Selection, Growth and the Size Distribution of Firms

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Employer Firms

• 5,697,759 U.S. firms with non-zero employment in 2002

• 913 firms with more than 10,000 employees

• 770,041 firms with zero employees in March 2002

• Small Business Administration

• Size categories 5,000-9,999 and 10,000 and over from Statistics of U.S. Businesses, U.S. Census Bureau
The Pareto Distribution

Suppose

\[ f(S) = \zeta S^{-(1+\zeta)}, \quad S \geq 1 \]

Then

\[ T(S) = \int_S^\infty \zeta X^{-(1+\zeta)}dX = S^{-\zeta} \]

and

\[ M(S) = \int_S^\infty \zeta X^{-\zeta}dX = \frac{\zeta}{\zeta - 1} S^{1-\zeta} \]

So:

\[ \ln[T(S)] = -\zeta \ln[S] \]

and:

\[ \ln[M(S)] = c - (\zeta - 1) \ln[S] \]
Number of Firms by Employment Size

log cumulative firm count

log firm employment

4
Cumulative Employment in Right Tail

fraction of all employees

firm employment
Employer Firms

log of number of employees

number of firms

$3 \times 10^6$

log of number of employees

0 1 2 3 4 5 6 7 8 9

0 0.5 1 1.5 2 2.5 3

0 0.5 1 1.5 2 2.5 3
Employer Firms

log of number of employees

number of firms in right tail

\(6 \times 10^6\)

log of number of employees
Employer Firms

log of number of employees

log of number of firms in right tail

log of number of firms in right tail vs. log of number of employees
Employer Firms

log of number of employees

log of number of firms in right tail

slope = −1.0629
\[
\text{log size} \quad \text{log right tail probability}
\]

+ : 2002
Firms versus Plants

log right tail probability

log employment
Estimated Pareto Density

log size
A Model That Seems to Fit Like a Dream...

\[ s = \ln(\text{employees}) \]

\[ \ln(\text{number of firms to right of } s) \]

log normal
Constant returns to scale—size arbitrary:

- Each employee hires new employee at rate $\lambda$ per unit of time
- Firms transition from $n$ to $n + 1$ at rate $\lambda n$ per unit of time
- New firms enter with $n = 1$ at a rate $(\gamma - \lambda) \sum_{n=1}^{\infty} n M_t(n)$ per unit of time
- $M_n(t) =$ measure of firms of size $n$ at time $t$

Then (Yule, 1925):

$$P_n = \frac{\gamma \Gamma(n) \Gamma \left(1 + \frac{\gamma}{\lambda}\right)}{\lambda \Gamma(n + 1 + \frac{\gamma}{\lambda})}$$

Observe:

$$\lim_{\lambda \to \gamma} P_n = \frac{1}{n(n + 1)}$$

and:

$$\sum_{k=n}^{\infty} \frac{1}{k(k + 1)} = \frac{1}{n}$$
Team of a manager with skill $z$ with $n$ workers:

- produce $X_t z A(n)$
- decreasing returns to $n$
- skill distribution $P(z)$

For example, $A(n) = n^\beta$, $P(z) = 1 - z^{-\alpha}$:

$$\Pr[N(z) \geq n] \propto n^{-\alpha(1-\beta)}$$

Size distribution of firms reflects skill distribution of managers.

Panel driven by $X_t$.

Identity of managers could change as they move through $P$. 
Goods in $[0, 1]$. Quality ladder. Incumbents and entrants attempt to innovate.

Poisson innovations, endogenous:

- firms transit from $n$ goods to $n + 1$ at a rate $\mu n$
- and from $n$ to $n - 1$ at a rate $\lambda n$
- entry with 1 good at rate $(\lambda - \mu)n$

Size distribution:

$$P_n \propto \frac{1}{n} \left( \frac{\mu}{\lambda} \right)^n$$
The implied Zipf plot, $x = \ln(n)$ (up to “size” $e^7 \approx 1000$):

Variance of firm growth rate: $(\lambda + \mu)/n$. 
The Variance of Growth Rates

The Yule process:

\[
\text{var} \left[ \frac{X_{t+\Delta} - X_t}{X_t} \mid X_t = N \right] \approx \frac{1}{N} e^{\mu \Delta} (e^{\mu \Delta} - 1)
\]

