1. Model

Partial equilibrium model of an industry

- \( P(Q) \) inverse demand function

- Production function \( q = \phi h(n) \), \( \phi \in [0, 1] \) productivity parameter, \( n \) employment.
  Assume \( h' > 0 \), \( h'' < 0 \), \( \lim_{n \to 0} h'(n) = \infty \).

- \( \phi \) follows a Markov process

  \[ \phi_{t+1} \text{ distributed } F(\cdot; \phi_t) \]

  where \( \frac{\partial F}{\partial \phi} < 0 \)

- Assume that for each \( \varepsilon > 0 \) and \( \phi_t \) there exists an \( n \) such that \( F^n(\varepsilon|\phi_t) > 0 \), where \( F^n(\varepsilon|\phi_t) \) is what the distribution of \( \phi_{t+n} \) would be if exit were infeasible.

- The exists a fixed cost \( c_f > 0 \) to remain in the market

- There is a cost of entry \( c_e > 0 \). Entrants draw from a distribution \( G \).

2. Timing

<table>
<thead>
<tr>
<th>Incumbent</th>
<th>Observes ( \phi_t )</th>
<th>Pays fixed cost ( c_f ) or</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Sets ( q ) to max ( \pi )</td>
</tr>
<tr>
<td></td>
<td>Stay in and draw ( \phi_{t+1} )</td>
<td></td>
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</tbody>
</table>

Exit and get 0

New entrant pays \( c_e \) same as incumbent
3. Stationary Equilibrium

Set of objects:

- Price \( p \)
- \( \mu \) measure of types \( \phi \) of incumbents at the beginning of the period
- \( M \) measure of new entrant to enter in the period

That satisfy

- Supply equals demand in the output market
- Firms maximize profits in output decisions and exit decisions
- Entry condition holds (return to entry is zero of \( M > 0 \) and otherwise nonpositive).
- The exit and entry behavior implies the invariant measure \( \mu \).

4. Individual Behavior

(1) Production decision:

\[
\max_n p\phi h(n) - wn - c_f
\]

The FONC is

\[
p\phi h'(n) - w = 0
\]

Let \( n(\phi, p) \) solve this problem. Let \( q(\phi, n) = p\phi h(n(\phi, n)) \) be the optimal quantity and let \( \pi(\phi, p) \) be the maximized profit.

(2) Exit decision

\[
v(\phi, p) = \pi(\phi, p) + \max \left\{ 0, \beta \int_0^1 v(\phi', p)f(\phi'|\phi)d\phi' \right\}
\]

Standard dynamic programing arguments show a solution \( v(\phi, p) \) exists and is strictly increasing in \( \phi \) and \( p \). (Note: this claim uses the fact that an increase in \( \phi \) shifts the distribution of \( \phi' \) in a first-order stochastic dominance fashion.) Let \( E(\phi, p) \) be the expected return to staying,

\[
E(\phi, p) = \beta \int_0^1 v(\phi', p)f(\phi'|\phi)d\phi'
\]
This is strictly increasing in $p$ and $\phi$. Suppose that $E(1, p) > 0$ and $E(0, p) < 0$. Then let $x(p)$ be the unique point in $(0, 1)$ satisfying

$$E(x(p), p) = 0$$

This is the value of $\phi$ where the individual is just indifferent to staying or leaving. If $E(1, p) \leq 0$, then let $x(p) = 1$ and if $E(0, p) > 0$ let $x(p) = 0$. It the cutoff $x(p)$ is not at a corner it is strictly increasing in $p$.

(3) Entry Decision. The return to entry is

$$\int_0^1 v(\phi, p) g(\phi) d\phi - c_e$$

The first term is plotted in figure 1. Let $p^*$ be the unique price where the above is zero.

5. The Stationary Distribution

Focus on case where $x^* = x(p^*) > 0$. (If $x(p^*) < 0$ there exist equilibria with no entry or exit. Equilibrium will depend upon the initial stock of firms) In the case where $x(p^*) > 0$ there is a unique stationary equilibirum. The stationary price is $p^*$ and the quantity is $Q^* = D(p^*)$.

What is the stationary distribution of firms?

