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A Perpetual Race to Stay Ahead

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This paper presents a model of dynamic competition between two firms that repeatedly engage in an innovative activity. The state of competition—measured by the difference between the number of innovations introduced by the firms—evolves stochastically according to their effort level. The structure of Markov perfect equilibria is identified. It is generally not true that competition is fiercest when firms are closest. Rather, firms invest under two distinct circumstances: while sufficiently ahead, to outstrip their rival and secure a durable leadership; while behind, to regain leadership and prevent the situation from worsening to the point where their rival outstrips them.

1. INTRODUCTION

In markets where the technological leader earns a substantial premium, competition takes on the characteristics of an endless race. The purpose of this paper is to explain the dynamics of such a race. It is important to understand the evolution of competition not only for its own sake, but also because this evolution is a critical determinant of industry growth and long-term market structure.

Research-intensive industries, such as the pharmaceutical and electronics industries, are typical examples of oligopolistic markets in which technological competition is characterized by a series of incremental innovations. The industry leader is selected primarily on the basis of quality, not price. To become a leader, firms frequently introduce innovations, ranging from simple improvements and upgrades, to new versions of the product. This innovation process is risky, costly and gradual, and often favours the leader, because of experience, learning-by-doing, and users' feedback, perhaps best exemplified by the practice of beta versions in the software industry.

Such a market structure cannot be adequately modelled by a repeated game since history obviously affects market opportunities. More precisely, previous outcomes determine the range of available actions and pay-offs. Economists have thus resorted to the study of races to gain insights into the strategic issues raised by such competition. With the notable exceptions of Aoki (1991), Harris (1991) and Budd, Harris and Vickers (1993), most of the literature has analysed formal races, displaying exogenous finishing lines or deadlines, as well as final prizes and rewards, and the corresponding results rely on backward induction. However, such assumptions do not seem to capture accurately the essential features of many markets. Often, the rivalry only ends, if at all, when one of the competitors withdraws, so that the deadline should be endogenized. Moreover, benefits and losses are continuously collected. Additionally, the emerging empirical literature on races stresses the inadequacy of standard models and the need to model R&D as a race with multiple prizes. In the words of Cockburn and Henderson (1994), "Our finding that the modern game theoretic literature is of only limited usefulness as an empirical guide points to the need (...) to model R&D as a race with multiple prizes."

The model analysed in this paper is an attempt to offer an adequate framework for this type of dynamic competition. The stochastic evolution of the state of competition between two firms depends on their respective effort levels. The state of competition is measured by the difference between successful innovations introduced by the firms. A higher effort level, though more costly, is conducive to innovative success. Success, in turn, makes it more likely that the state of competition evolves in the direction where the firm exerting high effort is leader and is rewarded with correspondingly larger revenues.

This model is an extension of Aoki (1991) in several respects. There, two firms of equal efficiency engage in R&D competition for an infinite number of periods.[1] Their R&D technology is deterministic: a firm either makes a costly effort and advances its state of knowledge by one step, or it does not and its state does not change. Only the firm at the frontier of knowledge sells the product and earns a monopoly profit which is independent of the actual level of the frontier. Firms maximize their expected discounted pay-off, and attention is restricted to equilibria in symmetric and stationary strategies. If costs are not too high relative to profits, both firms invest when they are even, and possibly randomize their investment decision when the follower is one step behind. In any case, the follower does not invest at all when it is two or more steps behind, and the outcome of the race is predictable—except possibly for the stochastic element introduced by mixed strategies. The gap between firms never exceeds two steps, and firms never alternate positions as leaders and followers.

In this paper, the technology is not restricted to being deterministic. That is, an investment level generates a probability distribution over outcomes. Second, the technology may depend on the respective positions of the players. Third, players may have different pay-off parameters and attention is therefore not restricted to symmetric strategies.

However, this model is highly stylized, and is meant to be so. It is as simple as possible given the main goal of this paper: to show that the strategic nature of the race generates an incentive to compete vigorously, a factor that is usually ignored by analyses of formal races or of non-strategic settings. Indeed, one might expect so-called (m, M)-strategies to be optimal, a situation in which high effort is exerted if and only if the firm's lead or lag is sufficiently small. This is indeed the case under circumstances described in what follows, namely when agents are relatively impatient and the leader enjoys a strong advantage in the innovative activity. This, however, is not true in general. The standard intuition is incomplete. Effort affects the state transition directly by improving one's own speed of innovation, but also indirectly through its effect on the rival's incentives to innovate. As a result, equilibrium behaviour is more intricate. To state this point as clearly as possible, I focus on a very simple pay-off structure, corresponding to what appears to be the most favourable scenario for equilibria in (m, M)-strategies: the flow pay-off from being ahead does not depend on the size of the lead, just as the flow pay-off from being behind does not depend on the size of the lag. Increasing one's lead of course remains valuable as it affects the expected duration of the lead. Even in this case, there are typically two distinct circumstances (or intervals of states) in which a player chooses to exert high effort: while sufficiently ahead, to finish off his opponent and secure a durable leadership, thereby deterring his opponent from racing with him; while behind, to try to regain the lead and prevent the situation from worsening to the point where it is his opponent who tries to finish him off. But in intermediate situations, as when the lead is moderate, low effort may be optimal, because there is neither immediate danger of losing the leadership nor a clear prospect of securing it by getting the laggard to loosen his grip. Thus, it is not necessarily true that competition is fiercest when competitors are close, as could be suggested by some existing models (see Aghion, Harris and Vickers, 1997). The outcome determined by (m, M)-strategies may be described as leapfrogging, in which the laggard actively tries to usurp the leader's position, while more general optimal strategies generate patterns corresponding to "frontier-hugging", since the laggard simply tries to close the

1. Aoki also develops two richer variations, but does not characterize their equilibria. The first of these variations is a particular case of the special case studied in Section 3, for which an equilibrium characterization is provided.

gap between himself and the technological leader. In fact, leapfrogging and frontier-hugging are the two kinds of catching-up behaviour identified by recent empirical investigations in the high-end computer industry (see Khanna, 1995). This paper does not claim, obviously, to identify the structure of equilibria for general pay-off structures. Naturally, for a more complicated pay-off structure, one should expect even more intricate equilibrium strategies. Indeed, once the effects highlighted in this model are understood, it becomes easy to construct pay-off structures in which there will be four, six or more distinct circumstances in which players exert high effort, based on the very two motivations identified in this paper.

To gain further understanding, two extreme cases are studied. In the first case, it is assumed that high effort always results in innovative success, while in the second low effort always results in failure. These two extreme cases are more tractable versions that exhibit optimal strategies with a similar, "bimodal" structure and straightforward interpretations. In the second case, a distinction arises relative to the firms' behaviour as the lead of one of them is increasing. When is the leader the first to exert low effort? The answer to this question determines whether the competition eventually comes to an end, or whether both firms perpetually compete for leadership, which changes hands over and over.

The next section introduces the model, presents the results in the general case, and examines two extreme cases. Section 3 further discusses the findings and relates them to the existing literature. Section 4 concludes.

2. THE RACE

2.1. *The model*

There are two players, Player 1 and Player 2. There are two possible *outcomes*, which I denote as Success (S) and Failure (F). A generic outcome is denoted Y. There are two *effort levels*, or *actions*, called high (H) and low (L). A generic effort level is denoted E. A player can be in one of two *positions*, either ahead or behind. Time is discrete and indexed by $t \in \mathbb{N}_0$.

In every period, both players simultaneously choose an action. An outcome results for each player, whose probability of Success depends only on his current effort choice and on his current position. In particular, it does not depend on the other player's action. The probability of Success of a player who is ahead and who exerts effort level E is denoted α^E. The probability of Success of a player who is behind and who exerts effort level E is denoted β^E. Hence, for instance, β^L refers to the probability of Success of a player who is behind and exerts low effort. High effort increases the probability of being successful for any given position: $\alpha^H > \alpha^L$, $\beta^H > \beta^L$. Of particular interest is the case in which position does not affect the probability of Success. In fact, all the results that follow hold for the more general case in which a Player is at least as likely to succeed while ahead as when behind, for a given effort choice: $\alpha^E \geqq \beta^E$, $\forall E \in \{H, L\}$. All probabilities belong to the interval $(0, 1)$.

Position is meant to capture relative overall success. To this end, I define the *state* of the game in period t, for any $t \in \mathbb{N}_0$, as the difference between the total number of Successes of Player 2 and those of Player 1, computed from period 0 to period $t - 1$, period $t - 1$ included. For $t = 0$, it is equal to zero. (This assumption is inessential, as attention is restricted to subgame perfect equilibria.) The state space is thus $\mathbb{Z}$, the set of integers, and overall, Player 2 has more successes in period t than does Player 1 if and only if the state is positive. The state in period t is denoted by k_t. The state determines position. Player 2 is ahead in period t whenever $k_t > 0$, while Player 1 is ahead whenever $k_t < 0$. When $k_t = 0$, Player 1 is ahead or behind with equal probability. Player $i \in \{1, 2\}$ is behind if and only if Player $j \neq i$ is ahead.

In period t, for any $t > 0$, both players observe all outcomes and actions up to and including period $t - 1$. A *Markov strategy* (or *Markov policy*) for Player $i = 1, 2$ is a mapping from $\mathbb{Z}$ to

$[0, 1]$, where $\tau_i(k)$ is the probability of high effort exerted by Player i at state $k \in \mathbb{Z}$. Abusing notation when actions are pure, let $\tau_i(k) = H$ (resp. $\tau_i(k) = L$) whenever the probability of high effort is 1 (resp. 0) at state k under strategy τ_i. The space of Markov strategies for Player i is denoted M_i. Attention is restricted to Markov strategies, hereafter simply referred to as strategies. Whenever possible, I write $\sigma_i(k)$ (or σ_i^k) for the probability of Success of Player i at state k under strategy τ_i: for instance, for $k > 0$, $\sigma_2(k) = \tau_2(k)\alpha^H + (1 - \tau_2(k))\alpha^L$. A pair of strategies $\tau \triangleq (\tau_1, \tau_2)$ uniquely maps $t \in \mathbb{N}_0$ into a probability measure π_t^τ on $(\mathbb{Z}, 2^{\mathbb{Z}})$. The probability of being in state k in period t, evaluated in period 0, under the strategy profile τ is denoted $\pi_t^\tau(k)$.

In every period, Player i obtains a reward $r_i(k, E)$ which depends only on his effort level E and position, as implied by $k \in \mathbb{Z}$. This reward is assumed to have the following separable form: it is equal to the difference between the *revenue*, which depends only on the player's position, and the *cost*, which depends only on his effort level. The revenue of Player i who is ahead is denoted by $R_i > 0$, which is larger than the revenue of Player i who is behind, normalized to $-R_i$.[2] This encompasses the case where the unnormalized revenue while behind is 0, as assumed in Aoki (1991), which can be interpreted as selling nothing. Let $c_i > 0$ be the cost of exerting high effort, which is strictly larger than the cost of exerting low effort, hereafter normalized to 0.

Player i discounts future rewards at rate $\delta_i \in (0, 1)$, and maximizes normalized, total discounted expected rewards, or overall *pay-off*. Given strategies τ_i and τ_j, the overall pay-off to Player i is denoted by $V_i(\tau_i, \tau_j)$. Finally, write $V_i(\tau_i, \tau_j; k)$ for the normalized, discounted expected future rewards of Player i starting at state $k \in \mathbb{Z}$ given strategies τ_i and τ_j. In particular, $V_i(\tau_i, \tau_j) = V_i(\tau_i, \tau_j; 0)$. When the reference to τ is understood, let $V_i(k)$ be the value for Player i of this pay-off evaluated from k on.

Player i's objective is thus to maximize:

$$V_i(\tau_i, \tau_j) \equiv (1 - \delta_i) \sum_{t=0}^{\infty} \sum_{k=-\infty}^{\infty} \pi_t^{\tau_i, \tau_j}(k) \delta_i^t r_i(k, \tau_i(k)),$$

where $r_i(k, \tau_i(k)) = \tau_i(k) r_i(k, H) + (1 - \tau_i(k)) r_i(k, L)$ is the expected reward of Player i given state k and Markov strategy τ_i. A strategy that attains the maximum given τ_j is said to be *optimal* given τ_j. The boundedness of rewards ensures that the objective is well-defined and that optimal strategies exist.

I focus attention on *Markov perfect equilibria* (or MPE), that is, on subgame perfect equilibria in Markov strategies. The focus on Markov strategies is restrictive. Although this model does not satisfy Dutta's (1995) sufficient conditions for a folk theorem, it is not difficult to find parameters for which more collusive outcomes can be supported by non-Markovian strategies, using reversion to an MPE play as a threat to enforce collusion.

Definition 1. A pair of strategies $\tau \equiv (\tau_1, \tau_2) \in M \triangleq M_1 \times M_2$ is an MPE if for any Player i, any state $k \in \mathbb{Z}$, and any strategy $\tilde{\tau}_i \in M_i$,

$$V_i(\tau_i, \tau_j; k) \geqq V_i(\tilde{\tau}_i, \tau_j; k).$$

Hence, an MPE is simply a pair of strategies such that each strategy is optimal given the other player's strategy, starting from any state. Notice, however, that an MPE is also a perfect

2. Indeed, adding or subtracting a constant to the revenue of a player (in every state) does not affect his incentives and simply shifts his value function. Therefore, if his revenue while ahead is R_i^A and his revenue while behind is R_i^B, then, after subtracting the average $(R_i^A + R_i^B)/2$ from both revenues, his revenue as a leader becomes $R_i = (R_i^A - R_i^B)/2$, whereas his revenue as a laggard is $(R_i^B - R_i^A)/2 = -R_i$.

equilibrium of the game where arbitrary (non-Markovian) strategies are allowed. Indeed, if a profitable non-Markovian deviation exists, then, given the Markovian nature of the game and of the opponent's strategy, there also exists a profitable Markovian deviation. If $\tau \equiv (\tau_1, \tau_2)$ is an MPE, $V_i(\tau_i, \tau_j, \cdot)$ is called the *value function* of Player i. The existence of an MPE follows from standard results (see, for instance, Federgruen, 1978). A direct proof is given in the Appendix, showing that, due to discounting, the game is effectively finite, and low effort is a dominant action for large enough states.

