Go from Single Agent Problem to Entry Game

- Like before: time $t = 1, 2, \ldots$.

- Two kinds of agents: Incumbents in industry already. Numbered $i = 1$ to $\omega_t$, where $\omega_t < \bar{n}$, maximum number of agents.
  
  - $a^I_{i,t} = 1$ stay in

  - $a^I_{i,t} = 0$, exit.

- Suppose one potential entrant each period. Entry feasible only if $\omega < \bar{n}$ (simplifies the state space)
- $a_t^E = 1$, enter

- $a_t^E = 0$, not enter.

**State variables at time $t$**

- State $\omega_t =$number of firms in industry at beginning of period.(since symmetric, can keep track just number of firms).

- A utility shock to each choice of incumbents $i = 1, \ldots n_t$ and potential entrant
  
  * $\epsilon_{i,t,0}$ utility shock to $a_{i,t}^I = 0$

  * $\epsilon_{i,t,1}$ utility shock to $a_{i,t}^I = 1$

  * analogous shocks for potential entrant
• Play of game.

  – Start with $\omega_t$ firms.

  – Random utility shocks are private information!

  – Simultaneously incumbents make exit decision and potential entrant makes exit decision. Let

    \[ n_t = \sum_{i=1}^{\omega_t} 1[a_{it}=1] + 1[a_t^E=1] \]

    this is the number of firms at end of period

• Payoffs (other than utility shock mentioned above

  – $\pi_n$ profit to each firm in the industry (including entrant, if it comes in)
- Expect $\pi_n > \pi_{n+1}$.

- Maybe $\pi_1 > 2\pi_2$.
  - Entry cost $\phi$
  - Exit value $\xi$

- Transition $\omega_{t+1} = n_t$
Equilibrium

• Will define a symmetric equilibrium where firms in the same state do the same thing

• Policy function of incumbent $\tilde{a}^I(\omega, \varepsilon^I_0, \varepsilon^I_1)$ choice given state and two draws

• Policy function of entrant is similar $\tilde{a}^E(\omega, \varepsilon^I_0, \varepsilon^I_1)$

• Value functions $V^I(\omega, \varepsilon^I_0, \varepsilon^I_1)$ and $V^E(\omega, \varepsilon^E_0, \varepsilon^E_1)$

• Equilibrium: Values solve the Bellman equations, given the policies. Policies are optimal given the Value functions and that other firms following the policy rules
• Look at choice specific value functions

\[ V^I(\omega, 0, \varepsilon_0) = \xi + \varepsilon_0 \]
\[ V^I(\omega, 1, \varepsilon_1) = E_n[\pi_n + \beta E_\varepsilon[V^I(n, \varepsilon)]|\omega, a^I_1 = 1] + \varepsilon_1 \]
\[ V^E(\omega, 0, \varepsilon_0) = \varepsilon_0 \]
\[ V^E(\omega, 1, \varepsilon_1) = -\phi + E_n[\pi_n + \beta E_\varepsilon[V^I(n, \varepsilon)]|\omega, a^E_1 = 1] + \varepsilon_1 \]

• Then

\[ V^I(\omega, \varepsilon) = \max\{V^I(\omega, 0, \varepsilon_0), V^I(\omega, 1, \varepsilon_1)\} \]
\[ V^E(\omega, \varepsilon) = \max\{V^E(\omega, 0, \varepsilon_0), V^E(\omega, 1, \varepsilon_1)\} \]

• Policy can be summarized by cutoff rules

\[ \tilde{x}^I_\omega < \varepsilon^I_{1,t} - \varepsilon^I_{0,t} \text{ then } \tilde{a}^I(\omega, \varepsilon^I_0, \varepsilon^I_1) = 1, \text{ otherwise } = 0, \]
\[ \tilde{x}^E_\omega < \varepsilon^E_{1,t} - \varepsilon^E_{0,t}, \text{ then } \tilde{a}^E(\omega, \varepsilon^E_0, \varepsilon^E_1) = 1, \text{ otherwise } = 0. \]
then incumbent (entrant) stays, where

\[
\begin{align*}
\tilde{x}^I_\omega & = \xi - E_n[\pi_n + \beta E_\varepsilon[V^I(n, \varepsilon)]|\omega, a^I_1 = 1] \\
\tilde{x}^E_\omega & = -\phi + E_n[\pi_n + \beta E_\varepsilon[V^E(n, \varepsilon)]|\omega, a^E_1 = 1]
\end{align*}
\]

