Auctions

• Introduction of Hendricks and Porter. To sell interest in the subject, they argue

  – auctions are prevalent

  – data better than the typical data set in industrial organization

  – auction game is relatively simple, well-specified rules.

          Highest Bid  Second highest
  – Open  Dutch  English
  Closed  FPSB  SPSB (Vikrey)

  – “Auctions offer close connection between theory and empirical work..”
* Question: How are the bidders behaving

* Question: what is the best auction mechanism (revenues, total surplus, etc.)
Milgrom and Weber

- Random variables upper case, realizations lower case

- $n$ bidders indexed by $i$, each observes a signal $x_i$. Here $X_i$ is private information. Plus some random variable $V$ influences value

- $(V, X_1, \ldots, X_n)$ drawn from joint distribution $F$, with density $f$

- $x_{-i} = (x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_n)$

- Bidders payoff $U_i = u_i(V, X_i, X_{-i})$ (assume increasing in each argument)
• reserve price $r$

• Primitives $n$, $F$ and $\{u_i\}_{i=1}^n$. Common knowledge

• Definition: $n$ random variables $X = (X_1, \ldots X_n)$ joint density $f$ are affiliated if for all $x$ and $y$ then $f(x \land y)f(x \lor y) \geq f(x)f(y)$ (implies nonnegatively correlated)

• Assume
  
  – $F$ symmetric in signals $X$
  
  – $u_i$ symmetric in signals
  
  – $(V, X_1, \ldots, X_n)$ are affiliated random variables
• Let $Y_1, \ldots, Y_{n-1}$ be ordering of largest through smallest signals from $(X_2, \ldots, X_n)$. From Theorem 2 of Milgrom and Weber, $(V, X_1, Y_1, \ldots, Y_{n-1})$ are affiliated.

• Let’s discuss restrictions. (Asymmetry...)

• A bidding (pure) strategy for bidder $i$ is a correspondence $\beta_i : X_i \rightarrow R^+$. Empirical work: relies on $\beta_i$ being strictly increasing and therefore invertible.
Two special cases

- Private values: $u_i(v, x_i, x_{-i}) = x_i$.

- Pure common values $u_i(v, x_i, x_{-i}) = v$

- Presence of common components does not imply valuations are not private.
  - If distribution of $V$ conditional on $x_i$ is independent of $X_{-i}$ then private values.
  - Just define private valuation to be $u_i(V, X_i; X_{-1})$ which equals $u_i(V, X_i)$ in this case, just renormalize signals, so $E[U_i|X_i = x_i]$ is the signal.
• Intermediate case (Wilson) \( u_i(\nu, x_i, x_{i-1}) = u(\nu, x_i) \).

• Look at winner’s curse issue in pure common value model

\[
\nu_1(x) = E[V|X_1 = x] = \int \nu f_{V|X_1}(\nu|x) d\nu
\]

compare with

\[
E[\max_i \{\nu_1(X_i)\}] \geq \max_i \nu_1(X_i)
\]

Given convexity of max operator and Jensen’s inequality. Winning is bad news

\[
E[V|X_1 = x, Y_1 < x] < E[V|X_1 = x]
\]
Section 3: Structural analysis of 2nd Price Auctions

• 2nd price are easier to model. But price also doesn’t do as much work (no competition effect in choice of bid)

• 3.1 Theory English auction (let’s do the “button” auction)

\[ w(x, y) = E(u(V, x)|x_1 = x, Y_1 = y) \]

• For button, bidding strategy is (using affiliation in argument)

\[ \beta(x) = E[u(V, x)|X_1 = x, Y_1 = x] = w(x, x) \]

Suppose bidding has reached \( b = \beta(y) \) (all else playing there), payoff positive if \( x > y \) and negative if \( y < x \).
• Reserve price

\[ x^*(r) = \inf \{ x \mid E[w(x, Y_1) \mid X_1 = x, Y_1 < x] \geq r \} \]

• Suppose Private values. Then equilibrium is unique (don’t need to assume independence, very general. See Vickry).