For given Markov strategies τ_1 and τ_2, the state $k \in \mathbb{Z}$ follows a nearest-neighbour random walk. Suppose, for example, that $k_t = k > 0$, $\tau_1(k) = E$ and $\tau_2(k) = E'$, with $E, E' \in \{H, L\}$, then $k_{t+1} = k + 1$ with probability $(1 - \beta^E)\alpha^{E'}$, $k_{t+1} = k - 1$ with probability $\beta^E(1 - \alpha^{E'})$, and $k_{t+1} = k$ with probability $\beta^E\alpha^{E'} + (1 - \beta^E)(1 - \alpha^{E'})$. The state moves up one unit whenever Player 2 is successful while Player 1 is not, moves down one unit whenever Player 1 is successful while Player 2 is not, and remains unchanged whenever both players are unsuccessful or successful. The transition probabilities for k obtain similarly for k negative or zero.

The assumption that rewards and transition probabilities depend only on the positions (as well as, of course, on the effort choices), as opposed to the absolute level of technology achieved, is very strong. Given the pay-off structure adopted in this paper, it is best to regard the demand as consisting of two distinct markets, the leader's market being more lucrative than that of the laggard; or, the leader having the lion's share of the market, as in the pharmaceutical industry (Cockburn and Henderson, 1994), or, more specifically—and fortunately—in the industry of implantable cardiac pacemakers (Banbury and Mitchell, 1995). Alternatively, one may assume that the leading product commands a significant mark-up, as is the case in the semiconductor industry (see Gruber, 1994). Another crucial assumption of the model is that leadership is conducive to success, in the sense that, for a given effort level, the firm that is ahead is at least as likely to experience a success as the firm that is behind. Of course, this includes the important case in which only effort levels, and not positions, influence the transition probabilities. However, this model extends to situations in which learning-by-doing (see Lerner, 1997), consumer feedback (as in the software industry or the pacemaker industry, see Banbury and Mitchell, 1995) or clinical experience (as in the pharmaceutical industry, see Cockburn and Henderson, 1994) exceed the spill-over effects that tend to favour the laggard. Observe also that the alternative assumption that, for a given effort level, the laggard is strictly more likely than the leader to succeed, implies that, if players do not use strictly dominated strategies, the random walk followed by the state will be recurrent. In particular, one can then use Dutta (1995) to show that his folk theorem for stochastic games holds.

The assumption that the effort's outcomes are uncorrelated across firms rules out some interesting phenomena (see Cabral, 1999 for the analysis of a race where variance and co-variance are strategic choices). Finally, while, as mentioned, the model encompasses the situation in which the laggard sells nothing, it would clearly be a desirable extension to allow the laggard to actually exit, as a third option available to firms. Exiting can be modelled as an effort choice commanding degenerate transitions. As such, the choice between exiting and competing is a special case of the current model requiring minor modifications, but the choice of effort for a firm which chooses to compete is then trivial. The importance of these assumptions is further assessed in the discussion in Section 3.

2.2. The equilibrium analysis

The structure of equilibrium is derived from three observations about optimal strategies. These observations are stated here from Player 2's point of view, but analogous statements obviously hold for Player 1.

Lemma 2 establishes that Player 2's effort level decreases with his lead whenever the leads considered exceed the largest state at which Player 1 still finds it optimal to exert effort. Lemma 3 states that, if Player 2 exerts high effort at some state within an interval (of either positive or negative integers) over which his opponent's effort level does not vary, then Player 2 must either exert high effort at all larger states within that interval, or at all smaller states. Finally, Lemma 4 establishes that the laggard's effort level increases whenever his lag decreases.

The proofs of Lemmas 3 and 4 are relegated to the Appendix. Although Lemma 2 can be derived from Lemma 3 and dominance arguments, its proof is simple and illustrates the role of the assumption that leadership is conducive to success. The arguments rely on dynamic programming and use the relationship between the variations of the value function and the incentives to exert effort. To see this connection, observe first that the value function $V_2(\cdot)$ is bounded below by $-R_2$ and bounded above by R_2. Moreover, these bounds are strict, since transition probabilities are strictly positive. Since the revenue of Player 2 is increasing in the state, it follows that $V_2(\cdot)$ is strictly increasing:

Lemma 1. *V_2 is increasing.*

Proof. Consider for instance states k and k', where $k > k' \geqq 0$. Instead of following the optimal strategy τ_2, Player 2 can, starting from state k, exert low effort until the state hits k' (if ever), at which point he reverts to τ_2. The value at k under τ_2 is at least as large as the value of following this alternative strategy, which yields at k a weighted average of R_2 and of $V_2(k')$, which is strictly larger than the latter. A similar argument applies for $k < k' \leqq 0$. ‖

It is standard to show that the value function satisfies the optimality equation. For instance, when $k > 0$ and Player 1's probability of winning at k given his effort level, is denoted as $\sigma_1(k)$, the value function V_2 satisfies:

$$V_2(k) = \max\{V_2^H(k), V_2^L(k)\},$$

where

$$\begin{aligned} V_2^H(k) = {} & (1-\delta_2)(R_2 - c_2) + \delta_2\sigma_1(k)(1-\alpha^H)V_2(k-1) + \delta_2\alpha^H(1-\sigma_1(k))V_2(k+1) \\ & + \delta_2(\alpha^H\sigma_1(k) + (1-\alpha^H)(1-\sigma_1(k)))V_2(k), \end{aligned} \tag{1}$$

and

$$\begin{aligned} V_2^L(k) = {} & (1-\delta_2)R_2 + \delta_2\sigma_1(k)(1-\alpha^L)V_2(k-1) + \delta_2\alpha^L(1-\sigma_1(k))V_2(k+1) \\ & + \delta_2(\alpha^L\sigma_1(k) + (1-\alpha^L)(1-\sigma_1(k)))V_2(k). \end{aligned} \tag{2}$$

Thus, it is evident that high effort at k is preferred by Player 2 to low effort at k if and only if:

$$\frac{1-\delta_2}{\delta_2}\frac{c_2}{\alpha^H - \alpha^L} \leqq \sigma_1(k)\triangle V_2(k-1) + (1-\sigma_1(k))\triangle V_2(k), \tag{3}$$

where $\triangle f(k) \triangleq f(k+1) - f(k)$, for any $k \in \mathbb{Z}$ and any $f \in \mathbb{R}^{\mathbb{Z}}$. That is, Player 2 exerts effort if, and only if, the combined loss in the case where he fails and the gain in the case where he succeeds, exceeds the appropriately normalized and discounted cost of effort. Of course, the expectation depends on Player 1's probability of success. If he were to exert a constant effort level on $\mathbb{Z}$, then one could show that the value function $V_2(\cdot)$ is S-shaped, with Player 2 exerting high effort exactly in an interval of integers including or immediately next to the origin. However, Player 1 will typically not choose to remain idle and identifying patterns of behaviour that are compatible with equation (3) becomes a non-trivial exercise.

Lemma 2 determines the shape of the tails of optimal strategies.

Lemma 2. (1) *Suppose* $\tau_1(k) = \tau_1(k+1) = L$ *and* $\tau_2(k) = L$*, for* $k > 0$*. Then* $\tau_2(k+1) = L$.

(2) *Suppose* $\tau_1(k) = \tau_1(k+1) = H$ *and* $\tau_2(k) = H$*, for* $k+1 < 0$*. Then* $\tau_2(k+1) = H$.

Proof. (1) Suppose that $\tau_1(k) = L$. Observe from (2) that $V_2^L(k)$ is a weighted average of R, $V_2(k+1)$ and $V_2(k-1)$. Obviously, R is an upper bound on $V_2(k+1)$. Also, since $\alpha^L \geqq \beta^L$, the weight on $V_2(k+1)$ is larger than the weight on $V_2(k-1)$. Together with monotonicity, this ensures that $V_2^L(k)$ is strictly larger than the average $(V_2(k-1)+V_2(k+1))/2$. However, $V_2(k) \geqq V_2^L(k)$. This guarantees that V_2 is strictly concave at k whenever $\tau_1(k) = L$. Under the assumptions of the lemma, Player 1 exerts low effort at k and $k+1$, and thus V_2 is strictly concave at k and $k+1$. Since low effort is optimal at k, it must be that $\frac{1-\delta_2}{\delta_2}\frac{c_2}{\alpha^H-\alpha^L} \geqq \beta^L \triangle V_2(k-1) + (1-\beta^L)\triangle V_2(k) > \beta^L \triangle V_2(k) + (1-\beta^L)\triangle V_2(k+1)$, where the second inequality follows from the strict concavity of V_2. This implies that low effort is optimal for Player 2 at $k+1$. The second part of the lemma is proved similarly. ‖

Lemma 2 gives a primary perspective on the structure of optimal policies. If Player 1 exerts high effort on some interval of negative integers, then whenever Player 2 exerts high effort at a state in this interval, he also does at all larger states in this interval. If Player 1 exerts low effort on some interval of positive integers, then whenever Player 2 exerts low effort at a state in this interval, he does so as well at any larger state in this interval.

Lemma 2 is an instance of a more general phenomenon. Consider an interval of states such that Player 2's revenue is constant over this interval, as is his rival's effort level. Then if Player 2 exerts high effort at some state within this interval, he must be motivated to do so either by the fear of getting the low continuation value associated with the lower extremity of the interval, or by the hope of getting the high continuation value associated with the upper extremity of the interval. It turns out that, in the first case, Player 2 also exerts high effort at lower states within this interval, while in the second case, he keeps on exerting high effort at larger states within the interval. This is precisely what the proof of Lemma 3 shows. However, it is sufficient for the present purposes to restrict attention to positive intervals.

Lemma 3. *Suppose that Player* 1 *exerts a constant effort level on some positive interval* $I = \{m, m+1, \ldots, n-1, n\}$*. If there exists* $k \in I$ *such that* $\tau_2(k) = L$ *and* $\tau_2(k+1) = H$*, then* $\tau_2(k') = H$ *for any* $k' > k$*,* $k' \in I$.

Proof. See Appendix. ‖

Thus, either high effort is employed to try to prevent the state from reaching the lower extremity of the interval, and incentives to do so decrease with the state, or high effort is employed to try to push the state towards the upper extremity of the interval, and incentives to do so increase with the state. Observe in particular that this implies (the first part of) Lemma 2, given that high effort is a dominated action for sufficiently large states.

It follows from Lemma 3 that an equilibrium involves strategies that specify constant actions over overlapping intervals on each half-line. However, it does not tell us how many such intervals may occur. Lemma 4 shows that a laggard switches action at one state at most.

Lemma 4. *For* $k < 0$*,* $\sigma_2(k) \geqq \sigma_2(k-1)$.

Proof. See Appendix. ‖

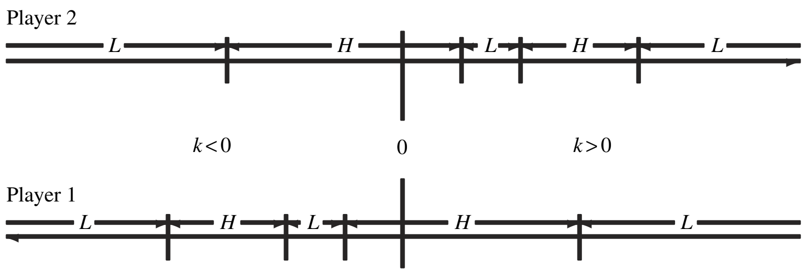

FIGURE 1
Structure of equilibria

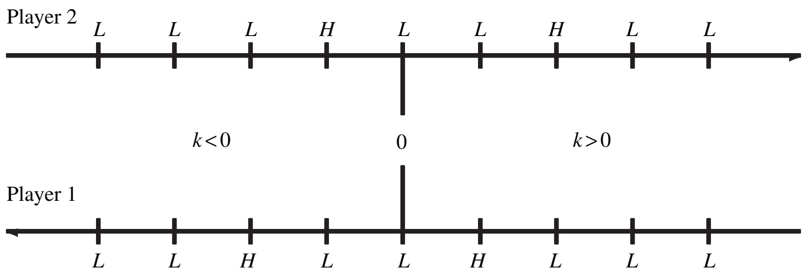

FIGURE 2
Equilibrium of Example 1

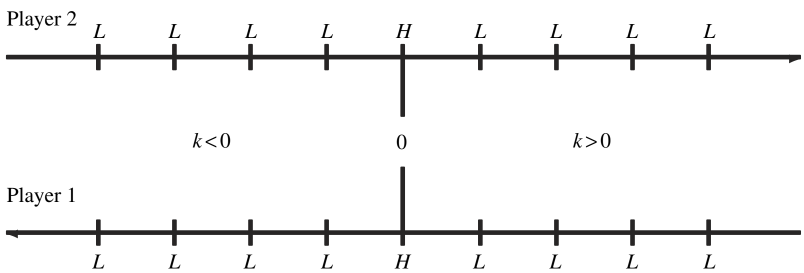

FIGURE 3
Absorbing equilibrium of Example 2

Lemmas 2, 3 and 4 completely identify the structure of the equilibria, as displayed in Figure 1. The optimal strategy of a player is not necessarily unimodal in the state variable: if Player i exerts high effort at two different states, he might exert low effort at some intermediate state, but if so, that intermediate state belongs to a unique interval of integers, contained in the half-line where Player i is ahead, and such that Player j exerts high effort at any state of this interval. Notice that strict randomization may only occur at the "switching" points, which are the extremities of the intervals on which monotonicity has been shown. Indeed, suppose that a player is indifferent between effort levels at a node contained in one of the intervals where his optimal strategy is monotonic by virtue of Lemmas 2, 3 or 4, and suppose that this node is not one of the extremities of the interval. Then his alternative strategy, obtained by modifying the effort level at this node in such a way that monotonicity is violated, achieves the same overall pay-off (at any node), and is thus also optimal, contradicting at least one of the monotonicity results.