Mean growth rate is \(\mu\) and:

\[
\lim_{\Delta \downarrow 0} \frac{1}{\Delta} \text{var} \left[ \frac{X_{t+\Delta} - X_t}{X_t} \mid X_t \right] = \frac{\mu}{X_t}
\]

Instead, if \(X_t = X_0 \exp \left( \left[ \mu - \frac{1}{2} \sigma^2 \right] t + \sigma W_t \right)\) then:

\[
\lim_{\Delta \downarrow 0} \frac{1}{\Delta} \text{var} \left[ \frac{X_{t+\Delta} - X_t}{X_t} \mid X_t \right] = \sigma^2
\]
THIS PAPER

Firms are monopolistic competitors.

Permanent shocks to preferences and technologies associated with firms.

Low productivity firms exit, new firms imitate and attempt to enter.

- Selection produces Pareto right tail rather than log-normal.
- Population productivity grows faster than mean of incumbents.
- Thickness of right tail depends on the difference.
- Zipf tail when entry costs are high or imitation is difficult.

Champernowne (1948), Gabaix (1999).

The Economy

Preferences:

- differentiated commodities with permanent taste shocks

Technologies:

- at a cost, entrants draw technologies from some distribution
- fixed overhead labor, asymptotic constant returns to scale
- random productivity, quality growth.

Two versions: (1) technology adoption and (2) endogenous growth.
Consumers

A population $H e^{\eta t}$ with preferences over aggregate consumption:

$$\left( E \left[ \int_0^\infty \rho e^{-\rho t} [C_t e^{-\eta t}]^{1-\gamma} dt \right] \right)^{1/(1-\gamma)}$$

where:

$$C_t = \left[ \int u^{1-\beta} c_t^\beta(u, p) dM_t(u, p) \right]^{1/\beta}$$

Real expenditures are:

$$\frac{pc_t(u, p)}{P_t} = (uC_t)^{1-\beta} c_t^\beta(u, p), \quad P_t = \left[ \int up^{-\beta/(1-\beta)} dM_t(u, p) \right]^{-(1-\beta)/\beta}$$
Firms

Firm-specific output and technologies.

Asymptotic constant returns to scale.

“Volatile memory:”

—continuation requires $\lambda_F$ units of labor per unit of time.

Unit arrival rate of new firms costs $\lambda_E$ units of labor per unit of time.
Production

Output:

\[ y_{t,a} = z_{t,a} L_{t,a} \]

Implied variable profits:

\[ \max_L \left\{ Z_{t,a}^\beta C_{t+a}^{1-\beta} L^\beta - w_{t+a} L \right\} \]

where:

\[ Z_{t,a} = \left( u_{t,a}^{1-\beta} z_{t,a}^\beta \right)^{1/\beta} \]

evolves according to the black-box process:

\[ Z_{t,a} = Z \exp (\theta_E t + \theta_I a + \sigma_Z W_a) \]

The initial condition \( Z \) is drawn from some distribution \( G \).
The Growth Rate

Balanced growth:

• wages $w_t = we^{\kappa t}$

• aggregate consumption $C_t = Ce^{(\kappa + \eta)t}$

• the number of firms $M_t = Me^{\eta t}$.

Distribution of $Z_{t,a}^\beta C_{t+a}^{1-\beta} L_{t,a}^\beta - w_{t+a} L_{t,a}$ must have a trend $e^{\kappa t}$.