• Common Values: not unique. Wallet game \( x_i \in [0, \bar{x}] \) bid for the pot (money in both wallets, 2nd price). Equilibrium parameterized by \( \alpha \)

\[ \beta_1(x_1) = (1 + \alpha) x_1, \quad \beta_2(x_2) = \left(1 + \frac{1}{\alpha} \right) x_2. \]
Estimation

• What is $F$? (distribution of $X_i$ and $V$)

• Button auction with IPV, so $\beta(x) = x$, $x \geq r$.

• Let $w_t$ be winning bid, given $n_t$ bidders and $r_t$ reserve.

• Data $\{w_t, n_t, r_t\}_{t=1}^{T}$, if $m_t \geq 1$ (where $m_t$ is active bidders)

• $w_t = \max \left\{ x_{2:n_t}, r_t \right\}$ (where subscript 2 denotes the 2nd order statistic)
Background, distribution of first and second order statistic

- Distribution of max of $n$ is

$$\Pr(X_1 \leq x_1) = F_X(x_1)^n$$

- Distribution of $X_1$ and $X_2$

$$\Pr(X_1 \leq x_1, X_2 \leq x_2 | n) = \Pr(X_1 \leq x_2) + \Pr(X_2 < X_1 \leq x_1, X_2 \leq x_2)$$

$$= F_X(x_2)^n + n(F_X(x_1) - F_X(x_2))F_X(x_2)^{n-1}$$

- Distribution just of $X_2$?

$$\Pr(X_2 \leq x_2) = nF_X(x_2)^{n-1} - (n-1)F_X(x_2)^n$$

(explain for case of $n = 2$, then this is

$$2F_X(x_2) - F_X(x_2)^2.$$
Take set where first draw less than $x_2$ plus events where second draw less than $x_2$. These include all cases where second highest is less than $x_2$. However, there is double counting, (events where both are less than $x_2$ are counted twice. Then density function for the distribution of the second order statistics is

$$f_2(x) = 2f_X(x) - 2F_X(x)f_X(x)$$
Donald and Paarsch (1996) (Parametric approach, IPV)

- \( m_t = 0 \), \( \Pr(m_t = 0) = F_X(r_t)^{n_t} \)

- \( m_t = 1 \). \( w_t = r_t \) and \( \Pr(m_t = 1) = n_tF_X(r_t)^{n_t-1}(1 - F_X(r_t)) \)

- \( m_t > 1 \), then \( w_t \sim h_t(w) = n_t(n_1 - 1)F_X(w)^{n_t-2}(1 - F_x(w))f_x(w) \) (reduces to formula above for \( n_t = 2 \))

- Define \( D_t = 1 \), if \( m_t = 1 \), \( = 0 \), if \( m_t > 1 \)

- Likelihood function

\[
L = \prod_t \left[ \frac{h_t(w_t)^{1-D_t} \Pr \{m_t = 1\}^{D_t}}{(1 - \Pr \{m_t = 0\})} \right]
\]
• Pick θ to maximize L given $F_X(\cdot|θ)$. With the answer can solve question of maximizing reserve

$$R = x_0 F_X(r)^n + rn F_X(r)^{r-1} [1 - F_X(r)] + \int_r^{\bar{x}} wn(n-1) F_X(w)^{n-2} (1 - F_X(w)) dw$$

with solution

$$r = x_0 + \frac{1 - F_X(r)}{f_X(r)}$$

which is independent of $n$. 
Nonparametric Approaches

• Take independent private values. Let’s take simplest case with no reservation price. We see $x_{2:n}$

• What next? Athey and Haile do statistics. Not surprising that can you back out an underlying distribution if see second highest of $n$ draws (obviously in great shape if you see all the bids!) Nonparametric approaches are straightforward here.

• Suppose private values by correlated. Then not in good shape.