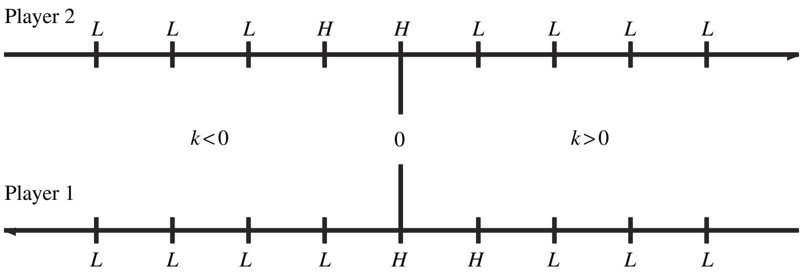

FIGURE 4
Reflecting equilibrium of Example 3

To understand why it is sometimes optimal to exert low effort in an intermediate interval while leading, it is helpful, though by no means necessary, to consider the case in which δ_1, R_1 and R_2 and β^H are large, while β^L, c_1, δ_2 are not. In this case, Player 1 has strong incentives to exert high effort while behind, since leadership is valuable and high effort is both cheap and effective, and Player 1 is sufficiently patient. As a result, Player 1 gives up only for large lags. Since R_2 is large, Player 2 finds it optimal to exert high effort when his lead is too small. When his lead approaches the level at which Player 1 gives up, he may also exert high effort since a short streak of luck would get Player 1 to loosen his grip, thereby leaving a lasting leadership to Player 2. But for moderate leads, Player 2 might prefer low effort. Indeed, since Player 1 exerts high effort and does so quite successfully, any attempt by Player 2 to secure his leadership is bound to take time and money. Since there is no immediate danger of losing leadership either, it may simply be best to make the most of the high rewards associated with leadership. Such subtleties do not arise for a player who trails since in his case the rewards he gets from exerting low effort are hardly a temptation, unless his situation becomes hopeless, for these rewards are a lower bound on his value function.

While the optimal strategies need not be unimodal in the state, this is the case under stronger conditions. A strategy τ_i is a (m, M)-*strategy* (Rath, 1977) if it specifies effort levels that are unimodal in the state, with either -1, 0 or 1 belonging to the mode. Loosely speaking, such a strategy specifies effort levels that are increasing on $\mathbb{Z}^-$ and decreasing on $\mathbb{Z}^+$ (but the threshold might be -1 or 1 rather than 0 in specific instances because of the tie-breaking rule at 0). It is called (m, M) because, whenever such a strategy is pure, it can be characterized by two integers, $m \leqq 1$ and $M > -1$, with $m \leqq M$, where Player i switches effort levels: for any integer strictly smaller than m and for any integer weakly larger than M, he exerts low effort, and exerts high effort otherwise. In particular, never exerting effort is a (m, M)-strategy with $m = M$. When a (m, M)-strategy specifies mixing, it is equivalent from the point of view of Player i to a pure strategy having the features just described. Moreover, it will be seen that mixing may only occur at those switching points. The concept of a (m, M)-strategy is intuitive, since it specifies that, whenever a player exerts high effort, he keeps doing so as long as his lead or his lag does not become too large. This captures the idea that high effort is exerted whenever the "stakes" are high enough. Lemma 4 provides sufficient conditions to guarantee that any MPE is in (m, M)-strategies.

Corollary 1. *If $\delta_i \leqq \alpha^L/\beta^H$, then every equilibrium is in (m, M)-strategies.*

Proof. It is a matter of simple algebra to verify that the L.H.S. of the inequality $(*)$ in the Appendix is negative if $\delta_2 \leqq \alpha^L/\beta^H$. Hence, from (strengthening of) Lemma 4 (Lemma 4′, in the Appendix), for $k > 0$, $\sigma_2(k+1) \leqq \sigma_2(k)$, and for $k < 0$, $\sigma_2(k) \geqq \sigma_2(k-1)$. The analogous results hold for Player 1. ‖

Hence, any equilibrium is in (m, M)-strategies whenever players are sufficiently impatient, or whenever being ahead provides better chances of Success no matter the effort levels. If $\delta_i > \alpha^L/\beta^H$ however, equilibria which are not in (m, M)-strategies exist. An example of such an equilibrium is given in the next section for particularly simple and extreme parameters, but perfectness and pay-off continuity in the parameters ensures that the features of this equilibrium hold for the corresponding equilibria with nearby parameters.

There are two extreme cases for which more specific results are available, when only one action generates noise. While the previous analysis has been derived under the assumption that both actions were noisy, it is straightforward to adapt the treatment to these extreme cases. When neither action is noisy, one is back to the model of Aoki (1991). In the first extreme case, high effort leads to success with probability one (for any state). In the second case, low effort leads to failure with probability one (for any state). The former may be relevant for applications in which the technical uncertainty can be insured against (see Freeman and Soete, 1997, pp. 242–245). The latter corresponds to situations where it is not possible to experience a success without a minimal investment (see Kamien and Schwartz, 1982, pp. 54–58).

2.3. *It is now or never: high effort leads to success*

Assume that high effort ensures success. That is, let $\alpha^H = \beta^H = 1$ and write $\alpha^L = \alpha \in (0, 1)$, $\beta^L = \beta \in (0, 1)$. Hence Player 2 can prevent the state from dropping below any level he wishes (provided the state is above that level), while Player 1 can prevent it from rising above any level he wishes. Such a threshold is called a *reflecting* threshold and Player i is said to *reflect* at k if his Markovian strategy specifies a reflecting threshold at state k, that is, if his strategy specifies high effort at k.

It is clear that whenever Player 2 chooses to reflect at $k > 0$, it cannot be optimal for Player 1 to reflect at $k' > k$, since there is no hope for him to ever be ahead, and hence no incentive for him to exert costly effort. Hence also Player 2 chooses low effort for any $k' > k$, since given his own reflection at k, he is guaranteed to be ahead forever (it is important here that Player 1 does not reflect for $k' > k$, for otherwise a higher reflecting threshold for Player 2 might be less costly in the long run given the frequency of visits at that threshold).

Further, suppose again that Player 2 chooses to reflect at $k > 1$. Then it cannot be that Player 1 reflects at k, for his overall pay-off evaluated at k would then be $-R_1 - c_1 < -R_1$ since the state would be trapped at k forever. Given this and given that $k > 1$, it must be that Player 1 reflects at $k - 1$, since otherwise Player 2 could do better by reflecting at $k - 1$ (exerting low effort at higher states) rather than by reflecting at k provided, of course, that reflection at positive states is worthwhile to begin with. Suppose in addition that $-R_2 > R_2 - c_2$. Then, since Player 1 reflects at $k - 1$, it cannot be optimal for Player 2 to reflect at $k - 1$ also. Hence, for $-R_i > R_i - c_i$, if a symmetric equilibrium exists where high effort is exerted at a state $k \notin \{-1, 0, 1\}$, both players must use strategies that are not unimodal in the state since, letting $k > 1$ be a state where Player 2 reflects, it must be that $\tau_2(k - 1) = L$, $\tau_2(k) = H$ and since the equilibrium is symmetric, $\tau_2(1 - k) = H$. The next example illustrates that such equilibria do occur and are not pathological, although equilibria with simpler structures may occur.

Example 1. $\alpha = \beta = 1/4$, $\delta_i \simeq 1$, $R_i/c_i = R/c \in [1/6, 2/3]$.

Consider the following profile of strategies $\tau = (\tau_1, \tau_2)$. $\tau_2(-1) = \tau_2(2) = H$, $\tau_2(k) = L$ for $k \notin \{-1, 2\}$, $\tau_1(-2) = \tau_1(1) = H$, $\tau_1(k) = L$ for $k \notin \{-2, 1\}$.

To verify that this profile forms an MPE, let us verify that Player 2 has no profitable deviations. If Player 2 exerts high effort at 1, the random walk corresponding to the evolution of the state is eventually absorbed at 1. Since Player 2 is patient, the value (at, say, the origin) of such a strategy equals $R - c$. If Player 2 exerts no effort at -1, 0 or 1, the state is eventually trapped below -2, and the value at zero to Player 2 of such an outcome is $-R$. If Player 2 exerts high effort at 0 but not at 1, the random walk spends roughly half of its time at 0, and half of its time at 1, which is worth $(R - c)/2$. If τ_2 is followed instead, the random walk spends 2/3 of its time at 0, $1/6$ at -1 and $1/6$ at 1, which yields a pay-off of $-c/6$. It is easy to check that $-c/6$ is larger than $R - c$, $(R - c)/2$ and $-R$ since $R/c \in [1/6, 2/3]$. Finally, given that whenever the state is equal to or below -2, the revenue to Player 2 will always be $-R$ whatever his effort level, it is optimal not to exert high effort at such states. Similarly, whenever the state is strictly above 2, Player 2 should not exert high effort. Finally, if Player 2 does not exert high effort at 2, given that $\alpha = \beta$, if the state is initially at 2, it will eventually be trapped within $\{-1, 0, 1\}$ with probability one, yielding a pay-off of $-c/6$. If Player 2 exerts high effort at 2 and the state is initially at 2, the probability that at any future date t the state is at 2 vanishes, and thus high effort at 2 yields a pay-off of R, which is larger than $-c/6$. See Figure 2.

Suppose now that Player 1 exerts low effort at all states except possibly state -1 and 0. Clearly then, Player 2 cannot exert high effort at states larger than 2, for if Player 2 wants to exert high effort at strictly positive states, it is best to exert it only at 1. Further, it can be shown that the value, evaluated at the origin for instance, of reflection at a state $k < -1$ is strictly convex in k (provided of course that Player 2 does not exert high effort for $n' < k$). This means that either reflection is worthwhile and then it should occur "as soon as possible", or it is not, and Player 2 should not exert high effort at all at negative states. From such considerations follows the next proposition.

Proposition 1. *Suppose that $\tau_1(k) = L$ for any $k \notin \{-1, 0\}$. Then $I = \{k \in \mathbb{Z} : \tau_2(k) = H\}$ is either empty or an interval containing 0 whose upper extremity is either 0 or 1.*

Proof. See Appendix. ‖

It follows by symmetry that $\tau_2(k) = L$ for any $k \notin \{0, 1\}$ implies that $\tau_1(k) = H$ if and only if $k \in I$, where I is either empty or an interval containing 0 whose lower extremity is either 0 or -1. Proposition 1 helps us understand when equilibria are in (m, M)-strategies. Suppose that Player 1 only exerts high effort, if anywhere, at -1 and 0. Then either Player 2 exerts high effort, if anywhere, in an interval with upper extremity 0 or 1, or he never exerts high effort. Suppose that this interval is contained in $\{0, 1\}$ (which includes the case where high effort is never exerted). Then Proposition 1 can be applied in turn to Player 1, who accordingly only exerts high effort, if at all, in an interval containing the origin and whose lower extremity is either 0 or -1. If this interval is similarly contained in $\{-1, 0\}$, then there exists an equilibrium in (m, M)-strategies, possibly involving mixing, where high effort is exerted at most at -1, 0 or 1. As has just been seen, this requires however that the interval I_i where Player i exerts high effort be of small length, as is the case, for instance, when Player i is sufficiently impatient. But the interval I_2 is of greater length, for instance, whenever Player 2 finds it worthwhile, at moderate negative states, to exert high effort in order to "speed things up". If indeed Player 2 exerts high effort in a larger interval I_2, Proposition 1 cannot be applied to Player 1 and there is accordingly no reason for Player 1 to restrict high effort to $\{-1, 0\}$. On the contrary, he might exert high effort both at states around 0 to prevent his position from deteriorating, and around the lower extremity of I_2, to preserve his position at such a state. In such a case, the only MPE might be of the type described in Example 1, where high effort is exerted by each player in two different intervals: when a player is ahead, to preserve his position, and when he is behind, to prevent his position from worsening.

It appears more important to understand in which cases Player 2 exerts high effort "around" 0 than to identify under which conditions high effort is exerted exactly up to 0 or exactly up to 1, a distinction which essentially depends on the tie-breaking rule at 0. The next lemma determines under which conditions Player 2 chooses to reflect near the origin.

Lemma 5. *For a given strategy τ_1 satisfying the assumptions of Proposition* 1*, there exists a threshold T_2 such that $I \neq \emptyset$ whenever $R_2/c_2 > T_2$, and $I = \emptyset$ whenever $R_2/c_2 < T_2$. This threshold increases with β and decreases with δ_2.*

Proof. See Appendix. ||

The decision depends on the ratio R_2/c_2, that is, on the stakes of being ahead normalized by the cost of exerting high effort. This ratio is low when effort is costly, or when the premium of being ahead is small. In this case, reflection is not an attractive policy. The threshold T_2 increases with the discount factor, since the immediate cost of high effort matters less for patient players. Finally, this threshold decreases with β. If β is low, the expected time before the state leaves the half-line where it sojourns is large, and failing to exert high effort around 0 has long-lasting consequences in this case.