This yields:

$$\kappa = \underbrace{\theta_E}_{\text{quantity and quality}} + \underbrace{\left(\frac{1 - \beta}{\beta}\right) \eta}_{\text{variety}}$$
The Firm-Specific State Variable

Variable profits:

\[ Z_{t,a}^\beta C_{t+a}^{1-\beta} L_{t,a}^\beta - w_{t+a}L_{t,a} = w_{t+a} \left[ S_{t,a}^{1-\beta} L_{t,a}^\beta - L_{t,a} \right] \]

where:

\[ S_{t,a} = \left( \frac{Z_{t,a}}{w_{t+a}} \right)^{\beta/(1-\beta)} \frac{C_{t+a}}{w_{t+a}} \]

Dynamics:

\[ S_a = \exp \left( s[Z] \right) \left[ \exp \left( [\theta_I - \theta_E]a + \sigma_Z W_a \right) \right]^{\beta/(1-\beta)} \]

where:

\[ e^{s[Z]} = \left( \frac{Z}{w} \right)^{\beta/(1-\beta)} \frac{C'}{w} \]
The Firm-Specific State Variable

\[ S_a = \exp(s[Z]) \left[ \exp \left( [\theta_I - \theta_E]a + \sigma_Z W_a \right) \right]^{\beta/(1-\beta)} \]

where:

\[ e^{s[Z]} = \left( \frac{Z}{w} \right)^{\beta/(1-\beta)} \frac{C'}{w} \]

So \( s_a = \ln(S_a) \) follows:

\[ ds_a = \mu da + \sigma dW_a \]

where:

\[ \begin{bmatrix} \mu \\ \sigma \end{bmatrix} = \frac{\beta}{1 - \beta} \begin{bmatrix} \theta_I - \theta_E \\ \sigma_Z \end{bmatrix} \]

Typically, \( \mu < 0 \), but can have \( \mu > 0 \) if \( \eta > 0 \).
Variable Profits per unit of Fixed Costs

Let $L(S)$ solve:

$$\lambda_F Q(S) = \max_L \left\{ S^{1-\beta} L^\beta - L \right\}$$

This gives

$$L(S) = \beta^{1-\beta} S$$

And

$$Q(S) = \frac{1}{\lambda_F} \left[ \frac{1}{\beta} - 1 \right] \beta^{1-\beta} S$$

Tractable extension:

$$y = zA(L), \text{ where } A \text{ is convex-concave and asymptotically linear.}$$
The Stopping Problem

The value of a firm with productivity $Z_{t,a}$ at time $t + a$ is:

$$w_{t+a} \lambda_F V \left( s \left[ Z_{t,a} e^{-\theta E t} \right] \right)$$

where:

$$V(s) = \max \left[ \mathbb{E} \left[ \int_0^\tau e^{-(r-\kappa)a} [Q(e^{s_0}) - 1] \, da \mid s_0 = s \right] \right]$$

The Bellman equation is ($\mathcal{A} = \text{Apply Ito}$):

$$rV(s) = \kappa V(s) + \mathcal{A}V(s) + Q(e^s) - 1$$

At the exit barrier $b$:

$$V(b) = 0, \, DV(b) = 0$$

A further boundary condition is $V(s) \leq e^s / (r - [\kappa + \sigma^2 / 2])$. 
Solution

The

\[ V(s) = \frac{1}{r - \kappa} \frac{\xi}{1 + \xi} \left( e^{s-b} - 1 - \frac{1 - e^{-\xi(s-b)}}{\xi} \right) \]

where

\[ e^b = \left( \frac{\xi}{1 + \xi} \right) \left( 1 - \frac{\mu + \frac{1}{2} \sigma^2}{r - \kappa} \right) \]

and

\[ \xi = \frac{\mu}{\sigma^2} + \sqrt{\left( \frac{\mu}{\sigma^2} \right)^2 + \frac{r - \kappa}{\sigma^2/2}} \]

(One of the roots of \( r - \kappa = \mu x + \frac{1}{2} \sigma^2 x^2 \) is \( x = -\xi \).)
Log of profitability $q = \ln[Q(b)]$ at exit, as a function of drift:

$$d = -\mu/(\sigma^2/2) \text{ and } [-\mu, \sigma] = [\theta_E - \theta_I, \sigma_Z] \beta/(1 - \beta).$$

Faster aggregate productivity growth: firms “throw in the towel” more quickly.
The Value Function

\[ V(s) \]

\[ \zeta = 1.11 \]

\[ \zeta = 1.33 \]

\[ \rho = 0.02 \]

\[ \sigma = 0.15 \]

\[ u(c) = \ln(c) \]
Entry

Labor cost of an arrival rate of $I_t$ entry opportunities per unit of time:

$$L_{E,t} = \lambda_E I_t$$

An entry opportunity yields a draw $Z$ from a distribution $G$.