• Suppose independent but common values. Have multiple equilibria issue.
• What about correlation? How does this affect reservation prices? (See Aradillas-Lopez, Gandhi, and Quint)
Symmetric First-Price Auctions (Hendricks and Porter Handbook Chapter)

• Assume $I$ bidders, using a common bid function $\beta$ with inverse $\eta$. Then bidder 1’s profits from bidding $b$ given a signal $x$ are

$$\pi(b, x) = \int_{\bar{x}}^{\eta(b)} [w(x, y) - b]dF_{Y_1|X_1}(y|x)$$

Differentiating w.r.t. $b$

$$[w(x, \eta(b)) - b]f_{Y_1|X_1}(\eta(b)|x) \frac{d\eta}{db} - \int_{\bar{x}}^{\eta(b)} dF_{Y_1|X_1}(y|x) = 0$$

and imposing symmetry and $x = \eta(b)$ and $\frac{d\eta}{d\beta} = 1/\beta'(x)$ yields

$$[w(x, x) - \beta(x)]f_{Y_1|X_1}(x|x) - \beta'(x)F_{Y_1|X_1}(x|x) = 0$$

The equilibrium bid solves this differential equation, subject to the boundary condition $\beta(x^*) = r$ (where $x^*(r)$ is the lowest
signal such that the expected value conditional on winning is at least the reserve price. The solution is

$$\beta(x) = rL(x^*|x) + \int_{x^*}^{x} w(s, s)dL(s|x)$$

where

$$L(s|x) = \exp \left\{-\int_{s}^{x} \frac{f_{Y_1|X_1}(t|t)}{F_{Y_1|X_1}(t|t)}dt\right\}$$

- The symmetric equilibrium exists under fairly weak regularity conditions on $F$. (Assumption of a binding reserve price ensure that there is a unique symmetric equilibrium)

- Special case of affiliated private values, the equilibrium bid function reduces to

$$\beta(x) = x - \frac{\int_{r}^{x} F_{Y_1|X_1}(s|x)ds}{F_{Y_1|X_1}(x|x)}$$
which further reduces to (in the case of independent private values)

\[ \beta(x) = x - \frac{\int_r^x F_X(s)^{n-1} ds}{F_X(x)^{n-1}} \]

the mark down \( x - \beta(x) \) is decreasing in the number of bidders \( n \) and increasing in the dispersion of the value distribution. In the case of private values, it can also be expressed as

\[ \beta(x) = E[\max \{r, Y_1\} | X_1 = x, Y_1 \leq x] \]

Very intuitive from Bertrand pricing
• Discuss MLE approach of Paarsch. Winning bid is observed if and only if $x_{1:n_t} \geq r_t$. Probability of no one winning is $F_X(r_t)^{n_t}$. The probability distribution function of $w$ is

$$h_t(w) = t \cdot F_X(\eta_t(w))^{n_t-1} f_X(\eta_t(w)) \eta'_t(\omega) = \frac{n_t F_X(\eta_t(w))^{n_t}}{(n_t - 1)(\eta_t(w) - w)}$$

• Can construct a likelihood function. (Then after estimation can calculate optimal reserve price, for example)

• Data on bids \( \left\{ \{b_{it}\}^{m_t}_{i=1}, n_t, r_t \right\}^T_{t=1} \)

• Recall in symmetric equilibrium

\[
(\eta(b) - b) f_{Y_1|X_1}(\eta(b)|\eta(b)\eta'(b) - F_{Y_1|X_1}(\eta(b), \eta(b)) = 0
\]

• Define \( M_1 = \beta(Y_1) \) as the maximum bid of bidder 1’s rivals and let conditional distribution of \( M_1 \) given bidder 1’s bid \( B_1 \) be denoted \( G_{M_1|B_1}(\cdot|\cdot) \) and density given by \( g_{M_1|B_1}(\cdot|\cdot) \). Monotonicity of \( \beta \) and \( \eta \) implies that for any \( b \in (r, \beta(\bar{\alpha})) \),

\[
G_{M_1|B_1}(m|b) = F_{Y_1|X_1}(\eta(m)|\eta(b))
\]
The associated density function is given by

\[ g_{M_1|B_1}(m|b) = f_{Y_1|X_1}(\eta(m)|\eta(b))\eta'(m) \]

Substituting the above into the first-order condition for bidder 1 yields

\[ (\eta(b) - b)g_{M_1|B_1}(b|b) - G_{M_1|B_1}(b|b) \]

or

\[ \eta(b) = b + \frac{G_{M_1|B_1}(b|b)}{g_{M_1|B_1}(b|b)} \]