To summarize, there are two robust features of Player 2's equilibrium strategy in this extreme case. As in the general model, if Player 2 exerts high effort at a negative state, he does so too at all larger negative states. Furthermore, there is at most one strictly positive state at which he exerts high effort, and Player 1 exerts high effort at all (strictly) smaller strictly positive states, and low effort at all (weakly) larger states. Hence, the equilibrium need not be in (m, M)-strategies. When it is, the effort level increases with the discount factor and decreases with the probability of Success of the laggard in the sense of Lemma 4.

2.4. *Low effort leads to failure*

The other extreme case occurs when $\alpha^L = \beta^L = 0$, while $\alpha^H = \alpha \in (0, 1)$, $\beta^H = \beta \in (0, 1)$. In this case, a player who exerts low effort experiences a failure. Hence, for Player 2, low effort induces an upper reflecting barrier, while it induces a lower reflecting barrier for Player 1. The interpretation of low effort depends on the situation. If a player exerts low effort while being behind, he thereby condemns himself to be behind forever. Accordingly, a player who exerts low effort while being behind is said to *give up* at that state. A player who exerts low effort while being ahead simply considers that his lead is so large that it is not worthwhile to keep on building it up. Hence, a player who exerts low effort while being ahead *relaxes* at that state.

What determines the main features of the equilibrium then is what happens "first" as the lead of one player increases: does the player who is ahead relax before his opponent gives up, or does the opposite happen instead? There are correspondingly two classes of "symmetric" equilibria: *absorbing equilibria*, in which giving up occurs first, and *reflecting equilibria*, in which relaxation occurs first. The terminology is motivated as follows. When players give up on the equilibrium path, the state is eventually absorbed at such a state. When players relax, the state bounces back and forth between two endpoints, both players switching positions infinitely often. Of course, "asymmetric" equilibria, where at one end a player relaxes while at the other end he gives up may also occur.

Notice that whenever a player gives up, his opponent has no further incentives to exert high effort at that point. If he exerts low effort, the state is bound to remain at that point. But this means that the player who is ahead remains ahead forever while exerting low effort, an enviable situation indeed. Thus, in an absorbing equilibrium, the player who is ahead relaxes "as soon as" his opponent gives up.

The following two examples illustrate this discussion.

Example 2 (Absorbing Equilibrium). ($\delta_i = 1/2, \alpha = \beta = 1/3, R_i/c_i = 3$).

The following strategies constitute an equilibrium: $\tau_i(0) = H$, $\tau_i(k) = L$ for $k \neq 0$, for $i = 1, 2$. Player 2 gives up at -1, while Player 1 gives up at 1. Accordingly, Player 2 has no incentives to exert high effort at states $k \geqq 1$, while Player 1 has no incentives to exert high effort at states $k \leqq -1$. See Figure 3.

Example 3 (Reflecting Equilibrium). ($\delta_i = 1/2, \alpha = \beta = 1/3, R_i/c_i = 4$).

The following strategies constitute an equilibrium: $\tau_i(0) = H$, $\tau_1(1) = \tau_2(-1) = H$, $\tau_1(-1) = \tau_2(1) = L$, $\tau_i(k) = L$ for $k \neq \{-1, 0, 1\}$, for $i = 1, 2$. Player 1 relaxes at -1, at which point Player 2 still exerts high effort, ensuring thereby that the state bounces back to 0. Similarly, Player 2 relaxes at 1. See Figure 4.

Notice that the parameters of both examples are almost identical. An absorbing equilibrium does obtain for low R_i/c_i, whereas a reflecting equilibrium does obtain for a larger value of this ratio.

Let us call an absorbing equilibrium in which Player 2 gives up at state $-m_2$, while Player 1 gives up at state m_1, a (m_2, m_1)*-equilibrium*. Similarly, a (M_1, M_2)*-equilibrium* is a reflecting equilibrium in which Player 2 relaxes at $-M_1$ while Player 1 relaxes at M_2. Who stops exerting high effort first in equilibrium? The next lemma provides a simple answer to that question.

Lemma 6. *Let $\delta_1 = \delta_2$. In a (m_2, m_1)-equilibrium, $m_2 \geqq m_1$ if and only if $R_2/c_2 \geqq R_1/c_1$. In a (M_1, M_2)-equilibrium, $M_2 \geqq M_1$ if and only if $R_2/c_2 \geqq R_1/c_1$.*

Proof. See Appendix. ||

Hence, in an absorbing equilibrium, a player with a high R/c ratio does not give up quickly. This is intuitive since a high R/c means that the stakes (of being ahead rather than behind) are high relative to the cost of staying in the race. Similarly, in a reflecting equilibrium, a player with a high R/c ratio relaxes late, since high effort is not very costly (as compared with the stakes R), and thus, relaxing is not very profitable, compared with the perspective of securing position.

Of course, equilibria might be neither reflecting nor absorbing. As an extreme example, suppose that c_1 is negligible, R_1 is not, and Player 2 has a much smaller ratio R_2/c_2. Player 1 then exerts high effort within a large interval around the origin: relaxing is not profitable for him, nor is giving up. All Player 1 then cares about is being ahead. In that case, the barriers determining the equilibrium's outcome are determined by R_2/c_2: the state evolves between an upper barrier where Player 2 relaxes, and a lower, absorbing, barrier, where Player 2 gives up.

To complete the description of equilibrium, it is necessary to understand the structure of optimal strategies outside of the interval between the largest negative barrier and the lowest positive one. In the case of absorbing equilibrium, this is fairly easy. Consider an absorbing barrier $m_1 > 0$, where Player 1 gives up. As a consequence, Player 2 also exerts low effort at that state. The value function of Player 1 is thus worth $-R_1$ at that state. At larger states, there are no incentives for Player 1 to exert high effort, for given the optimal actions specified at m_1, he is bound to be behind forever after. Hence also, Player 2 exerts low effort at larger states. Thus, in the case of an absorbing (m_2, m_1)-equilibrium, optimal strategies are (m, M), with switching points given for both players by m_2 and m_1.

The situation is more complicated in the case of a reflecting (M_1, M_2)-equilibrium. Consider positive states. For states large enough, Player 1 exerts low effort. Let the smallest of these states be m_1. Of course, low effort by Player 1 at m_1 triggers low effort by Player 2 at that state, and, by the previous argument, low effort is then exerted by both players at all states larger than m_1. Since the equilibrium is reflecting, $m_1 > M_2$. In the interval $\{M_2, \ldots, m_1\}$, Player 1 exerts high effort, by definition of m_1. Player 2 exerts low effort at M_2 (since this is the reflecting barrier), and low effort at m_1. However, Player 1 need not exert low effort at all states in the interval. Rather, depending on the parameters, he might prefer to exert low effort in an interval $\{M_2, \ldots, T\}$ and high effort in the interval $\{T + 1, m_1 - 1\}$, whenever these intervals are well defined, for $T \in \{M_2 + 1, \ldots, m_1 - 1\}$. The point is that m_1 might be much larger than M_2, so that for states close to M_2, reflection is preferred. For states close to m_1 however, the chances of reaching state m_1 are high enough so that Player 2 finds it profitable to exert high effort there. In this case, once more, optimal strategies are not (m, M). This is likely to happen if the normalized cost $c/(\alpha^H - \alpha^L)$ is low, so that it is desirable to exert high effort when one's rival is on the brink of giving up. Hence, while absorbing equilibria necessarily specify (m, M)-strategies, there is a natural sense in which reflecting equilibria correspond to the other kinds of equilibria which have emerged so far in all the models investigated.

It is desirable to understand what determines whether equilibria are absorbing or reflecting. The role of heterogeneity in pay-offs having already been addressed, suppose for simplicity that $R_1/c_1 = R_2/c_2$, $\delta_1 = \delta_2 = \delta$, and focus on symmetric equilibria. It can be shown that absorbing equilibria exist while reflecting equilibria do not when the discount factor is sufficiently low and I also suspect that they do not exist when the discount factor is sufficiently high. When players are very impatient, it is trivial that players always exert low effort and thus, that the equilibrium is absorbing. On the other hand, when players are very patient, they care mainly about the long-run fraction of time spent on each half-line. Not surprisingly, the band between the lower and the upper barrier is large when δ is close to one. Reflection is only profitable at one state, the barrier, which is obviously visited much fewer times than each half-line. (Of course, a larger value of $\alpha - \beta$, that is, a stronger drift away from the origin makes such visits more frequent. But a larger drift also makes abandon more attractive, since it makes it harder to come back once behind.) Finally, there is the issue of multiplicity of equilibria. The proof of the previous lemma shows that absorbing equilibria are "essentially" unique: uniqueness would result if the barriers were real variables, so that multiplicity only occurs because of the discrete features of the game. In fact, it is shown in the Appendix that if Player 2 gives up at $m_2 < 0$ in an absorbing equilibrium, which implies that Player 1 exerts low effort at m_2, it must be that Player 2 would have given up at m_2 even if Player 1 had exerted high effort at m_2 and lower states, implying that m_2 is the maximizer (or the largest of two adjacent maximizers, in rare circumstances) of a simple quasi-concave programme. To understand why he would have given up at m_2 anyway, notice that, if m_2 is the lower barrier of an absorbing equilibrium, it must be that giving up earlier is not worthwhile, while if giving up later was optimal for Player 2 when Player 1 exerts high effort at lower states, he would prefer high effort at m_2 to his equilibrium choice, since under high effort, the state would be reflected at m_2.

3. RELATED LITERATURE AND DISCUSSION

The early literature on innovative competition either assumed away uncertainty or borrowed the framework of a formal race. The most notable exceptions are Harris (1991), Budd *et al.* (1993) and Dutta and Rustichini (1995).

Dutta and Rustichini study a continuous time stochastic game in which agents can make costly discrete or discontinuous changes in the pay-off-relevant state, whose space is identified with the real line. If unhindered, the state evolves according to a Brownian motion. Player 1 prefers higher states to lower ones, while Player 2 has the opposite preference. They prove the existence of MEP which are characterized by two-sided s-S rules. Strategies are completely determined by four parameters, L, $L + \theta$ (for Player 1) and U, $U - \mu$ (for Player 2), where

$L \leqq L + \theta < U$ and $L < U - \mu \leqq U$. If the state is ever at or below L, Player 1 jumps up to $L + \theta$, whereas if it ever gets up to U or above, Player 2 jumps down to $U - \mu$. In some of these equilibria either or both players might be passive. The cartel (cooperative) solution is also s-S, but the symmetric cartel solution has a wider band than the symmetric Markov equilibrium. The crucial assumption here is that the cost of these changes does not depend on their size. Players can change the state by any amount at a fixed cost. Of course, the fear of immediate retaliation forces them into moderation. The main conceptual difference between the present paper and Dutta and Rustichini's lies in the nature of the control exerted by the players. In the former, players can *influence* the evolution of the state, *step by step*. In the latter, players can *dictate* this evolution, imposing arbitrary *discrete* changes upon the state. Which setting is relevant depends, of course, on the particular application, but the analysis shows that the predictions differ. While simple s-S strategies are optimal, simple (m, M)-strategies need not be. A continuous version of the model developed in this paper is available from the author.

An important branch of the literature followed Futia (1980) and Reinganum (1985). Futia's framework has some similarities with the discrete-time model, but its focus is on explaining some stylized facts and a comparison is therefore difficult to make. Reinganum's model shares with the present one the distinction between leader and follower. There is, however, no measure of the gap between firms, and the leader cannot build his lead up, and therefore has fewer incentives than his rival, since protecting his leadership is his sole motivation, while his opponent hopes to gain the lead. As a consequence, action–reaction equilibrium appears, in which leadership frequently changes hands.

Much closer are the papers by Harris (1991) and Budd *et al*. (1993). The first is roughly an undiscounted version of the second. Competition is modelled as a continuous process, so that innovative success by the laggard does not catapult him into the technological lead, leapfrogging the leader, but simply narrows the gap between the two firms before overtaking possibly occurs. The state is modelled as market share which evolves according to a Brownian motion whose drift depends on the firms' effort choices. The state space is identified with the unit interval, and different boundary conditions are investigated. Given the generality of their model, the authors resort to asymptotic expansions in the interest rate and in the level of uncertainty as measured by the diffusion to study how effort varies with the state. The effects underscored by both expansions are fortunately comparable. Most important is a *joint profit effect*. If joint profits from the product market are higher on average if the gap between firms grows rather than shrinks, then the leader tends to make greater effort than the laggard. *Joint cost effects* are of two kinds. First, there are profit-incentive cost effects, which occur if, for instance, each firm's profit function is steeply sloped at state s. Then each firm has a strong incentive to advance at s and exerts high effort at such a state. The state of competition thus tends away from points where effort levels are high. Second, there are endpoint effects. Firms in some cases obtain relief from points at or near the endpoints. Finally, there is an effect only observed in simulations, a *self-reinforcing cost effect*, illustrating that the pattern of joint effort costs and the pattern of evolution of industry structure may interact in a mutually reinforcing manner.