Zero-profit condition:

$$\lambda_E = \lambda_F \int V(s[Z]) G(dZ)$$

Technology adoption: $G$ exogenous.
Kolmogorov Forward Equation

\[ y_{t+h} = y_t + \begin{cases} \mu h + \sigma \sqrt{h} & \text{w.p. } \frac{1}{2} \\ \mu h - \sigma \sqrt{h} & \text{w.p. } \frac{1}{2} \end{cases} \]

Let \( f(t, y) \) be the density at time \( t \):

\[ f(t + h, y) = \frac{1}{2} f(t, y - \mu h - \sigma \sqrt{h}) + \frac{1}{2} f(t, y - \mu h + \sigma \sqrt{h}) \]

Therefore:

\[
\frac{1}{h} \left[ f(t + h, y) - f(t, y) \right] = \frac{1}{h} \left[ f(t, y - \mu h) - f(t, y) \right] + \\
\frac{1}{2} \frac{\sigma^2}{(\sigma \sqrt{h})^2} \left[ f(t, y - \mu h - \sigma \sqrt{h}) - 2 f(t, y - \mu h) + f(t, y - \mu h + \sigma \sqrt{h}) \right]
\]

Taking limits:

\[ D_t f(t, y) = -\mu D_y f(t, y) + \frac{1}{2} \sigma^2 D_y^2 f(t, y) \]
Aside on Exit Rates

Suppose:
\[ dy_t = \mu dt + \sigma dW_t \]

together with an exit barrier at \( b \), so that \( f(t, b) = 0 \).

Measure of a cohort:
\[ m(t) = \int_b^\infty D_t f(t, y) dy \]

Then, using integration-by-parts twice:
\[ Dm(t) = \int_b^\infty D_t f(t, y) dy = \int_b^\infty \left[ -\mu D_y f(t, y) + \frac{1}{2} \sigma^2 D_{yy} f(t, y) \right] dy = -\frac{1}{2} \sigma^2 D_y f(t, b) \]
Firm Population Dynamics

Density of firms:

\[ k(t, a, s) = m(a, s) I e^{nt} \]  (!)

Kolmogorov:

\[ D_t k(t, a, s) = -D_a k(t, a, s) - \mu D_s k(t, a, s) + \frac{1}{2} \sigma^2 D_{ss} k(t, a, s) \]

Therefore:

\[ D_a m(a, s) = -\eta m(a, s) - \mu D_s m(a, s) + \frac{1}{2} \sigma^2 D_{ss} m(a, s) \]

At age zero:

\[ \lim_{a \downarrow 0} \int_b^s m(a, x)dx = F(s) - F(b) \]

where \( G(Z) = F(s[Z]) \). At the exit boundary, \( m(a, b) = 0 \).
Integrate forward equation over age:

\[-m(0, s) = \int_0^\infty D_a m(a, s) da = -\eta m(s) - \mu D_s m(s) + \frac{1}{2} \sigma^2 D_{ss} m(s)\]

Thus, assuming \( F \) has a density:

\[\eta m(s) = -\mu D_s m(s) + \frac{1}{2} \sigma^2 D_{ss} m(s) + D F(s)\]

Linear, second-order, inhomogeneous ODE with constant coefficients.
The Stationary Density of Age and Size

\[ m(a, s) = \int_b^\infty e^{-\eta a} \psi(a, s | x) F(dx) \]

where:

\[ \psi(a, s | x) = \frac{1}{\sigma \sqrt{a}} \left[ \phi \left( \frac{s - x - \mu a}{\sigma \sqrt{a}} \right) - e^{-\frac{(x-b)^2}{2}} \phi \left( \frac{s + x - 2b - \mu a}{\sigma \sqrt{a}} \right) \right] \]

and where \( \phi \) is the standard normal probability density.