In the special case of IPV, \( G_{M_1|B_1} = G^{n-1} \), where \( G \) is the marginal bid distribution of individuals bidders, also \( g_{M_1|B_1} = (n - 1)gG^{n-2} \), so the inverse bid function is

\[ \eta(b) = b + \frac{G(b)}{(n - 1)g(b)} \]
• Estimation proceeds in two steps

– Step 1: Estimate \( \hat{g}_{M_i|B_i}(m|b) \), \( \hat{G}_{M_i|B_i}(m|b) \) either parametrically or nonparametrically. Then form data

\[
\hat{x}_{it} = b_{it} + \frac{\hat{G}_{M_i|B_i}(b_{it}|b_{it})}{\hat{g}_{M_i|B_i}(b_{it}|b_{it})}
\]

– Step 2: Estimate \( f_X(\hat{x}_{it}) \) and \( F_X(\hat{x}_{it}) \). Then can run counterfactuals (i.e. change in reserve price, change in number of bidders, of course we have kept number of bidders as exogenous here. If bidders endogenous, need to do more):
Putting these ideas to work. Hortacsu and Kastl

- Study auction of 3-month and 12-month treasury bill in Canada

- Dealers submit bids for themselves, as well as customers. So see what customers do. What do dealers get out of the additional information (what is value)
  - And what is this info? Fundamental value, or info about competition?

- Start with a null hypothesis of private value (so not learning about fundamental value)
  - Create a test statistic and can’t reject private values
– Then estimate the value of the information about the competition.
Data

- Who are these dealers and customers?
  - Desjardin Securities, JPMorgan not dealers, but customers
  - BNP Paribas and Bank of America are "customers"...
  - Major thing left unexplained is why we have customers "customers are typically large banks that for some reason choose not to be registered as dealers."
  - Look at summary statistics. Look at # of customers, pretty small. Would have expected the dealers to be small in comparison to number of customers.
Preliminary Evidence

- 3M sample, out of 660 dealer bids also accompanied with customer bid, 216 previously placed dealer bids were updated.

- 12M sample, 275 out of 659 are updated.

- Info that bids are correlated with updated bids. (But would have been nice to see the correlation with updated bids from other customers that they don’t see)

- What about cases where not updated? Argue there isn’t much time left. Takes about 5 minutes to put a new bid in and when customer submits bid with less then 5 minutes to go, can’t react. (Could use as a benchmark reactions to customer bids from dealers.)
Model of Bidding

- Two classes of bidders, $N_d$ number of potential dealers, $N_c$ number of potential customers

- Discriminatory auction (pay as bid)

- Bid for shares of $Q$ which is random

- Before bidding, players observe private signals $S_1^c, S_2^c, ..., S_{N_c}^c, S_1^d, ..., S_{N_d}^d$.

- Stage 1 of bidding: dealers may submit $y_1^d(p|S^d)$, specifies for each pcie $p$ how big a share of the securities dealer $i$ demands given signal (Assume constrained to use step function, with finite steps)
• Stage 2: customers get matched to a dealer, bid $y^c(p|S^c)$

• In stage 3, dealers may submit a late bid, $y^{3d}(p, S^d, Z^d)$, where $Z^d$ contains all new information (including bid and anything else)

• Rationalize early bidding, assume some probability won’t be able to make a second bid, so put something in. Has some complicated stuff about observing a signal about whether the dealer will or will not have a chance to submit late bid...

• Assumption 1: $S^c_1, S^c_2, \ldots S^c_{N_c}, S^d_1, \ldots S^d_{N_d}$, independent and drawn from common support (symmetric private values), c.d.f. $F^d(S^d)$ and $F^c(S^c)$
• Assumption 2: Supply $Q$ random. Per bidder supply $\bar{Q} = \frac{Q}{N_c + N_d}$ is independent of private signals.