The joint profit effect is by now relatively well understood. It is the driving force of the traditional patent race analysis, as in Fudenberg, Gilbert, Stiglitz and Tirole (1983), Grossman and Shapiro (1987), Harris and Vickers (1987) or Lippman and McCardle (1987). In these papers, this effect tends to favour increasing dominance, the leader tending to get further and further ahead. In the present paper, it is difficult to interpret the results as being driven by such an effect. Indeed, revenues are constant over each half-line, and so are their sums. However, the sum is larger on the half-line where the player with the larger R_i is ahead. When the intensity of effort can be measured by simple indices like thresholds, as in the extreme cases studied, there is a natural sense in which the player with the larger R_i/c_i exerts more effort. This can indeed be interpreted as a joint profit effect, where profits need to be normalized by the cost of effort. The structure of the equilibria can also be interpreted in terms of the joint cost effect. One way to understand why (m, M)-equilibria need not be optimal is precisely this one. Sometimes, it is simply too costly to exert high effort when the opponent does, for it may be better to lose some ground than to remain stuck at a state where the cost of effort is incurred. Rather, players might prefer to exert high effort just "before" or just "after" their opponent. Efforts are strategic substitutes in this model: an increase in the rival's effort level tends to decrease a firm's own effort level. Accordingly, strategies are typically not (m, M). Hence, unless the state is expected to spend only a little time at points where effort levels are high, both players tend not to exert high effort simultaneously in equilibrium.

The assumptions of the model are rather strong. Instantaneous revenue is constant on each half-line, probabilities of success given the effort choice are also constant, while the cost of effort is assumed constant on $\mathbb{Z}$. Some of the assumptions can be relaxed while preserving some features of the equilibrium as described in this paper. To ensure monotonicity of the value function, for instance, it is sufficient that instantaneous revenue be increasing on $\mathbb{Z}$, strictly increasing at some state $k \in \mathbb{Z}$, and bounded. Also, the assumption that the cost of effort be constant on all the integers can be relaxed. If one assumes that the cost is constant on each half-line, then all the structural results of the paper still hold. The second part of Proposition 2, namely the monotonicity of Player 2's optimal effort level in the state remains valid as long as the cost of effort does not grow too fast (*i.e.* letting c_2^k denote the cost of effort of Player 2 at state k, monotonicity obtains on $\mathbb{Z}_-$ as long as $c_{k+1}/c_k \leqq \frac{\alpha^L(1-\beta^L)}{\beta^L(1-\alpha^L)}$ for all $k \leqq 0$). Similarly, Lemma 2 remains valid as long as the cost does not decrease too fast (*i.e.* if $c_{k+1}/c_k \geqq \frac{\beta^L(1-\alpha^L)}{\alpha^L(1-\beta^L)}$ for all $k \leqq 0$).

Other assumptions are harder to relax. However, weaker assumptions are unlikely to yield a simpler structure of equilibrium. For instance, it would be interesting to understand what happens when the laggard is more likely to succeed than the leader, for a given effort level. In this case, besides monotonicity (and the dominance of low effort for states that are either very large or very small), most of the structural features of equilibrium depend on the parameters. Nevertheless, an extension of Lemma 3 can be used to show that, as long as Player 1's optimal effort level is increasing on the negative half-line, Player 2's optimal effort level is also increasing on this half-line. This suggests that the optimal policy of the laggard typically exhibits a "nicer" structure than the leader's policy.

Another strong assumption of the model is the "bang-bang" structure of the revenue, which jumps from one extreme to the other as soon as leadership changes hands. How much can this assumption be relaxed? As mentioned, monotonicity of the rewards is sufficient for monotonicity of the value function. As can be seen from the proofs of Lemmas 2 and 4, as proved in the Appendix, the difference $R_2^k - V_2^k$ (where k refers to the state) plays an important role. Specifically, it is important for the results of this paper to be generalized that this difference be negative for k negative, and positive for k positive. It is easy to provide specific, sufficient restrictions on the collection $\{R_2^k\}_{k\in\mathbb{Z}}$ for given parameters δ, α^H, α^L, β^H, β^L and c. Finally, the generalization of Lemma 3 requires additionally that $\{R_2^k\}$ be convex on $\mathbb{Z}_-$ and concave on $\mathbb{Z}_+$. The results extend thus to rewards which are "sufficiently" increasing and "sufficiently" S-shaped, where "sufficiently" depends on the other parameters of the model.

4. CONCLUSION

This paper is an attempt to unravel some of the features that characterize equilibria in a model of dynamic competition between two firms. The stochastic evolution of the state of competition depends on the respective effort levels of the firms, which try to take the lead in order to enjoy higher flow pay-offs. Under some assumptions, equilibria involve only simple, intuitive strategies, where firms only exert high effort when their lead or their lag is small enough, and choose to exert low effort otherwise. In general however, such a simple pattern does not arise. Rather, a firm exerts high effort in two distinct situations. When a firm's lead is sufficiently large, such an effort level is motivated by the tangible prospects of securing the position, thereby leaving its rival durably behind, which results in it loosening its grip. When a firm's lead is very small, or its lag not too large, high effort helps to defend or regain the higher revenues associated with leadership and to avoid falling by the wayside. However, in intermediate situations, as when the lead is moderate, low effort may be the best choice since there is neither an urgent need for an active defence of the leadership, nor hope to see the laggard quickly let go. Hence, it is generally not true that the struggle is fiercest when firms are shoulder to shoulder. These effects are even more pronounced in the case in which high effort guarantees success.

When low effort entails failure, a closely related distinction arises. When a firm's lead increases, does this firm stop exerting high effort before or after its rival? If the firm which is ahead stops first, then leadership bounces back and forth between the firms. If the firm which is behind stops first, then chance determines which firm eventually becomes the leader forever. When firms are very patient or very impatient, the latter absorbing equilibrium occurs. In any case, a firm with a high revenue over cost of effort ratio keeps on exerting high effort longer than does its rival.

There are many possible extensions to this work. First, this paper assumes that the heterogeneity among firms is given and common knowledge. It seems interesting to study the case in which firms' characteristics are private information. Obviously, issues of reputation emerge. A firm can try, through its behaviour, to convince its opponent that it is of a type that it really is not, but that it prefers to be believed to be. Although its opponent is not naive, such considerations ought to change the conclusions of the model considerably.

It seems also worthwhile to investigate what happens when the state is a measure of a reputation. Firms which know their type and are matched every period with clients who do not know their types but prefer to experience a success, and therefore care about the state as it allows them to make inferences about the firms. This imposes further constraints on equilibrium, for reputation needs to be valuable. That is, it must be that clients indeed expect the firm which is ahead to be more likely to be successful than its rival, given both the inferences about types and about the effort levels exerted under the equilibrium strategies. The model developed in this paper helps one understand how an asset such as a technological position should be managed. It remains to be understood how such an asset is valued.

APPENDIX

A.1. Proof of existence of MPE

For $N \in \mathbb{N}$, define the truncated (finite state) stochastic game Γ_N as follows. Players are Player 1 and Player 2. The state space is $\{-N, -N+1, \ldots, N-1, N\}$, the action space for each player is [0, 1] corresponding to the probability that high effort is exerted, with low effort being exerted with complementary probability, the probability of transition and rewards at each state except $-N$ and N are as in the game defined in the paper. States $-N$ and N are absorbing. That is, once the random walk reaches one of these states, it remains in that state forever. Rewards at state $-N$ are R_1 $(-R_2)$ for Player 1 (2), while rewards at state N are $-R_1$ and R_2 respectively. Total pay-offs to Player i starting in state k, given a pair of strategies τ, are denoted $R^{\tau}_{i,N}(k)$. Existence of MPE (that is, subgame perfect equilibrium in stationary Markov strategies) in the game Γ_N follows from a standard application of Kakutani's fixed point theorem (see Theorem 4.6.4, Filar and Vrieze, 1996, p. 219). For each $N \in \mathbb{N}$, pick an MPE $\tau_N = (\tau_{1,N}, \tau_{2,N})$ of the game Γ_N. Let $N_0 \in N$ be such that $N_0 > \max\left\{\ln \frac{(1-\delta_1)c_1}{2R_1} / \ln \delta_1, \ln \frac{(1-\delta_2)c_2}{2R_2} / \ln \delta_2\right\}$. This ensures that for any $n \geqq N_0$, $(1-\delta_2)(R_2-c_2)+\delta_2 R_2 < (1-\delta_2^n)R_2+\delta_2^n(-R_2)$, and $(1-\delta_1)(-R_1-c_1)+\delta_1(1-\delta_1^{n-1})(-R_1)+\delta_1^n R_1 < R_1$, that is, that at any state k larger than N_0, it is a dominant action to have $\tau_i(k) = 0$ for $i = 1, 2$. Similarly, low effort is strictly preferred to high effort by both players at states $k \leqq -N_0$. Consider the sequence of equilibrium action vectors $\{\{\tau_N(-N_0), \tau_N(-N_0+1), \dots, \tau_N(N_0-1), \tau_N(N_0)\}, N \geqq N_0\}$. Pick a convergent subsequence (in the product topology) and define $\tau : \mathbb{Z} \to [0, 1]$, such that $\tau(k)$ equals its limit for $k \in \{-N_0, \dots, N_0\}$ and equals 0 otherwise. It remains to show that τ constitute an MPE of the game defined in the paper. Obviously, τ are Markov (stationary) strategies by construction. If τ do not constitute an equilibrium, there exists $\varepsilon > 0$, a Player i, a state $k \in Z$ and a strategy τ_i' such that $R_i^{(\tau_i',\tau_j)}(k) \geqq R_i^{\tau}(k) + \varepsilon$. Define $\tau_{1,N}'$ as the restriction of τ_1' to states $\{-N, \dots, N\}$. By continuity, there exists N large enough, $N > k$ such that $R_{i,N}^{(\tau_{i,N}',\tau_{j,N})}(k) \geqq R_{i,N}^{\tau}(k) + \varepsilon/2$, a contradiction.

A.2. *Proof of Lemma* 3

The strategy of the proof is as follows. Suppose for the sake of contradiction that there exists a positive interval $I = \{m, m+1, \dots, n-1, n\}$, in which Player 1 exerts a constant effort level, and that there exists $k \in I$, $k' \in I$, $k \leqq k'$, such that the optimal strategy of Player 2 specifies $\tau(k) = L$, $\tau(k+1) = H$, $\tau(k') = H$, $\tau(k'+1) = L$. Let β be the probability of success of Player 1 at any state of I. Let $V_I = V_2(k-1)$, $W_I = V_2(k+2)$, $V_{II} = V_2(k'-1)$, $W_{II} = V_2(k'+2)$. Since the value function of Player 2 is strictly increasing, $V_I < W_I < V_{II} < W_{II}$ (adapting the argument for the case $k+2 \geqq k'-2$ is straightforward). It is easy to determine the average of $V_2(k')$ and $V_2(k'+1)$ under τ, denoted V, as a function of R_2, c_2, V_{II}, W_{II}. Similarly, one can determine $V_{H,H}$ ($V_{L,L}$) as the average of the values at k' and at $k'+1$ obtained by exerting high (low) effort at both k' and $k'+1$, holding the values at $k'-1$ and at $k'+2$ fixed at V_{II} and at W_{II}. Since τ is optimal, it must be that $V \geqq V_{H,H}$ and that $V \geqq V_{L,L}$. These two inequalities give an upper and a lower bound of W_{II} as a function of V_{II}, R_2, c_2. Because the upper bound must be larger than the lower bound, this inequality provides in turn an upper bound on V_{II} as a function of R_2 and c_2. Proceeding similarly at k and $k+1$, one obtains a lower bound on W_I. Clearly, the upper bound on V_{II} should be larger than the lower bound on W_I, but this yields the desired contradiction. Details follow. Since the effort level is decreasing at k', holding V_{II} and W_{II}, exerting "early effort" (that is, $\tau(k') = H$, $\tau(k'+1) = L$) should yield a higher average $\frac{1}{2}(V(k') + V(k'+1))$ than exerting "always effort" (that is, $\tau(k') = H$, $\tau(k'+1) = H$). Computing the difference of the corresponding average values gives an expression which is linear in W_{II},

$$-(\alpha^H - \alpha^L)(1 - \delta_2 + 2\delta_2\alpha^H(1-\beta) + \delta_2\beta(1-\alpha^H))$$
$$\times(\delta_2^2(1-\alpha^H)(1-\beta)\beta^2 + \delta_2(1-\delta_2)(1-\beta)(\alpha^H + 2\beta(1-\alpha^H)) + (1-\delta_2)^2(1-\beta)),$$

which is negative. Accordingly, this necessary condition yields an upper bound to W_{II}. Similarly, early effort should give an average value larger than exerting "no effort" (that is, $\tau(k') = L$, $\tau(k'+1) = L$). Computing the difference of the corresponding average values gives an expression which is linear in W_{II}, with the same sign as:

$$(\alpha^H - \alpha^L)(1-\beta)^2\delta_2^2\alpha^L(1 - \delta_2(1-\beta))(1 - \delta_2 + 2\delta_2\beta(1-\alpha^L) + \delta_2\alpha^L(1-\beta)),$$

which is positive. We thus obtain a lower bound on W_{II}. The upper bound being necessarily larger than the lower bound, the corresponding difference should be positive. This difference is easily seen to be proportional to a term linear in V_{II}, with the same sign as:

$$-\delta_2^2(1-\delta_2)\beta(1-\beta)(\alpha^H - \alpha^L)^2\begin{pmatrix}(1-\delta_2)^2 + \delta_2(1-\delta_2)((1-\beta)(\alpha^L + \alpha^H) + \beta(2 - \alpha^L - \alpha^H)) + \\ \delta_2^2(\beta^2(1-\alpha^L)(1-\alpha^H) + \alpha^L\alpha^H(1-\beta)^2 + \alpha^L(1-\alpha^L)\beta(1-\beta))\end{pmatrix},$$

which is negative. We have thus an upper bound on V_{II}. In fact, we obtain that:

$$V_{II} \leqq R_2 - \frac{(1-\delta_2)^2 + \delta_2(1-\delta_2)\beta(2-\alpha^L) + \delta_2^2\beta(\beta - \alpha^L)}{\delta_2\beta(\alpha^H - \alpha^L)} \triangleq v_{II}.$$