Interpretation of \( \psi(a, s | x) \):

the density of survivors at age \( a \) with profitability \( s \) of the cohort that entered with the same initial profitability \( x \) (not a p.d.f.)
The Life of a Cohort
The Hazard Rate Conditional on Fixed Characteristics

![Graph showing the hazard rate conditional on fixed characteristics. The graph plots exit rate against age. There are two curves on the graph, one labeled $x - b = 1$ and another labeled $x - b = 0.75$. The x-axis represents age, ranging from 0 to 10, and the y-axis represents the exit rate, ranging from 0 to 0.5.](image)
The Size Marginal

Integrating over age gives:

\[ m(s) = \int_b^\infty \pi(s|x) \left( \frac{1 - e^{-\zeta(s-b)}}{\eta} \right) F(dx) \]

where:

\[ \pi(s|x) = \zeta e^{-\zeta(s-b)} \left( \frac{e^{\zeta_*(s-b)} - 1}{\zeta_*} \right)^{-1} \min \left\{ \frac{e^{[\zeta+\zeta_*(s-b)]} - 1}{\zeta + \zeta_*}, \frac{e^{[\zeta+\zeta_*(x-b)]} - 1}{\zeta + \zeta_*} \right\} \]

and:

\[ \zeta = -\frac{\mu}{\sigma^2} + \sqrt{\left( \frac{\mu}{\sigma^2} \right)^2 + \frac{\eta}{\sigma^2/2}} \quad \zeta_* = \frac{\mu}{\sigma^2} + \sqrt{\left( \frac{\mu}{\sigma^2} \right)^2 + \frac{\eta}{\sigma^2/2}} \]
The Power Law

The size marginal is a weighted average of:

$$\int_{0}^{\infty} e^{-\eta a} \psi(a, s \mid x) da \propto e^{-\zeta(s-b)} \left( \min \left\{ e^{[\zeta+\zeta^*](s-b)}, e^{[\zeta+\zeta^*](x-b)} \right\} - 1 \right)$$

If $\eta = 0$ then $\zeta^* = 0$ and:

$$\zeta = -\frac{\mu}{\sigma^2/2} = \frac{\theta_E - \theta_I}{\frac{1}{2} \left( \frac{\beta}{1-\beta} \right) \sigma_Z^2}$$

where:

$$\theta_E = \text{growth rate in population}$$

$$\theta_I = \text{growth rate among incumbents}$$
Stationary Size Density Conditional on Fixed Characteristics

\[ \pi(s|x) \propto 1 - e^{-\zeta(s-b)} \]

\[ \beta \propto e^{-\zeta(s-b)} \]

\[ \beta \propto e^{-\zeta(s-b)} \]
Suppose $F$ is a point mass at $x > b$ and $\eta = 0$.

Exit rate per unit of time:

$$\frac{1}{2}\sigma^2 D\pi(b|x) = \frac{1}{2}\sigma^2 \left(\frac{\zeta}{x - b}\right)$$

Contribution of selection to average size:

$$\frac{1}{2}\sigma^2 D\pi(b|x) \times (x - b) = \frac{1}{2}\sigma^2 \zeta = -\mu$$

In terms of productivity:

$$\theta_E = \theta_I + \frac{1}{2}\sigma^2 \left(\frac{\beta}{1 - \beta}\right) \zeta$$
Technology Adoption – Assumptions

A 1: The value of human capital is finite:
\[ \rho + \gamma \kappa > \kappa \]

A 2: The value of a firm conditional on initial conditions is finite:
\[ \rho + \gamma \kappa > \kappa + \mu + \frac{1}{2} \sigma^2 \]

A 3: The value of entry is well defined and finite:
\[ \int Z^{\beta/(1-\beta)} G(dZ) < \infty \]

A 4: The stationary distribution of firm characteristics has a finite mean:
\[ \eta > \mu + \frac{1}{2} \sigma^2 \]
Technology Adoption – Equilibrium Conditions

Zero profit:

\[ \lambda_E = \lambda_F \int V(s[Z])G(dZ), \quad e^s[Z] = \frac{1 - \beta}{\lambda_F} \left( \frac{C}{w} \right) \left( \frac{\beta Z}{w} \right)^{\beta/(1-\beta)} \]

Goods market:

\[ \frac{C}{w} = \frac{\lambda_F I}{1 - \beta} \int_b^\infty e^s m(s) ds \]

Labor market:

\[ \begin{bmatrix} L_E & L_F & L \end{bmatrix} = \begin{bmatrix} \lambda_E & \lambda_F \int_b^\infty m(s) ds & \left( \frac{\beta}{1-\beta} \right) \lambda_F \int_b^\infty e^s m(s) ds \end{bmatrix} I \]

\[ H = L_E + L_F + L \]
Some Comparative Statics

Per capita consumption $C/H$ is a function of $(\lambda_E, \lambda_F)/H$.