- Let \(\tilde{x} = (\tilde{x}^I_1, \tilde{x}^I_2, \ldots \tilde{x}^I_n, \tilde{x}^E_1, \tilde{x}^E_2, \ldots \tilde{x}^E_n)\) be a vector of cutoff rule.

- Can use recursive methods for solve for an equilibrium.
Data

- See state of industry each period $\omega_t$ and $n_t$

- Let $exit_t$ the number of firms exiting each period, and $entry_t$ the number of entrants

\[ n_t = \omega_t - exit_t + entry_t \]

- So have data set $(\omega_t, exit_t, entry_t)t = 1, \ldots, T$

- Parameter vector $\theta = (\pi_1, \pi_2, \ldots, \pi_{\bar{n}}, \phi, \xi)$

- What next?
- How implement a nested-fixed point approach?

- How implement a two-step approach? (Assumptions? Advantages?)
Multiple Equilibria Issue

- Related Game. Two incumbent firms, simultaneously decide whether or not to produce in a period. But no exit issue. If don’t produce, still around to decide tomorrow. Also no entrant.

- Throws out the dynamics. No state to keep track of.

- Suppose firm 1 takes as given firm 2 is using rule $\tilde{x}_2$. For $x_2 = \varepsilon_{2,1} - \varepsilon_{2,0}$. Define logistic distribution

$$F(x_2) = \frac{\exp(x_2)}{1 + \exp(x_2)}$$
Let $V_1(a, \varepsilon_{1,a})$ be the choice specific value function

\[
V_1(0, \varepsilon_{1,0}) = \varepsilon_{1,0} \\
V_1(1, \varepsilon_{1,1}) = F(\tilde{x}_2)\pi_1 + (1 - F(\tilde{x}_2))\pi_2 + \varepsilon_{1,1}
\]

So

\[
\tilde{x}_1 = \tilde{X}_1^*(\tilde{x}_2) = -F(\tilde{x}_2)\pi_1 - (1 - F(\tilde{x}_2))\pi_2
\]

Get a mapping from $\tilde{x}_2$ to $\tilde{x}_1$,

\[
\frac{d\tilde{x}_1}{d\tilde{x}_2} = f(\tilde{x}_2)(\pi_2 - \pi_1)
\]

• Possibility of multiple equilibria?

• Possibility of asymmetric equilibria (more below)?
Big Picture

• Entry model above, can estimate vector $\pi_1, \pi_2, \ldots \pi_{\bar{n}}$

• What is potentially interesting economics?

• Bring in market size. Think of a broader data set. Write $\pi(n, \text{pop})$ for $n$ firms and population.

• $(\omega_{j,t}, \text{exit}_{j,t}, \text{entry}_{j,t}, \text{pop}_{j})$ for markets $j$, and time periods $t = 1, \ldots T$

• Bresnahan and Reiss idea
Now Generalize the Setup (Quick Overview of Ericson-Pakes Full Model)

Give incumbents more to do. Not just staying in or out. But get better somehow. But let’s discretize it. Make a continuous decision $x$ of investment. Let’s put in some aggregate state that moves around (demand, aggregate costs)

- Each firm is in discrete state $\omega_i$

- Have some stage game with reduced form profits $\pi_i(\omega_i, \omega_{-i})$.

