Lecture 2

Evolution of Market Concentration

- Examine concentration in a structure with long-run constant returns to scale

- Static Cournot Duopoly (do this quickly!)

- Dynamic Duopoly (Deterministic, wait till Tues for stochastic structure)
Technology

- $K_i$ capital of firm $i$
- $Q_i$ output of firm $i$
- $q = \frac{K}{Q}$ output per unit of capital
- $c(q)$ cost per unit of capital when output intensity is $q$. $c' > 0$, $c'' > 0$.
- $C(q) = Kc(q)$ is total cost
Example:

- Cobb-Douglas $Q = L^{\alpha}K^{1-\alpha}$.

- Suppose $L$ is $1$ per unit.

\[
C(Q) = \left[ \frac{Q}{K^{1-\alpha}} \right]^{\frac{1}{\alpha}}
\]
\[
c(q) = q^{\frac{1}{\alpha}}
\]
Static Cournot

- $K_i$ fixed

- $P(Q)$ industry demand where $P'(Q) < 0$.

- Cournot problem. Firm 1 takes $q_2$ as fixed. Maximize profits per unit of capital

$$\max_{q_1} P(K_1 q_1 + K_2 q_2) q_1 - c(q_1)$$

- FONC

$$P(K_1 q_1 + K_2 q_2) + P'(K_1 q_1 + K_2 q_2) K_1 q_1 - c'(q_1) = 0$$
• SOC

\[ 2P'(K_1q_1 + K_2q_2)K_1 + P''(K_1q_1 + K_2q_2)K_1^2q_1 - c''(q_1) < 0 \]

• Reaction function \( q_1 = R(q_2) \) solves above.

• If \( K_1 = K_1 \), then weak conditions get existence of symmetric equilibrium (if reaction function continuous. \( (P'' \leq 0 \) is sufficient)

• Let \( q^c \) solve \( q^c = R(q^c) \).
Infinitely Repeated Game (supergame)

- $K_1 = K_2 = 1$ fixed over time.

- $\beta$ discount factor

- Can collusion be supported?

- Let $q^m$ solve the above

\[
\max_{q_1, q_2} P(q_1 + q_2)(q_1 + q_2) - c(q_1) - c(q_2)
\]

\[
FONC : P + P' - c'(q_i) = 0
\]
\[
\pi^c = P(q^c)q^c - c(q^c) \\
\pi^m = P(q^m)q^m - c(q^m)
\]

- Can show \(\pi^c < \pi^m\). So have standard prisoner’s dilemma.

- Can collusive solution be supported?
Trigger Strategies

- If deviate play Cournot forever, otherwise $q^m$

- Return to cooperation

\[
\frac{1}{1 - \beta} \pi^m
\]

- Return to deviating

\[
\max_{q_1} P(q_1 + q^m)q_1 - c(q_1) + \frac{\beta}{1 - \beta} \pi^c = \pi^{dev} + \frac{\beta}{1 - \beta} \pi^c
\]
• Won’t deviate iff

\[ \pi^{dev} - \pi^m \leq \frac{\beta}{1 - \beta} (\pi^m - \pi^c) \]

so get cooperation for sufficiently high \( \beta \).

• More complicated solutions if there is uncertainty, imperfect monitoring, etc. (Abreu, Pearce, and Staccetti).
Markov Perfect Equilibria (Maskin and Tirole)

- Equilibrium policy functions depend only on payoff relevant states. Let $s$ be a vector of such states.

- $\pi_i(a_1, a_2, s)$ current period payoff to player $i$ given actions $a_1$ and $a_2$ in the current period and state $s$. $\pi_1$

- $s' = f(a_1, a_2, s)$ be transition function

- Let $\tilde{a}_i(s)$ be policy function and suppose $\tilde{v}_i(s)$ satisfies

$$\tilde{v}_1(s) = \max_{a_1} \pi(a_1, \tilde{a}_2(s), s) + \beta \tilde{v}_1(f(a_1, \tilde{a}_2(s), s))$$

and let $\tilde{a}_1$ be the solution. Suppose $\tilde{v}_2(s)$ and $\tilde{a}_2(s)$ satisfy the analogous relationships. Then $(\tilde{a}_1, \tilde{a}_2, \tilde{v}_1, \tilde{v}_2)$ is a Markov-perfect equilibrium.
Cournot Duopoly

- Suppose

\[ K_1 = K_2 = 1 \]

fixed over time.