• 3rd stage, joint distribution at dealer’s stage $F^3e((S^d_1, Z_1), \ldots (S^d_{N_d}, Z_{N_d})|S^c, \{y\})$

• Winning $q$ units has marginal valuation $v(q, S)$ (same for all). Increasing in $S$, decreasing in $q$. (Independent private values)

• Notation: $\theta^k_i$ private information of bidder $i$ in stage $k$
  
  $- \theta^2_i = S^c_i$
  
  $- \theta^1_i = S^d_i$, $\theta^3_i = (S^d_i, Z^d_i)$ (Note $Z^d_i$ includes customer signal because can back out the bidding function $y^c_i(p|S^c_i)$)
• Equations 1 and 2

- Dealer at stage 1

\[
\Pr(p \geq P^c|s_i) = \text{EI} \left( Q - \sum_{j \in C} y^c(p|S_j) - \sum_{k \in D \setminus i} y^d(p|S_k, Z_k) \geq y^{1d}(p|s_i) \right)
\]

- Dealer at stage 3

\[
\Pr(p \geq P^c|s_i, z_i) = \text{EI} \left( Q - \sum_{j \in C \setminus m} y^c(p|S_j) - \sum_{k \in D \setminus i} y^d(p|S_k, Z_k) \geq y^{3d}(p|s_i, z_i) \right)
\]

• Proposition 1 (Kastl (2011). If no learning about fundamentals (includes IPV case), bid functions satisfy

\[
v(q_k, S_i) = b_k + \frac{\Pr(b_{k+1} \geq P^c|\theta_i)}{\Pr(b_k > P^c > b_{k+1}|\theta_i)}(b_k - b_{k+1})
\]
• Discussion of result. Write equation as (pretty intuitive)

\[ v((q_k, S_i) - b_k) \Pr(b_k > P^c > b_{k+1} | \theta_i) = \Pr(b_{k+1} \geq P^c | \theta_i)(b_k - b_{k+1}) \]

This is the FONC of changing cutoff quantity \( q_k \). One issue is it seems firms leave this cutoff the same, vary bid. Not a FONC for fixed cutoff, varying bid.
Estimating marginal distributions

- Start with discussion of previous literature (Hortescu and McAdams, Kastl) where symmetric (no dealers)
  - if $N$ bidders are ex-ante symmetric, private information is independent across bidders and data generated by a symmetric Bayesian Nash equilibrium.
  - Fix bidder. Draw randomly $N-1$ actual bid functions submitted by bidders.
  - Simulates one state of the world, so one potential realization of residual supply
  - Given bid, and residual supply can obtain a market clearing price. Get a full distribution of market clearing price given
a fixed bid. For a given bid (step function) we can use
the distribution of the market price and

\[ v(q_k, S_i) = b_k + \frac{\Pr(b_{k+1} \geq P^c|\theta_i)}{\Pr(b_k > P^c > b_{k+1}|\theta_i)}(b_k - b_{k+1}) \]

to back out marginal value at each step.

- Now two classes of bidders. Dealer bidding at stage 1
  - Draw customer bid (equals zero means non-participation)
  - If no customer, draw from pool of dealer bids with no
customer participation
  - If have customer, draw from pool of dealer bids with a
similar customer (why not just give the dealer that went
with the customer?)
To estimate the probability for state 3, draw $N_c - 1$ bids and take observed customer bid to plug in.
Value of Information

- $\Pi^{3d}(s_i, z_i)$ expected profit stage 3, including info from customer

- $\Pi^{1d}(s_i)$ before arrive.

\[
V I^d = \int_0^\infty \Pi^{3d}(s_i, z_i)dH(P^c, y^{3d}(s_i, z_i)) - \int_0^\infty \Pi^{1d}(s_i)dH(P^c, y^{1d}(s_i))
\]

- For stage 3 case, see one customer, integrate out the others, for stage 1 case integrate out all $N^c$ customers.

- Note, cannot calculate value of the system through this exercise (i.e., what happens if stage 3 bids are banned)
• Estimate the $V I^d = .45$ basis point. (.64 when use 4 auctions, .46 when 2, .26 when 1). (1.35 million per dealer per year)

• Compare to average expected profit of 1.65 basis points.