Focusing now on states k, $k+1$, it must be the case that exerting "late effort" (that is, $\tau(k) = L$, $\tau(k+1) = H$) should yield a higher average $\frac{1}{2}(V(k) + V(k+1))$ than exerting "always effort". Computing the difference between the corresponding average values gives an expression which is linear in V_I, with the same sign as:

$$\delta_2\beta(\alpha^H - \alpha^L)(1 - \delta_2 + 2\delta_2\alpha^H(1-\beta) + \delta_2\beta(1-\alpha^H))$$
$$\times(\delta_2^2(1-\alpha^H)(1-\beta)^2 + \delta_2(1-\delta_2)(1-\beta)(1 - \alpha^H\beta + \alpha^H(1-\beta)) + (1-\delta_2)^2),$$

which is always positive, yielding thus a lower bound on V_I. Similarly, exerting "late effort" should yield an average value larger than exerting "no effort" (at k and $k+1$). The difference between the values is linear in V_I, with the same sign as:

$$-\delta_2^2\beta^2(1-\alpha^L)(1-\delta_2\beta)(\alpha^H - \alpha^L)((1-\delta_2) + 2\delta_2\alpha^L(1-\beta) + \delta_2\beta(1-\alpha^L)),$$

which is negative, yielding thus an upper bound on V_I. An upper bound being necessarily larger than a lower bound, it must be that:

$$W_I \geqq R_2 + \frac{1-\delta_2+\delta_2\alpha^L(1-\beta)-\delta_2\beta(1-\delta_2\beta)}{\delta_2(1-\beta)(\alpha^H-\alpha^L)} \triangleq w_I.$$

It is now a matter of simple computations to verify that $w_I \geqq v_{II}$, yielding the desired contradiction.

A.3. *Proof of Lemma* 4

We will show the following strengthening of Lemma 4, which is the version implying Corollary 1. (In addition to proving that an optimal strategy is non-decreasing on $\mathbb{Z}_-$, it also gives sufficient conditions for an optimal strategy to be non-increasing on $\mathbb{Z}_+$.)

Lemma 4′. *Suppose that, given σ_1, σ_2 is an optimal strategy.*

(1) *Let $k>0$. If $V_2(k-1)$ satisfies*

$$\frac{\sigma_1^{k+1}(1-\delta_2\sigma_1^{k+1})}{(1-\sigma_1^{k+1})(1-\delta_2\sigma_1^k+\delta_2\sigma_1^{k+1})} - \frac{1-\delta_2}{\delta_2(1-\sigma_1^{k+1})} - \alpha^L < \frac{R_2-V_2(k-1)}{c_2/(\alpha^H-\alpha^L)}, \tag{*}$$

then $\sigma_2(k+1) \leqq \sigma_2(k)$. If the strict inequality is reversed, then there exists $V > V_2(k-1)$, such that, provided $V_2(k+2)=V$, $\sigma_2(k+1) > \sigma_2(k)$.

(2) *For $k<0$, $\sigma_2(k) \geqq \sigma_2(k-1)$.*

Remark. Lemma 4′ shows that $(*)$ is a sufficient condition for optimal strategies σ_2 to be non-increasing on the positive integers. It is also necessary to the extent that, if this condition is violated, and if $V=V_2(k+2)$ is treated as exogenous, there exists $V > V_2(k-1)$ for which the optimal strategy is non-decreasing on states k and $k+1$. The corresponding equation for negative states is trivially satisfied, yielding the second conclusion.

Proof. Suppose that the optimal strategy of Player 2 is increasing at $k>0$, that is, $\sigma_2(k)=L$ and $\sigma_2(k+1)=H$. In what follows, subscripts for Player 2 are dropped when no confusion is possible. Let us write $V_{E,E'}$ for the average of $V(k)$ and $V(k+1)$ under the strategy consisting of effort level E at k, and E' at $k+1$ and assigning the same actions as the optimal strategy does at all other states. The proof consists of comparing the average of the values achieved at k and $k+1$ by the different strategies which assign pure actions at states k and $k+1$, and assign the same actions at all other states as the strategy assumed to be optimal. Necessary and sufficient conditions will follow (vacuously satisfied for $k<0$). Obviously, the average under the optimal (increasing) strategy should equal $V_{L,H}$. Let $\Delta_{E,E'}=V_{H,H}-V_{E,E'}$. It is tedious but easy to show that $\Delta_{H,H}$ is a linear function of c, R, V_{k-1} and V_{k+2}. Let $V=V_{k+2}$ and Π be equal to R_2-V. That is, Π measures by how much the value at $k+2$ differs from the maximal possible value. Let also D be equal to $V_{k+2}-V_{k-1}$, that is, the increase in the value from state $k-1$ to state $k+1$ under the proposed strategy. In fact, $\Delta_{E,E'}$ is a linear function in c, R and D. Since the strategy we consider is optimal, it must be that $\Delta_{E,E'} \leqq 0$ for all E, E'. As a function of c, it is obvious that $\Delta_{H,H}(0)$ is strictly positive: when effort is not costly, then the best strategy is evidently always to exert high effort. If c increases, low effort becomes more attractive and accordingly, one would expect that $\Delta_{H,H}(c)$ becomes negative. Indeed, the algebra confirms this intuition, since:

$$\begin{aligned}\Delta_{H,H}(c)-\Delta_{H,H}(0) &= -(1-\delta_2)(1-\delta_2+2\delta_2\sigma^{E'}(1-\alpha^H)+\delta_2\alpha^H(1-\sigma^{E'}))c \\ &\times \begin{pmatrix} (1-\delta_2)^2+\delta_2(1-\delta_2)(\sigma^E(1-\alpha^L)+\sigma^{E'}(1-\alpha^H)+\alpha^H(1-\sigma^{E'})+\alpha^H(1-\sigma^E)) \\ +\,\delta_2^2(\sigma^E(\alpha^H-\alpha^L)(1-\sigma^E)+\sigma^E\sigma^{E'}(1-\alpha^L)(1-\alpha^H)+\alpha^H(1-\sigma^{E'})(\sigma^E(1-\alpha^L)+\alpha^L(1-\sigma^E))) \end{pmatrix},\end{aligned}$$

where $\sigma_1(k)=\sigma^E$ and $\sigma_1(k+1)=\sigma^{E'}$. $\Delta_{H,H}$ is thus a decreasing (linear) function of c. Similarly, $\Delta_{L,L}$ is a linear form in R_2, V, D and c, and as a function of c, it is an increasing function, for:

$$\begin{aligned}\Delta_{L,L}(c)-\Delta_{L,L}(0) &= (1-\delta_2)(1-\delta_2+2\delta_2\alpha^L(1-\sigma^E)+\delta_2\sigma^E(1-\alpha^L))c \\ &\times \begin{pmatrix} (1-\delta_2)^2+\delta_2(1-\delta_2)(\sigma^E(1-\alpha^L)+\sigma^{E'}(1-\alpha^L)+\alpha^L(1-\sigma^E)+\alpha^L(1-\sigma^{E'})) \\ +\,\delta_2^2(\sigma^E\sigma^{E'}(1-\alpha^L)^2+(\alpha^L)^2(1-\sigma^E)(1-\sigma^{E'})+\sigma^E(1-\alpha^L)((\alpha^H-\alpha^L)\sigma^{E'}+\alpha^L(1-\sigma^{E'}))) \end{pmatrix}.\end{aligned}$$

Variations with respect to D are intuitively less obvious. It turns out, however, that:

$$\begin{aligned}\Delta_{H,H}(D)-\Delta_{H,H}(0) &= \delta_2\sigma^E(\alpha^H-\alpha^L)(1-\delta_2+2\delta_2\sigma^{E'}(1-\alpha^H)+\delta_2\alpha^H(1-\sigma^{E'})) \\ &\times((1-\delta_2\sigma^E)(1-\delta_2+\delta_2\alpha^H(1-\sigma^{E'}))+\delta_2(1-\delta_2)\sigma^{E'}(1-\alpha^H))\cdot D,\end{aligned}$$

and hence $\Delta_{H,H}$ is increasing in D. As for $\Delta_{L,L}$,

$$\Delta_{L,L}(D)-\Delta_{L,L}(0)=-\delta_2^2\sigma^E\sigma^{E'}(\alpha^H-\alpha^L)(1-\alpha^L)(1-\delta_2\sigma^{E'})\\ \times(1-\delta_2+2\delta_2\alpha^L(1-\sigma^E)+\delta_2\sigma^E(1-\alpha^L))\cdot D,$$

which shows that it is a decreasing function in D. An immediate consequence of these monotonicity properties is that if $\Delta_{L,L}(D)=\Delta_{H,H}(D)$ implies that this value is strictly positive, then for any D, $\max_D\{\Delta_{H,H}(D),\Delta_{L,L}(D)\}>0$, so for the strategy considered to be optimal, this should not be the case. Because $\Delta_{E,E'}$ is linear in D, $\Delta_{L,L}(D)=\Delta_{H,H}(D)>0$ is equivalent to $\frac{\Delta_{H,H}(D)}{D}\cdot\Delta_{L,L}(0)>\frac{\Delta_{L,L}(D)}{D}\cdot\Delta_{H,H}(0)$. Tedious algebra shows that this is true if and only if:

$$\begin{aligned}&(1-\delta_2)\delta_2\sigma^E(\alpha^H-\alpha^L)(1-\delta_2+2\delta_2\alpha^L(1-\sigma^E)+\delta_2\sigma^E(1-\alpha^L))(1-\delta_2+2\sigma^{E'}(1-\alpha^H)+\delta_2\alpha^H(1-\sigma^{E'}))\\ &\quad\times\begin{pmatrix}(1-\delta_2)^2+\delta_2(1-\delta_2)(\alpha^H(1-\sigma^{E'})+\alpha^L(1-\sigma^E)+\sigma^E(1-\alpha^L)+\sigma^{E'}(1-\alpha^H))\\ +\,\delta_2^2((1-\alpha^L)(1-\alpha^H)\sigma^E\sigma^{E'}+\alpha^L\alpha^H(1-\sigma^E)(1-\sigma^{E'})+\alpha^H(1-\alpha^L)\sigma^E(1-\sigma^{E'}))\end{pmatrix}\\ &\quad\times(\delta_2(\alpha^H-\alpha^L)(1-\sigma^{E'})(1-\delta_2\sigma^E+\delta_2\sigma^{E'})\Pi\\ &\quad+((1-\delta_2\sigma^E)(1-\delta_2+\delta_2\alpha^L(1-\sigma^{E'}))-\delta_2^2\sigma^{E'}(1-\sigma^{E'})(1-\alpha^L))c)\end{aligned}$$

is positive. Noticing that all terms except possibly the last one are positive, this is equivalent to:

$$\frac{\delta_2^2\sigma^{E'}(1-\sigma^{E'})(1-\alpha^L)-(1-\delta_2\sigma^E)(1-\delta_2+\delta_2\alpha^L(1-\sigma^{E'}))}{\delta_2(\alpha^H-\alpha^L)(1-\sigma^{E'})(1-\delta_2\sigma^E+\delta_2\sigma^{E'})}<\frac{\Pi}{c}=\frac{R_2-V}{c}. \tag{A.1}$$

That is, if this condition holds, it must be that, for any $D>0$, either $\Delta_{H,H}$ or $\Delta_{L,L}$ is strictly positive. This shows that, under this condition, an optimal strategy σ_2 cannot be strictly increasing at $k>0$. A similar analysis can be done for $k+1<0$. The corresponding condition guaranteeing that σ_2 is non-decreasing at such a state (that is, that it cannot be that $\tau_2(k)=H$ and $\tau_2(k+1)=L$) is then:

$$-\frac{(1-\delta_2)^2+\delta_2(1-\delta_2)(\sigma^E(1-\beta^L)+\sigma^{E'})+\delta_2^2\sigma^E(\sigma^{E'}(1-\beta^L)-\beta^L(1-\sigma^E))}{\delta_2\sigma^E(\beta^H-\beta^L)(1-\delta_2\sigma^E+\delta_2\sigma^{E'})}<\frac{R+V}{c}, \tag{A.2}$$

where $\sigma_1(k)=\sigma^E$ and $\sigma_1(k+1)=\sigma^{E'}$. Since $R_2+V>0$, this inequality is trivially satisfied provided $(\sigma^{E'}(1-\beta^L)-\beta^L(1-\sigma^E))\geqq 0$, or, rearranging, $\beta^L\leqq\sigma^{E'}/(1-\sigma^E+\sigma^{E'})$. It is easy to see that this condition is satisfied as long as $\beta^L\leqq\alpha^L$.