—What is the set of Markov-perfect equilibria?

—What is the set of payoff-relevant states?

- Suppose

\[ K_{i,t} = Q_{i,t-1}(1 - \delta) \]

—Interpretation: use capital to make new capital.
—Adjustment costs (Lucas 1967, Prescott and Visscher (1980))

• Can separate output and investment. Add an output stage after the investment state. Assume $Q_i$ is capital and $Y_i$ is output. Suppose $Y_i \leq Q_i$ and zero marginal cost up to capacity. Suppose demand is elastic. Then firms always produce up to capacity.

• Define a Markov-perfect equilibrium

• What is a steady state?
Dynamics with $\beta = 0$

- Given $(K_1, K_2)$, solve the (asymmetric) Cournot duopoly problem

- Claim: if $K_1 > K_2$ then $q_1 < q_2$, but $q_1 K_1 > q_2 K_2$.

—FONC for two firms

\[
\begin{align*}
P + P'q_1K_1 - c'(q_1) &= 0 \\
P + P'q_2K_2 - c'(q_2) &= 0
\end{align*}
\]

Suppose instead that $q_1 \geq q_2$.

$\Rightarrow c'(q_1) \geq c'(q_2)$
⇒ $P'q_1K_1 \geq P'q_2K_2$

⇒ $K_1 \leq K_2$, a contradiction.

- Claim market shares converge to equality.

- 

$$\frac{K_1'}{K_2'} = \frac{q_1K_1(1-\delta)}{q_2K_2(1-\delta)} = \frac{q_1K_1}{q_2K_2} < \frac{K_1}{K_2}$$

But

$$1 < \frac{K_1'}{K_2'}$$
• So converge to 50-50 monotonically.

—Kydland, Dominant firm literature

• Intuition?

• Suppose $\beta > 0$

—analytic results difficult

—will go to computer and work this out

—Suppose commit to sequence of outputs. Does this matter? Look at $T = 2$ case.
Comment About the Role of Commitment

- MPE equilibrium very different from outcome of simultaneous move game where firm one and two pick vectors \((q_{11}, q_{12}, q_{13}, \ldots)\) and \((q_{21}, q_{22}, q_{23}, \ldots)\)
Benchmark Case of Perfect Competition Steady State

• Suppose agents take as given a constant price $p$.

• Let $v$ be the discounted value of owning one unit of capital at the beginning of a period

$$v = \max_q pq - c(q) + \beta \sigma qv$$

where

$$\sigma = 1 - \delta$$

• FONC

$$p - c'(q) + \beta \sigma v = 0 \quad (1)$$
• In a stationary equilibrium,
\[
\begin{align*}
\sigma q & = 1 \\
q^* & = \frac{1}{\sigma}
\end{align*}
\]

• \(v^*\) solves
\[
\begin{align*}
v^* & = pq^* - c(q^*) + \beta \sigma q^* v^* \\
& = pq^* - c(q^*) + \beta v^*
\end{align*}
\]
so
\[
v^* = \frac{pq^* - c(q^*)}{1 - \beta}
\]

• From the FONC
\[
p = c'(q^*) - \beta \sigma v^*
\]
• Plugging in the formula for $v^*$ yields

\[ p = c'(q^*) - \beta \sigma \frac{pq^* - c(q^*)}{1 - \beta} \]

Solving for $p$ yields the stationary competitive price

\[ p_C^* = (1 - \beta)c'(q^*) + \beta \sigma c(q^*). \]

• $Q^*_C$ be the stationary competitive output

• $x^*_C = \sigma Q^*_C$ be the stationary competitive capital level.
Pure Monopoly.

- The state variable is $K$ at the beginning of period capital. Let $w(K)$ be discounted maximized monopoly profit. This solves

$$w(K) = \max_q P(Kq) Kq - Kc(q) + \beta w(\sigma Kq)$$

- The FONC is

$$PK + P'K^2q - Kc' + \beta\sigma K \frac{dw}{dK} = 0$$

- Dividing by $x$,

$$P + P'Kq - c' + \beta\sigma \frac{dw}{dK} = 0$$
• Use the envelope theorem to verify that
\[ \frac{dw}{dK} = qc'(q) - c(q) \]
(Think of \( Q \) as the choice variable....).