- Let $\mu_t$ be the distribution of types at time $t$.
- $\gamma$ the distribution of entrants given a unit measure of entry.
- $M\gamma$ distribution of entrant given a mass $M$ of entry.
- $\hat{P}_x$ mapping that first truncates all $\phi < x$ and then runs it through $F$

The equilibrium distribution of firms must satisty the stationarity condition:

$$\mu^* = \hat{P}_x^*\mu^* + M^*\gamma$$

Or, rewriting, it solves:

$$[\hat{P}_x^* - I] \mu^* = M^*\gamma$$
or

\[ \mu^* = \left[ \hat{P}_x^*, I \right]^{-1} M^* \gamma \]

It also must satisfy the product market equilibrium condition

\[ p^e(\mu^*) = p^* \]

where \( p^e(\mu) \) is defined as the price solving

\[ \int_{0}^{1} q(p, \phi) \mu(\phi) d\phi = D(p) \]

In summary, to solve for the equilibrium do the following: (1) Take \( p^* \) as the price solving the free-entry condition. Then find the flow of entrants \( M^* \) so that the following holds:

\[ p^e(M^* \left[ \hat{P}_x^*, I \right]^{-1} \gamma) = p^* \]
6. Example

Suppose two types $\phi_1 = 0$, $\phi_2 = 1$. Suppose the distribution function satisfies

$$
\begin{pmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 - f_{22} \\
  0 & f_{22}
\end{pmatrix}
$$

In this example, type 1 always exits.

$$v_1(p) = \pi_1(p) = -c_f$$

Assume that demand is strong enough so that in equilibrium type 2 stays in and there is positive entry each period.

$$v_2 = \pi_2 + \beta(1 - f_{22})v_1 + \beta f_{22}v_2$$

Or

$$v_2 = \frac{1}{1 - \beta f_{22}} \pi_2 + \frac{\beta(1 - f_{22})}{1 - \beta f_{22}} (-c_f)$$

The equilibrium $p^*$ can be found from figure 3.

For this special case, $\hat{P}_{x^*}$ mapping is

$$
\hat{P}_{x^*} =
\begin{pmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{pmatrix} \begin{pmatrix}
  0 & 0 \\
  0 & 1
\end{pmatrix}
= \begin{pmatrix}
  1 & 1 - f_{22} \\
  0 & f_{22}
\end{pmatrix} \begin{pmatrix}
  0 & 0 \\
  0 & 1
\end{pmatrix} = \begin{pmatrix}
  0 & 1 - f_{22} \\
  0 & f_{22}
\end{pmatrix}
$$

Recall there are two parts of this mapping. The first part is the selection part. Firms with $\phi = \phi_1$ are shut down. This is accounts for the second term above. The second part is the firm goes through the $F$ processing mapping states this period to states next period. This is the first term above.
7. Applications of the Model

A. Firm Dynamics

Fact: Examine a cohort of entering firms and follow survivors. The average size of the survivors increases. The probability of discontinuance decreases.

Model: Look at special case.

<table>
<thead>
<tr>
<th>Period</th>
<th>Measure in state</th>
<th>Prob survive</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$M_\gamma_1$</td>
<td>$M_\gamma_2$</td>
</tr>
<tr>
<td>2</td>
<td>$(1 - f_{22})M_\gamma_2$</td>
<td>$f_{22}M_\gamma_2$</td>
</tr>
</tbody>
</table>

To be consistent with the empirical literature need $f_{22} > \gamma_2$. This also implies average size increases.

In the general model analogous mechanical conditions are needed. The distribution of new entrants can’t be too good compared with the transition function $F$.

B. A Cross Section of Industries

Study effects of changes in $c_e$ and $c_f$ on equilibrium variables:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>$\Delta c_e &gt; 0$</th>
<th>$\Delta c_f &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>price</td>
<td>$p$</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>cutoff</td>
<td>$x$</td>
<td>-</td>
<td>+ (under condition)</td>
</tr>
<tr>
<td>average firm size</td>
<td>$\int_0^1 \frac{q(p,\phi)\mu(\phi)d\phi}{\int_0^1 \mu(\phi)d\phi}$</td>
<td>+ (under condition)</td>
<td></td>
</tr>
<tr>
<td>$k$ concentration</td>
<td>$\int_0^1 \frac{q(p,\phi)\mu(\phi)d\phi}{\int_0^1 q(p,\phi)d\phi}$</td>
<td>where $\phi_k$ defined by $k = \int_0^1 \frac{\mu(\phi)d\phi}{\int_0^1 \mu(\phi)d\phi}$</td>
<td></td>
</tr>
<tr>
<td>profit</td>
<td>$\frac{\int_0^1 \pi(p,\phi)d\phi}{\int_0^1 \mu(\phi)d\phi}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tobin’s q</td>
<td>$\frac{\int_0^1 \pi(p,\phi)d\phi}{\int_0^1 \mu(\phi)d\phi}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Condition referred to above: Condition U.2 The profit function is separable $\pi(p, \phi) = y(\phi)z(p)$.
Figure 3
Special case

\[ \delta_{1} \left[ -\phi \right] + \sum_{2} \nu_{\alpha}(\phi) \]