A proportional reduction in $(\lambda_E, \lambda_F)/H$:

- no change in $s[Z]$ or $m(\cdot)$
- a proportional increase in $I$
- no change in $C/w$
- an increase in $C$ and $w$ with elasticity $(1 - \beta)/\beta$

A shift from $G(Z)$ to $G(Z/x)$, $x > 1$:

- no change in $s[Z]$, $m(\cdot)$ or $I$
- an increase in $C$ and $w$ by a factor $x$. 
But...

How come it so happens that:

$$\eta > (\theta_I - \theta_E) \left( \frac{\beta}{1 - \beta} \right) + \frac{1}{2} \sigma_Z^2 \left( \frac{\beta}{1 - \beta} \right)^2$$

and only just?

Make the distribution of $Z_{t,0} = Z e^{\theta_E t}$, as defined by $G(\cdot)$ and $\theta_E$, endogenous.

Say, $\theta_I$ is difficult to move around.
Imitation

Endogenous trend productivity $\theta_E$ and entry distribution $G$:

— entrant draws random firm from incumbent population

— productivity of randomly drawn incumbent at $t$: $X e^{\theta_E t}$

— entrant can imperfectly imitate this firm: $s[Z] = s[X] - \delta$

Additional equilibrium condition: $G$ replicates itself (modulo shift $\delta$.)

Initial conditions determine the level of the balanced growth path.
Entrants Imitate Incumbents

\[ f(x + \delta) \]

\[ f(x) \]
Endogenous Growth – Imitation Dynamics

Fix a candidate growth rate $\theta_E$ [implies $\mu, b, V(\cdot)$].

Fix attempted entry rate per incumbent, $\varepsilon_A > \eta$.

Imperfect imitation gives rise to entry from shifted density of incumbents:

$$\eta f(s) = -\mu Df(s) + \frac{1}{2} \sigma^2 D^2 f(s) + \varepsilon_A f(s + \delta)$$

This has two solutions, except for a unique $\varepsilon_A > \eta$ for which the following “characteristic equation” has only one solution:

$$\eta = \mu z + \frac{1}{2} \sigma^2 z^2 + \varepsilon_A e^{-\delta z}$$

A stability argument suggests this $\varepsilon_A$.

The solution is a Gamma:

$$f(s) = \alpha^2 (s - b) e^{-\alpha(s - b)}$$

where $\alpha$ is increasing in $\theta_E$. 
Endogenous Growth - Imitation Dynamics

Specifically:

\[ f(s) = \alpha^2(s - b)e^{-\alpha(s-b)} \]

where:

\[ \alpha = -\left( \mu + 1 \right) + \sqrt{\left( \mu \right)^2 + \frac{1}{\delta^2} + \frac{\eta}{\sigma^2/2}} \]

and:

\[ -\frac{\mu}{\sigma^2/2} = \frac{\theta_E - \theta_I}{\frac{1}{2}\left( \frac{\beta}{1-\beta} \right)\sigma_Z^2} \]

For large \( \delta \):

\[ \alpha \approx -\frac{\mu}{\sigma^2/2} \]
Endogenous Growth – Imitation Dynamics and Zero Profits

Determine $\theta_E$ using the zero profit condition:

$$\lambda_E = \lambda_F \int_b^\infty V(x)f(x + \delta)dx$$

where imitation and population dynamics gives:

$$f(s) = \alpha^2(s - b)e^{-\alpha(s - b)}$$

and:

$$\alpha = -\left(\frac{\mu}{\sigma^2} + \frac{1}{\delta}\right) + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{1}{\delta^2} + \frac{\eta}{\sigma^2/2}}$$

$V$ increasing in $\mu$ and recall:

$$-\frac{\mu}{\sigma^2/2} = \frac{\theta_E - \theta_I}{\frac{1}{2} \left(\frac{\beta}{1-\beta}\right) \sigma^2 Z}$$
The Zero-Profit Condition

\[ \frac{\lambda_E}{\lambda_F} = 0 \]

\[ \delta = 2 \ln(10) \]

\[ \delta = 4 \ln(10) \]
Why the Asymptote at $\alpha = 1$?