  - e.g. if have differentiated products oligopoly with demand $q_i(p_i, p_{-i}, \omega_i, \omega_{-i})$ cost $c_i(\omega_i, \omega_{-i})$, and some reduced from equilibrium price function $p_i^e(\omega_i, \omega_{i-1})$, then

$$
\pi_i(\omega_i, \omega_{-i}) = \big[ p_i^e(\omega_i, \omega_{-i}) - c_i(\omega_i, \omega_{-i}) \big] \\
\times q_i(p_i(\omega_i, \omega_{-i}), p_{-i}(\omega_i, \omega_{-i}), \omega_i, \omega_{-i})
$$
• Make decision to exit and take $\xi$ (let this be random now) or invest at $x$ and move state for next period.

• Ericson Pakes particular investment model:

$$\Pr(\nu|x_i) = \begin{cases} \frac{\alpha x_i}{1+\alpha x_i} & \text{if } \nu = 1 \\ \frac{1}{1+\alpha x_i} & \text{if } \nu = 0 \end{cases}$$

and

$$\omega'_i = \omega_i + \nu - \eta \text{ (where } \eta \text{ market)}$$

• Incumbent’s problem, takes a given that other firms $k$ are obeying Markov policy rules $x = \sigma_k(\omega)$, and let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ (if $N$ firms)

$$V(\omega_i, \omega_{-i}, \phi) = \pi(\omega_i, \omega_{-i}) + \max \left\{ \xi, \max_{x_i} H(x_i, \omega_i, \omega_{-i}, \xi) \right\}$$
where
\[
H(x_i, \omega_i, \omega_{-i}, \xi) \equiv -x_i + \beta E[V(\omega'_i, \omega'_{-i}, \xi') | \omega_i, \omega_{-i}, x_i, \sigma_{-i}(\omega)]
\]
and the solution to the above is \(x_i = \sigma_i(\omega)\).

- Entrant (suppose for simlicity):
  \[
  V^e(\omega, \phi) = \max \left\{ 0, \max_{x_i^e} -\phi - x_i^e + \ldots \right\}
  \]

- A **Symmetric** Markov-Perfect Equilibrium: Investment Policy functions \(\tilde{x}_i(\omega_i, \omega_{i-1})\), Entry rules, \(Entry(\omega, \phi^e)\), Exit rules \(Exit(\omega_i, \omega_{-i}, \xi)\), and value functions \(V(\omega_i, \omega_{-i}, \xi)\), satisfying the Bellman equations and optimality conditions above.

- Symmetry has bite, all differences in behavior coming through differences in states.
Illustrate Multiplicity with Example of Cournot Model with Fixed Cost

- Look at Cournot example, \( P = 12 - Q \) and suppose have a fixed cost equal to \( F \) (paid if \( q > 0 \), otherwise if \( q = 0 \) is avoided). Given output \( q_2 \), firm 1 solves the problem:

\[
\text{Either } q_1 > 0 \text{ and } q_1 = \arg \max_{q_1} [12 - q_1 - q_2] q_1 - F \\
of \quad q_1 = 0 \text{ and } \pi = 0.
\]

So if \( q_1 > 0 \), then reaction is

\[
\max (\text{given positive}) = 6 - \frac{q_2}{2}
\]
So compare

\[ pq_1 = [12 - q_1 - q_2] \left[ 6 - \frac{q_2}{2} \right] = \left[ 6 - \frac{q_2}{2} \right]^2 \]

with fixed cost

\[ \left[ 6 - \frac{q_2}{2} \right]^2 = F \]

Let

\[ 6 - \frac{q_2}{2} = F^{\frac{1}{2}} \]

\[ \hat{q}_2 = 12 - 2F^{\frac{1}{2}} \]

So

\[ \tilde{R}_1(q_2) = 6 - \frac{q_2}{2}, \quad q_2 < 12 - 2F^{\frac{1}{2}} \]

\[ = 0, \quad q_2 < 12 - 2F^{\frac{1}{2}}, \]

- If \( F = 0 \), then unique equilibrium is \( q_1 = q_2 = 4 \). Both firms
earn 16.

• Suppose $F = 15$: Then have three pure strategy equilibria, one symmetric, two asymmetric

• Suppose $F = 17$?