Sufficiency: We show that, if the inequality in (A.1) or (A.2) is strictly reversed, there exists $V>0$ (or equivalently, given $V(k-1)$, $D>0$) such that no optimal strategy satisfies the monotonicity holding if (A.1) or (A.2) holds. We prove the case $k<0$ (the proof for the case $k>0$ is analogous, step by step). To establish equations (A.1) and (A.2), we determined under which condition $\max_{D\in\mathbb{R}^+}\{\Delta_{L,L},\Delta_{H,H}\}\geqq 0$ holds. Given the variations of $\Delta_{L,L}$ and of $\Delta_{H,H}$ as a function of D, this condition holds if it holds at the intersection of these functions. Sufficiency holds if there exists D such that $\max_{D\in\mathbb{R}^+}\{\Delta_{L,L},\Delta_{H,H},\Delta_{L,H}\}$ is negative whenever equation (A.2) is violated. To do so, we evaluate $\Delta_{L,H}$ at the value of D for which $\Delta_{L,L}=\Delta_{H,H}$ and show that $\Delta_{L,H}$ is negative for this D whenever $\Delta_{L,L}$ and $\Delta_{H,H}$ are. Tedious but straightforward computations yield that, at this D, $\Delta_{L,H}$ equals:

$$\begin{aligned}&(1-\delta_2)\begin{bmatrix}(1-\delta_2)^2+\delta_2(1-\delta_2)[\sigma^E(1-\beta^H)+\sigma^{E'}(1-\beta^L)+\beta^H(1-\sigma^E)+\beta^L(1-\sigma^{E'})]\\ +\,\delta_2^2[(1-\sigma^E)(1-\sigma^{E'})\beta^L\beta^H+\sigma^E\sigma^{E'}(1-\beta^L)(1-\beta^H)+\sigma^E(1-\sigma^{E'})\beta^L(1-\beta^H)]\end{bmatrix}\\ &\quad\times T_1\cdot\left.\begin{bmatrix}(\delta_2\sigma^E(\beta^H-\beta^L)(1-\delta_2\sigma^E+\delta_2\sigma^{E'}))(R+V)\\ +\,((1-\delta_2)^2+\delta_2(1-\delta_2)(\sigma^E(1-\beta^L)+\sigma^{E'})+\delta_2^2\sigma^E(\sigma^{E'}(1-\beta^L)-\beta^L(1-\sigma^E)))c\end{bmatrix}\right/T_2\end{aligned}$$

where

$$T_1=\begin{aligned}&2(1-\delta_2)^2+\delta_2(1-\delta_2)[(\beta^L+\beta^H)(2(1-\sigma^E)+1-\sigma^{E'})+(2-\beta^H-\beta^L)(2\sigma^{E'}+\sigma^E)]\\ &+\delta_2^2\begin{bmatrix}2((\beta^L)^2+(\beta^H)^2)(1-\sigma^E)(1-\sigma^{E'})+2\sigma^E\sigma^{E'}((1-\beta^H)^2+(1-\beta^L)^2)\\ +(\beta^H(1-\beta^H)+\beta^L(1-\beta^L))(4\sigma^{E'}(1-\sigma^E)+\sigma^E(1-\sigma^{E'}))\end{bmatrix},\end{aligned}$$

and T_2 equals

$$(1-\delta_2)^3+\delta_2(1-\delta_2)^2(\sigma^{E'}+3\beta^H(1-\sigma^E)+2\sigma^E(1-\beta^H)+\beta^L(1-\sigma^E))$$
$$+\delta_2^2(1-\delta_2)\begin{bmatrix}(\sigma^E(1-\beta^H)+\beta^H(1-\sigma^E))(2\sigma^{E'}+2\beta^H(1-\sigma^E)+\sigma^E(1-\beta^H))\\ +\beta^L(1-\sigma^E)(\sigma^E+2\sigma^{E'}(1-\beta^L)+\beta^L(1-\sigma^{E'}))\end{bmatrix}$$
$$+\delta_2^3\begin{bmatrix}\sigma^E\sigma^{E'}(1-\beta^H)(2\beta^H(1-\sigma^E)+\sigma^E(1-\beta^H))\\ +2\sigma^{E'}(1-\sigma^E)\beta^L(1-\beta^L)+(1-\sigma^E)(1-\sigma^{E'})(\beta^L)^2\end{bmatrix}.$$

Notice that all terms in $\Delta_{L,H}(D)$ are obviously positive, except possibly the term immediately before the denominator. But this one is easily seen to be proportional to the term involved in equation (A.2), so that $\Delta_{L,H}(D)$ is positive if and only if equation (A.2) is satisfied. ‖

Proof of Proposition 1. I show that, provided that Player 1 exerts high effort only at states included in $\{-1,0\}$, it is optimal for Player 2 to exert only high effort, if any, in an interval with upper extremity contained in $\{0,1\}$.

Let us focus throughout on Player 2. First, it is easy to see that Player 2 does not exert high effort at states $k>1$. To see this, suppose to the contrary that Player 2's optimal strategy τ specifies high effort at $k>1$ and suppose without loss of generality that this is the largest such k. Consider the alternative strategy $\tilde{\tau}$ where $\tilde{\tau}(k)=L$, $\tilde{\tau}(k-1)=H$ and $\tilde{\tau}(k')=\tau(k')$ for $k'\notin\{k-1,k\}$. Notice that $V(k-1,\tilde{\tau})=V(k,\tau)>V(k-1,\tau)$, yielding the desired contradiction. Suppose next that it is optimal for Player 2 to exert high effort at state $k<-2$ (the case $k=-2$ is trivial) and consider the largest such k. That is, $\tau(k+i)=L$ for any $i=1,\dots,-2-k$, where τ denote the strategy of Player 2. Let us write V for $V(-1,\tau)$. Pick some state $i=1,\dots,-2-k$. It is easy to determine $V(i,\tau)$. Consider the alternative strategy $\tilde{\tau}$ such that $\tilde{\tau}(k+1)=H$, $\tilde{\tau}\,(k')=\tau(k')$ for $k'\neq k+1$, and determine similarly $V(i,\tilde{\tau})$ fixing $V(-1,\tilde{\tau})$ at V. The difference $V(i,\tilde{\tau})-V(i,\tau)$ can then be seen to be positive if and only if M_n is positive, where M_n equals:

$$(1-x)x^ny^3(1-\delta_2)(\delta_2(1-\alpha)-x(1-\alpha\delta_2))(1-\delta_2+\delta_2\beta(1-\alpha)+\delta_2\alpha(1-\beta)(1-y))c$$
$$-(1-y)y^nx^3(1-\delta_2)(\delta_2(1-\alpha)-y(1-\alpha\delta_2))(1-\delta_2+\delta_2\beta(1-\alpha)+\delta_2\alpha(1-\beta)(1-x))c$$
$$-(xy)^n(y-x)(1-\alpha)\beta\delta_2(\delta_2(1-\alpha)-y(1-\alpha\delta_2))(\delta_2(1-\alpha)-x(1-\alpha\delta_2))(R+V),$$

where x and y are the roots of $\delta_2\alpha(1-\beta)x^2-(1-\delta_2(\alpha\beta+(1-\alpha)(1-\beta)))x+\delta_2\beta(1-\alpha)$, and $0<x<1<y$. It is straightforward to verify that $(1-\delta_2+\delta_2\beta(1-\alpha)+\delta_2\alpha(1-\beta)(1-y))>0$, $(\delta_2(1-\alpha)-x(1-\alpha\delta_2))>0$, and $xy<1$. The first summand of M_n is thus positive, the second (including the minus sign) negative and the third positive. Given that $0<x<xy<1<y$, it then follows that all three terms are strictly decreasing in n, and thus so is M_n. By assumption, τ being optimal, $M_n\leqq 0$. But then $M_{n+1}<M_n$, and thus, defining $\hat{\tau}$ as the strategy such that $\hat{\tau}(k-1)=H$, $\hat{\tau}(k)=L$ and $\hat{\tau}\,(k')=\tau(k')$ for $k'\neq\{k-1,k\}$ and letting $V(i,\hat{\tau})$ denote its value at state i given that $V(-1,\hat{\tau})=V$, we have that $V(i,\tau)-V(i,\hat{\tau})<0$, contradicting the optimality of τ. Hence, either high effort is never exerted, or it is exerted in an interval whose upper extremity is at least -1. Since Player 2 does not exert high effort at states $k>1$, it remains to show that it cannot be optimal to reflect at -1 while exerting low effort at 0. Three possibilities arise: either Player 1 exerts high effort at -1, or he exerts high effort only at 0, or he never exerts high effort. If he exerts high effort at -1, the result is trivial, since Player 2 had better not exert high effort at strictly negative states. If he exerts high effort only at 0, then the policy τ_2 of Player 2 consisting in reflecting at -1 but not at 0, is compared with two alternative strategies: reflection at 0 (τ_2^0), or low effort at all states (τ_2^N). It can immediately be verified that τ_2^0 is better than τ_2 if R_2/c_2 is larger than some threshold T_0. Similarly, τ_2^N is better than τ_2 if R_2/c_2 is smaller than some threshold T_1. Easy algebra shows that $T_0<T_1$, implying that τ_2 is never optimal. The same procedure is used to prove that τ_2 is never optimal when Player 1 never exerts high effort, but the algebra being much more difficult, it may be useful for the courageous reader to realize that $T_0<T_1$ can be rewritten as:

$$0<(2-\alpha-\beta)(\alpha+\beta)(1-\delta_2)\delta_2^2(1-\delta_2+\delta_2\beta(1-\alpha)+\delta_2\alpha(1-\beta)+\sqrt{\Delta})$$
$$\times(\delta_2(\alpha-\beta)^2+(1-\delta_2)(\alpha(1-\beta)+3\beta(1-\alpha))+(\alpha-\beta)\sqrt{\Delta})$$
$$\times\begin{pmatrix}\delta_2^2(\alpha-\beta)(2-\alpha-\beta)+\delta_2(1-\delta_2)(2-\alpha-\beta+2\alpha(1-\beta)+2\beta(1-\alpha))\\ +2(1-\delta_2)^2+(2-\delta_2(\alpha+\beta))\sqrt{\Delta}\end{pmatrix}^{-1}\Bigg/$$

$$\begin{pmatrix}\delta_2^3(1-\alpha)(\alpha-\beta)(2-\alpha-\beta)(\alpha+\beta)+2\delta_2(1-\delta_2)^2(2-\alpha^2-\beta^2+4\beta(1-\alpha)+2\alpha(1-\beta))\\ +\delta_2^2(1-\delta_2)(2(\alpha-\beta)^2(1+\beta)+(1-\alpha)(\alpha+\beta)(3(2-\alpha-\beta)+4\alpha(1-\beta)+2\beta(\alpha-\beta)))\\ +4(1-\delta_2)^3+(4(1-\delta_2)+2\delta_2(1-\delta_2)(2(1+\beta)-(\alpha+\beta)^2)+\delta_2^2(1-\alpha)(2-\alpha-\beta)(\alpha+\beta))\sqrt{\Delta}\end{pmatrix},$$

which is always satisfied, as the reader may verify.

Proof of Lemma 5. Given that Player 1 never reflects, Player 2 has several possibilities: either he reflects at -1, or at 0, or at 1 (in which case he also finds it worthwhile to exert high effort at 0), or he never exerts high effort. Given the symmetry of the game, if he exerts high effort at no states at all, his overall pay-off (evaluated at state 0) is 0. To see then when Player 2 chooses to reflect, it is sufficient to study when reflection at the aforementioned states yields a positive pay-off (when evaluated at state 0). Let $\Delta = (1-\delta_2+\alpha(1-\beta)+\beta(1-\alpha))^2 - 4\delta_2^2\alpha\beta(1-\alpha)(1-\beta)$, which is positive. Reflection at 0 gives a positive pay-off if and only if:

$$R/c \geqq \frac{1-\delta_2-\delta_2(\alpha-\beta)+\sqrt{\Delta}}{\delta_2(2-\alpha-\beta)} \triangleq T_2.$$

It is straightforward to verify that $T_2 > 0$. Next, reflection at 1 yields a positive pay-off whenever:

$$R/c \geqq \frac{4(1-\delta_2)(1-\alpha)+\delta_2(2-\alpha-\beta)((1-\delta_2)(1-\alpha-\beta)-(\alpha-\beta)+\sqrt{\Delta})}{2\delta_2(1-\alpha)(2-\alpha-\beta)}.$$

Although the R.H.S. term, T_3, is positive, it can be smaller or larger than T_2 depending on the parameters. Reflection is thus preferred to low effort if and only if:

$$R/c \geqq \min\{T_2, T_3\}.$$

Fortunately, it is a matter of algebra to verify that $dT_2/d\delta_2 < 0$, $dT_3/d\delta_2 < 0$, $dT_2/d\beta > 0$, $dT_3/d\beta > 0$, so that comparative statics conclusions do not hinge upon the ranking of T_2 and T_3. Further, one can check that $dT_2/d\alpha$ and $dT_3/d\alpha$ are positive when $\beta = 0$, and that $dT_2/d\alpha$ and $dT_3/d\alpha$ are negative when β approaches α and δ_2 approaches 1.

Suppose now that Player 1 reflects at 0. Two possibilities arise. Either Player 2 reflects at 0, or he never reflects. Algebra yields that reflection at 0 is preferred to low effort whenever:

$$R/c \geqq \frac{1-\delta_2+2\delta_2(1-\alpha)+\sqrt{\Delta}}{\delta_2(2-\alpha-\beta)} \triangleq S_1.$$

It is readily verified that $S_1 > 0$. Thus, reflection at some point is preferred by Player 2 to no effort ever if and only if $R/c \geqq S_1$. It is then immediately verified that $dS_1/d\delta_2 < 0$, $dS_1/d\beta > 0$, $dS_1/d\alpha > 0$ for β close to 0, and $dS_1/d\alpha < 0$ when β approaches α while δ_2 approaches 1. Finally, if Player 1 reflects at -1 and 0, it is straightforward to show that reflection is preferred by Player 2 to low effort whenever:

$$R/c \geqq 1 + \frac{1-\delta_2}{\delta_2\left(1-\frac{\alpha+\beta}{2}\right)},$$

threshold that increases with α and β and decreases with δ_2.