Faster aggregate productivity growth:

— lowers $V(x + b)$
— shifts $f(x + b + \delta)$ to the left (first-order stochastic dominance)

This lowers the value of entry.

The value of attempting entry has a vertical asymptote at $\alpha = 1$ since:

$$V(s) \sim e^{s-b}, \quad \int_b^{\infty} e^{s-b} f(s)ds = \left(\frac{a}{a-1}\right)^2$$

Key:

mechanism relates the value of attempting entry to the average size (profitability) of incumbents—the latter grows without bound near 1.
Growth and Selection \((\eta = 0)\)

\[
\frac{1}{2}\sigma^2Df(b) \times \frac{\int_b^\infty (s - b)f(s + \delta)ds}{\int_b^\infty f(s + \delta)ds} = \frac{1}{2}\sigma^2\alpha = \frac{1}{2} \left( \frac{\beta \sigma Z}{1 - \beta} \right)^2 \alpha
\]

The equilibrium growth rate is:

\[
\theta_E = \theta_I \text{ incumbents} + \frac{1}{2}\sigma^2 \left( \frac{\beta \alpha}{1 - \beta} \right) \text{ selection}
\]

and \(\alpha\) is a decreasing function of \(\lambda_E/\lambda_F\) in equilibrium.

Every firm is an experiment, costly to set up and costly to run.

Growth speeds up when it becomes easier to replace failing experiments with new experiments that are based on successful existing experiments.
A Back-of-the-Envelope Calculation

Successful entry rate $\varepsilon_S$:

$$
\varepsilon_S = \eta \frac{\text{growth # of firms}}{\text{exit rate}} + \frac{1}{2} \sigma^2 D f(b) = \eta + \frac{1}{2} \sigma^2 \alpha^2
$$

Recall:

$$
\alpha \approx -\frac{\mu}{\sigma^2/2}
$$

Then:

$$
\sigma^2 = \frac{\varepsilon_S - \eta}{\alpha^2/2} = \frac{.116 - .01}{(1.06)^2/2} = (.43)^2 \quad \ldots
$$

and:

$$
-\mu \approx \frac{1}{2} \sigma^2 \alpha = \frac{\varepsilon_S - \eta}{\alpha} = \frac{.116 - .01}{1.06} = .1
$$

At $\beta = .9$:

$$
\kappa = \frac{\theta_I}{.006} + \frac{\theta_E - \theta_I}{-(1-\beta) \mu = .0130} + \left(\frac{1-\beta}{\beta} \right) \frac{\eta}{.001} = .02
$$
The Hazard Rate

Proposition:

If $\delta = 0$ then the hazard rate is constant

If $\delta > 0$ then the hazard rate $h(a)$ declines with age and:

$$\lim_{a \downarrow 0} h(a) = \infty$$

$$\lim_{a \to \infty} h(a) = \frac{1}{2} \left( \frac{-\mu}{\sigma} \right)^2$$
Predicted Survival Rates

\[ \delta = \infty \]
\[ \delta = 0 \]
\[ \delta = 1.98 \]

- Audretsch (1991)
- Meta and Portugal (1994)
- Headd (2002)

Surviving firms as fraction of age cohort vs. years since entry.
Heterogeneous Sectors

Cobb-Douglas utility:

\[ C_t = \prod_{n=1}^{N} C_{n,t}^{\nu_n} \quad \text{and} \quad C_{n,t} = C_n e^{(\kappa_n + \eta)t} \]