• Plugging this into the first-order condition and evaluating at the steady state output level \( q^* = \frac{1}{\sigma} \) yields
\[ p + P'qK - c' + \beta \sigma [qc' - c] = 0 \]
or
\[ p + P'q^*K = (1 - \beta)c' + \beta \sigma c = P_C^*. \]

• Let \( K \) solving the above be denoted \( K^*_M \).

• Now calculate the equilibrium off the steady state
A Technical Aside

Numerical Solutions of Dynamic Programming Problems

Monopoly Problem

• Statement of problem. \( w(K) \) value function and \( q(K) \) is policy function. Contraction mapping: Let \( w_0 \) be value function beginning next period. Then

\[
    w_1(K) = \max_q P(Kq)Kq - Kc(q) + \beta w_0(\sigma Kq).
\]

A solution is where \( w_1(K) = w_0(K) \) for all \( K \).

• Iterate
• How do numerically? Need an approximation for $w_0$.

• Discretize? Works well with single agent decision theory. For duopoly problem though continuity is useful.

• Polynomial approximation.
Example with Linear Approximation

1. Start with approximation

\[ \hat{w}_0(K) = \alpha_0 + \beta_0 K \]

2. Take a set of \( m \) evaluation points \( \tilde{K} = \{\tilde{K}_1, \tilde{K}_2, ..., \tilde{K}_m\} \)

3. Solve problem at each of this points with \( \hat{w}_0(K) \) instead of \( w_0(K) \).

\[ \tilde{w}_{1,i} = \max_q P(\tilde{K}_i q) \tilde{K}_i q - \tilde{K}_i c(q) + \beta \hat{w}_0(\sigma \tilde{K}_i q) \]

4. Yields a vector \( \tilde{W}_1 = (\tilde{w}_{1,1}, \tilde{w}_{1,2}, ..., \tilde{w}_{1,m}) \)
5. Use OLS to determine a new approximation

\[
\begin{pmatrix}
\alpha_1 \\
\beta_1
\end{pmatrix}
= (X'X)^{-1} X' \tilde{W}_1
\]

\[
X = 1^\sim \hat{K}
\]

6. Iterate until obtain convergence in \((\alpha_t, \beta_t)\)
General Polynomial Approximation

- Chebyshev polynomials (in class of orthogonal polynomials)

- Defined on range $x \in [-1, 1]$

$$T_n(x) = \cos(n \cos^{-1} x)$$
Figure 1:
Recipe in Judd

- **Step 1:** Evaluation points
  
  \[ z_k = - \cos\left(\frac{2k - 1}{2m} \pi\right), \quad k = 1, \ldots, m \]

- **Step 2:** Adjust the notes to the \([a,b]\) interval (here \(a = .5K^*_M, \ b = 1.5K^*_M\))

  \[ x_k = (z_k + 1) \left(\frac{b - a}{2}\right) + a, \quad k = 1, \ldots, m \]

- **Step 3:** Evaluate \(w(x)\) at the approximation nodes

  \[ \tilde{w}_k = w(x_k), \quad k = 1, \ldots, m \]
• Step 4: Compute the Chebyshev coefficients (remember $T_i$ orthogonal)

$$a_i = \frac{\sum_{k=1}^{m} \tilde{w}_k T_i(z_k)}{\sum_{k=1}^{m} T_i(z_k)^2}$$

• To arrive at the approximation

$$\hat{w}(x) = \sum_{i=0}^{n} a_i T_i(2\frac{x-a}{b-a} - 1)$$
Hints for Duopoly Problem

• \((a_0, \ldots a_n)\) coefficient vector for the value function \(v_1(K_1, K_2)\) approximation

• \((b_0, \ldots, b_n)\) coefficient vector for the policy function \(q_1(K_1, K_2)\) approximation.

• Use Judd’s techniques for approximation in \(R^2\) (page 238)

• You need to iterate on \(q_1\) as well as \(v_1\) since firm 1 takes firm 2’s action as given in the problem (and \(q_2(x, y) = q_1(y, x)\)).