Proof of Lemma 6. As an example on how to derive the relevant value function, suppose that Player 2 reflects at state $m > 1$ and gives up at state $-n < 1$, while Player 1 does not set up any barrier in the interval $\{-n, \dots, m\}$. Let z_1 and z_2 $(0 < z_1 < z_1 z_2 < 1 < z_2)$ be the roots of $\delta_2\alpha(1-\beta)x^2 - (1-\delta_2+\delta_2\alpha(1-\beta)+\delta_2\beta(1-\alpha))x + \delta_2\beta(1-\alpha)$. In what follows, subscripts i are dropped when no confusion is possible. The value $V \triangleq V_2(0)$ of such a policy solves the following difference equations along with the relevant boundary conditions:

$$\begin{cases} (1-\delta_2+2\delta_2\frac{\alpha+\beta}{2}(1-\frac{\alpha+\beta}{2}))V \\ \quad = (1-\delta_2)(-c)+\delta_2\frac{\alpha+\beta}{2}(1-\frac{\alpha+\beta}{2})(R-c+\theta_1 z_1+\theta_2 z_2)+\delta_2\frac{\alpha+\beta}{2}(1-\frac{\alpha+\beta}{2})(-R-c+\varphi_1 z_1+\varphi_2 z_2), \\ \quad\; (1-\delta_2+\delta_2\alpha(1-\beta)+\delta_2\beta(1-\alpha))(R-c+\theta_1 z_1+\theta_2 z_2) \\ \quad = (1-\delta_2)(R-c)+\delta_2\alpha(1-\beta)(R-c+\theta_1 z_1^2+\theta_2 z_2^2)+\delta_2\beta(1-\alpha)V, \\ \quad\; (1-\delta_2+\delta_2\alpha(1-\beta)+\delta_2\beta(1-\alpha))(-R-c+\varphi_1 z_1+\varphi_2 z_2) \\ \quad = (1-\delta_2)(-R-c)+\delta_2\alpha(1-\beta)(-R-c+\varphi_1 z_1^2+\varphi_2 z_2^2)+\delta_2\beta(1-\alpha)V, \\ \quad\; (1-\delta_2+\delta_2\alpha(1-\beta)+\delta_2\beta(1-\alpha))(-R-c+\varphi_1 z_1^{n-1}+\varphi_2 z_2^{n-1}) \\ \quad = (1-\delta_2)(-R-c)+\delta_2\alpha(1-\beta)(-R)+\delta_2\beta(1-\alpha)(-R-c+\varphi_1 z_1^{n-2}+\varphi_2 z_2^{n-2}), \\ \quad\; (1-\delta_2+\delta_2\alpha(1-\beta)+\delta_2\beta(1-\alpha))(R-c+\theta_1 z_1^{m-1}+\theta_2 z_2^{m-1}) \\ \quad = (1-\delta_2)(R-c)+\delta_2\alpha(1-\beta)V_m+\delta_2\beta(1-\alpha)(R-c+\theta_1 z_1^{m-2}+\theta_2 z_2^{m-2}), \\ \quad\; (1-\delta_2+\delta_2\beta)V_m = (1-\delta_2)R+\delta_2\beta(R-c+\theta_1 z_1^{m-1}+\theta_2 z_2^{m-1}) \end{cases}$$

where the unknowns are V, V_m, φ_1, φ_2, θ_1 and θ_2. However, the following changes of variable significantly reduce the complexity of this system. Let $p = \frac{\delta_2\alpha(1-\beta)}{(1-\delta_2+\delta_2\alpha(1-\beta)+\delta_2\beta(1-\alpha))}$, $q = \frac{\delta_2\beta(1-\alpha)}{(1-\delta_2+\delta_2\alpha(1-\beta)+\delta_2\beta(1-\alpha))}$, $\lambda = \frac{\delta_2(\alpha+\beta)/2}{1-\delta_2+2(\delta_2(\alpha+\beta)/2)(1-(\alpha+\beta)/2)}$ and $\mu = \frac{\delta_2\beta}{(1-\delta_2+\delta_2\beta)}$. Notice that z_1 and z_2 solve $px^2 - x + qx = 0$, and that the system of equations, upon rearranging, reduces to:

$$\begin{cases} V = -c + \lambda(\theta_1 z_1 + \theta_2 z_2 + \varphi_1 z_1 + \varphi_2 z_2), \\ \theta_1 + \theta_2 = V - R + c, \\ \varphi_1 + \varphi_2 = V + R + c, \\ \varphi_1 z_1^n + \varphi_2 z_2^n = c, \\ \theta_1 z_1^n + \theta_2 z_2^n = (1-\mu)c + \mu(\theta_1 z_1^{n-1} + \theta_2 z_2^{n-1}), \end{cases}$$

which can readily be solved. As a first step, it is useful to show that—assuming Player 1 always exerts high effort—the value of giving up at state $-n < 0$ is single-peaked in n. It is actually not even necessary to solve for the previous system. Instead, it is sufficient to consider how the value of giving up (as measured at state -1 as a convention), depends on n, given that the random walk is absorbed at the origin with pay-off from absorption given by (exogenous) $V \in (-R, R)$. Denote this value V_n. It is straightforward to show that $V_{n+1} - V_n$ is of the same sign as:

$$(z_2 - z_1)(z_1 z_2)^n (R + c + V) - (z_2^n (z_2 - 1) + z_1^n (1 - z_1))c,$$

which is strictly decreasing, so that V_n is single-peaked, and thus, admits at most two adjacent maximizers. From the previous expression, one can easily show that these maximizers are non-decreasing in V. This makes sense, since a larger terminating value at the origin makes Player 2 more reluctant to give up. Using exactly the same procedure for reflection at m and denoting V_m the value at state 1 of reflecting at $m > 0$, given terminating value of V at state 0, one obtains that $V_{m+1} - V_m$ is of the same sign as:

$$(z_2 - z_1)(z_2 - \mu)(\mu - z_1)(z_1 z_2)^m (R - c - V) - (1-\mu)(z_1(z_2 - 1)(z_2 - \mu) z_2^m - z_2(1 - z_1)(\mu - z_1) z_1^m)c,$$

which is possibly first increasing and then decreasing (it is easy to show that $z_1 < \mu < z_2$). Hence also V_m admits at most two adjacent maximizers, and it can be shown from the previous expression that these maximizers are non-increasing in V. This also makes sense: if the terminating value is more attractive, it is less costly to reflect earlier. To save on notation, write $K \simeq f(n)$ when $f(n) \geqq K \geqq f(n+1)$, or $f(n+1) \geqq K \geqq f(n)$. First, suppose that Player 1 gives up at $-n < 0$. In any absorbing equilibrium, this implies that Player 1 relaxes at $-n$. However, given this, Player 2 gives up at $-n$ only if $-n$ is the smallest maximizer of V_n, the value (at -1) of reflecting at $-n$. To prove this, it is easy to show that giving up at $-n$ is preferred (by Player 2) to high effort at that state (the particle being then reflected), if and only if $V \leqq -R - c + m_1 c$, where m_1 is some lengthy expression and V is the terminating value at 0, $V \in (-R, R)$. Also, giving up at $-n$ is preferred to reflection at $-n-1$ if and only if $V \leqq -R - c + m_2 c$, for some m_2 easy to determine. If $m_1 < m_2$, then $V \leqq -R - c + m_1 c$ implies $V < -R - c + m_2 c$, and since Player 2 by assumption prefers to give up at $-n$ rather than at $1 - n$, this would conclude the claim. It is easy to show that $m_2 - m_1$ is of the same sign as:

$$\begin{aligned} &(1 - \delta_2 + \delta_2 \alpha(1-\beta)(1 - z_2) + \delta_2 \beta(1-\alpha)) z_1^n z_2^3 ((1-\alpha)\beta\delta_2(1 - z_1) - z_1(1 - \delta_2) - \alpha(1-\beta)\delta_2 z_1) \\ &\quad - (1 - \delta_2 + \delta_2 \alpha(1-\beta)(1 - z_1) + \delta_2 \beta(1-\alpha)) z_1^n z_2^3 ((1-\alpha)\beta\delta_2(1 - z_2) - z_2(1 - \delta_2) - \alpha(1-\beta)\delta_2 z_2). \end{aligned}$$

Algebra shows that the last factors of each summand are negative, while the first are positive. It follows that this expression is increasing in n, but for $n = 1$, it reduces to $\frac{(1-\alpha)^2\beta^2\delta_2}{\alpha(1-\beta)}\sqrt{\Delta} > 0$, proving that indeed $m_2 > m_1$. Consider a reflecting equilibrium with reflection occurring at $-n < -1$ and at $m > 1$. It is straightforward to compute the value $V_m = V_2(0)$ induced by such strategies, where the subscript m refers to the state at which Player 2 reflects. The threshold m at which Player 2 reflects must maximize this value given threshold $-n$. Hence, the threshold m is the largest integer such that $V_{m+1} - V_m$ is positive. But $0 \simeq V_m$ is equivalent to $R_2/c_2 \simeq T_2^R$, for some positive T_2^R easily computable. Let $\triangle_i f$ be the difference operator with respect to variable i. One can show that:

$$\triangle_n T_2^R = \frac{(1-\mu)(z_2 - z_1)\lambda[(\mu - z_1)(1 - z_1)(1 - 2z_2\lambda) z_2 z_1^m - (z_2 - \mu)(z_2 - 1)(1 - 2z_1\lambda) z_1 z_2^m]}{[(z_2 - \mu)(1 - 2z_1\lambda) z_1 z_2^n + (\mu - z_1)(1 - 2z_2\lambda) z_2 z_1^n][(z_2 - \mu)(1 - 2z_1\lambda) z_2^n + (\mu - z_1)(1 - 2z_2\lambda) z_1^n]},$$

which is negative. Similarly, one obtains:

$$\triangle_m T_2^R = \frac{(1-\mu)(1 - z_1)(z_2 - 1)\begin{pmatrix} z_2^n z_1 (z_2 - \mu)(z_2^m (z_2 - \mu)(1 - 2z_1\lambda) + z_1^m (\mu - z_1)(1 - (z_1 + z_2)\lambda)) \\ + z_1^n z_2 (\mu - z_1)(z_2^m (z_2 - \mu)(1 - (z_1 + z_2)\lambda) + z_1^m (\mu - z_1)(1 - 2z_2\lambda)) \end{pmatrix}}{(\mu - z_1)(z_2 - \mu)(z_2 - z_1)(z_1 z_2)^m [(z_2 - \mu)(1 - 2z_1\lambda) z_1 z_2^n + (\mu - z_1)(1 - 2z_2\lambda) z_2 z_1^n]},$$

which is positive. Hence, $\triangle_n T_2^R < 0$ and $\triangle_m T_2^R > 0$. Also, by symmetry, $\triangle_n T_1^R > 0$ and $\triangle_m T_1^R < 0$. Of course, when $m = n$, $T_2^R = T_1^R$. Hence it must be that $R_2/c_2 \geqq R_1/c_1$ if and only if $m \geqq n$. Since $\triangle_n T_2^R < 0$ and $\triangle_m T_2^R > 0$, multiple reflecting equilibria may exist, since larger m implies larger n for both players (in fact, in continuous time, one can show that there are at most two equilibria). Consider now an absorbing equilibrium with absorption occurring at $-n < -1$ and at $m > 1$. It is straightforward to compute the value $V_n = V_2(0)$ induced by such strategies, where the subscript n refers to the state at which Player 2 gives up. The threshold $-n$ at which Player 2 gives up must maximize this value given threshold m (recall that we have seen that this needs to be true in an absorbing equilibrium). Hence, the threshold $-n$ is the smallest integer such that $V_{n+1} - V_n$ is positive. But $0 \simeq V_n$ is equivalent to $R_2/c_2 \simeq T_2^A$, for some positive T^A easily computable. One can show that:

$$\triangle_m T_2^A = \frac{\lambda(z_2 - z_1)[z_1^{m+n}(z_2^m + z_2^n)(1 - z_1)(1 - 2z_2\lambda) + z_2^{m+n}(z_1^m + z_1^n)(z_2 - 1)(1 - 2z_1\lambda)]}{(z_1 z_2)^n (z_2^{m+1}(1 - 2z_1\lambda) - z_1^{m+1}(1 - 2z_2\lambda))(z_2^m(1 - 2z_1\lambda) - z_1^m(1 - 2z_2\lambda))},$$

which is positive. Also:

$$\triangle_n T_2^A = \frac{(1 - z_1)(z_2 - 1)\begin{pmatrix} z_2^{n+1}(z_2^m(1 - 2z_1\lambda) - z_1^m(1 - (z_1 + z_2)\lambda)) \\ -z_1^{n+1}(z_2^m(1 - (z_1 + z_2)\lambda) - z_1^m(1 - 2z_2\lambda)) \end{pmatrix}}{(z_1 z_2)^{n+1}(z_2 - z_1)(z_2^m(1 - 2z_1\lambda) - z_1^m(1 - 2z_2\lambda))},$$

which is positive too. However, $\triangle_n T_2^A > \triangle_m T_2^A > 0$, as straightforward computation establishes. Symmetrically then, $0 < \triangle_n T_1^A < \triangle_m T_1^A$. Since $T_1^A = T_2^A$ on the diagonal $m = n$, this establishes that $R_2/c_2 \geqq R_1/c_1$ if and only if $n \geqq m$. These variations also imply that multiple equilibria may only arise on adjacent states. Finally, it is a matter of tedious but straightforward computation to extend these results to the boundaries m and/or $n \in \{-1, 0, 1\}$.

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