industry output $X$, if firm $i$ produces $q_i$ units in the current period, its total cost in the period is $q_i c(X)$. Note the learning by doing here occurs at the industry level. There is a knowledge spillover here since firm 2’s production cost next period are smaller when firm 1 produces more in the current period.

Suppose the discount factor is $\beta < 1$.

Assume in each period $t$ the two firms simultaneously choose output levels $q_{1,t}$ and $q_{2,t}$ in a Cournot fashion.

(a) Define a Markov-perfect equilibrium in this model. Define a stationary equilibrium.

(b) Suppose $\beta = 0$. Determine the transition equation mapping last period’s industry output $X$ to this period’s industry output $Q$. Under what condition does there exist a unique stationary equilibrium?

(c) Consider a two-period version of the model, $t \in \{1, 2\}$ and let $X_1 = 0$ be the initial state. If is possible that an equilibrium path for $\beta > 0$ would be the same as the equilibrium path when $\beta = 0$?
Question 2

Take the dynamic industry model discussed in class. Assume the parameterization
\[
\begin{align*}
    c(q) & = \frac{q^2}{2} \\
    D(p) & = p^{-\varepsilon_D} = p^{-2} \\
    P(Q) & = Q^{-\frac{1}{\sigma}}
\end{align*}
\]
where \(\varepsilon_D\) is the elasticity of demand. Assume \(\beta = .5\) and \(\sigma = 1 - \delta = .5\). Following the class notes:
\[
\begin{align*}
    q^* & = \frac{1}{\sigma} = 2 \\
    p_C^* & = (1 - \beta)c'(q^*) + \beta\sigma c(q^*) \\
    & = .5q^* + .25q^{*2} = 1.5 \\
    p_M^* & = \frac{\varepsilon_D}{\varepsilon_D - 1}p_C^* = 3 \\
    Q_C^* & = p_C^{*2} \\
    Q_M^* & = p_M^{*2} \\
    K_C^* & = \sigma Q_C^* \\
    K_M^* & = \sigma Q_M^*
\end{align*}
\]

(a) Use value function iteration and Chebyshev approximation (page 223 in Judd) to calculate the equilibrium value function for the monopoly problem.

Use \(n = 5\) (the order of the polynomials) and \(m = 10\) (the number of grid points). Let \(a = .5K_M^*\) and \(b = 1.5K_M^*\); these are the endpoints of the grid using Judd’s notation.

Iterate on the vector \((a_0, a_1, ..., a_n)\) which is the vector determining the approximation of \(w(K)\) (Sorry for the awkward notation where \(a\) denotes two things; this is Judd’s fault). Start with \(a_i = 0\) for all \(i\) and stop when
\[
\max_{i \in \{0,n\}} |a_{i+1} - a_i| < .000001
\]
where \(t\) denotes a particular iteration.

After the value function converges approximate the policy function \(q(K)\). Let the initial capital level be \(K_0 = a = .5K_M^*\) and calculate for periods 1-25 the following variables: \(K_t, q_t, P_t\) and \(w_t(K_t)\). Compare with \(K_M^*, q^*, P_M^*\) and \(w_M^*\), the stationary monopoly levels.
(b) Let \((a_0, \ldots, a_n)\) be the coefficient vector for the value function \(v_1(K_1, K_2)\) approximation and \((b_0, \ldots, b_n)\) the coefficient vector for the policy function \(q_1(K_1, K_2)\) approximation. Use Judd’s techniques for approximation in \(R^2\) (page 238) to approximate the Markov perfect equilibrium. Note you need to iterate on \(q_1\) as well as \(v_1\) since firm 1 takes firm 2’s action as given in the problem (and \(q_2(x, y) = q_1(y, x)\)).

Let \(a = .25K_1^*\) and \(b = K_2^*\) be the end points of the grid.

Solve for the equilibrium path for the first 25 periods starting at \(K_{1,0} = b\) and \(K_{2,0} = a\). Print out \(q_{1,t}, q_{2,t}, K_{1,t}, K_{2,t}\), and \(P_t\). What happens to market share over time?

Let \((K_{1D}^*, K_{2D}^*)\) be the stationary duopoly capital stocks. Let \(MR_1^*\) be the marginal revenue of firm 1 at the stationary level,

\[
MR_1^* = P_D^* + P_{D'}^* q_1^* K_{1D}^*
\]

Verify that

\[
MR_1^* < p_C^*
\]

(Note the marginal revenue at \(t = 25\) is a sufficiently close approximation). Thus marginal revenue is less than stationary marginal cost. What is the intuition for why this is the case?

To help with the intuition for why this is the case, consider the following alternative duopoly problem. Suppose that at time 0 the initial state is \((K_{1,0}, K_{2,0})\) where \(K_{1,0} = K_{2,0}\). Suppose at time 0 each firm selects an output choice \(q_1\) and \(q_2\) that is constant over time. The environment is therefore one of a one-shot game. Suppose that firm 1 takes as given the \(q_2 = \frac{1}{\sigma} = q^*\). Solve for the initial capital levels \(K_{1,0} = K_{2,0}\) such that it is optimal for firm 1 to choose \(q_1 = q^*\) taking as given that \(q_2 = q^*\). In what way does the analysis of this alternative environment help with the intuition mentioned above?