Growth rates:

\[ \kappa = \sum_{n=1}^{N} \nu_n \kappa_n \quad \text{and} \quad \kappa_n = \theta_n + \left(\frac{1 - \beta_n}{\beta_n}\right) \eta \]

Profitability state variables:

\[ e^{s_n[Z]} = \frac{\nu_n(1 - \beta_n)}{\lambda_{F,n}} \left(\frac{\beta_n Z P_n / P}{w}\right) C \]

\[ P = \prod_{n=1}^{N} \left(\frac{P_n}{\nu_n}\right)^{\nu_n} \]

Economy-wide size distribution:

\[ f(s) = \sum_{n=1}^{N} p_n f_n \left(s - \ln \left(\frac{\lambda_{F,n}}{1 - \beta_n}\right)\right) \quad \text{and} \quad p_n \propto \nu_n \left(\frac{\lambda_{F,n}}{1 - \beta_n} \int_{b_n}^{\infty} e^s f_n(s) ds\right)^{-1} \]
Equilibrium Conditions – Technology Adoption

Zero profit:

\[ \lambda_{E,n} = \lambda_{F,n} \int V_n(s_n[Z])G_n(dZ), \quad e^{s_n[Z]} = \frac{\nu_n(1 - \beta_n)}{\lambda_{F,n}} \left( \frac{C'}{w} \right) \left( \frac{\beta_n Z P_n}{w} \right)^{\beta_n/(1-\beta_n)} \]

Goods market:

\[ \nu_n \left( \frac{C'}{w} \right) = \frac{\lambda_{F,n} I_n}{1 - \beta_n} \int_{b_n}^{\infty} e^s m_n(s) ds \]

Labor market:

\[ H = \sum_{n=1}^{N} (L_{E,n} + L_{F,n} + L_n), \quad \begin{bmatrix} L_{E,n} \\ L_{F,n} \\ L_n \end{bmatrix} = I_n \begin{bmatrix} \lambda_{E,n} \\ \lambda_{F,n} \int_{b_n}^{\infty} m_n(s) ds \\ \frac{\beta_n}{1-\beta_n} \lambda_{F,n} \int_{b_n}^{\infty} e^s m_n(s) ds \end{bmatrix} \]

\[ w = \prod_{n=1}^{N} \left( \frac{\nu_n w}{P_n/P} \right)^{\nu_n} \]
Some Comparative Statics

Per capita consumption $C/H$ is a function of $\{(\lambda_E, \lambda_F,n)/H\}_{n=1}^N$.

A proportional reduction in $\{(\lambda_E, \lambda_F,n)/H\}_{n=1}^N$ leads to:

— no change in $s_n[Z]$ or $m_n$
— a proportional increase in $I_n$
— no change in $C/w$
— an increase in $w/(P_n/P)$ with elasticity $(1 - \beta_n)/\beta_n$
— an increase in $C$ and $w$ with elasticity $\sum_{n=1}^N \nu_n(1 - \beta_n)/\beta_n$
— a shift of resources to industries with a stronger taste for variety.

A common shift from $G_n(Z)$ to $G_n(Z/x)$, $x > 1$

— no change in $s_n[Z]$, $m_n$ or $I_n$
— an increase in $C$ and $w$ by a factor $x$. 
A Model That Seems to Fit Like a Dream ...

\[ s = \ln(\text{employees}) \]

\[ \ln(\text{number of firms to right of } s) \]

log normal
ANALYTICALLY TRACTABLE EXTENSIONS

• physical capital

• an initial growth phase of exponentially distributed duration

• shape of $A(L)$ can be used to interpret deviations from Gibrat’s Law

• international trade:
  
  — fixed cost of entering and maintaining a foreign market [Opromolla and Irrarrazabal, 2005]

• advertising and market access [Arkolakis, 2006]
Concluding Remarks

Key ingredients:

— iso-elastic utility functions
— asymptotic constant returns
— permanent idiosyncratic shocks to preferences and technology
— some mechanism that connects the value of attempting entry to the average size of incumbents

Pareto-like right tails arise even if there is a lot of fixed heterogeneity.

Zipf’s Law follows if:

— entry costs are large relative to fixed costs
— imitation is difficult.