Notes on the Dominant Firm Model
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1. Description of the Model

Time is discrete, \( t = 0, 1, ... \)

At the beginning of the period a firm begins with a stock of capital \( k \). The firm’s capital stock can be augmented by using existing capital to grow new capital. The cost of expanding one existing unit of capital at rate \( i \) is \( H(i) \). So if a firm starts with \( k \) units of existing capital and grows this capital at rate \( i \), the firm ends up with \( k(1 + i) \) units of capital and this costs \( kH(i) \). Assume that \( H(i) \) is strictly increasing and strictly concave. Assume further the Inada conditions \( \lim_{i \to 0} H'(i) = 0 \) and \( \lim_{i \to \infty} H'(i) = \infty \) hold.

The end-of-period capital stock is used to produce output. One unit of end-of-period capital produces one unit of output. There is no free disposal of output. Hence if a firm begins the period with \( k \) units of capital and invests at rate \( i \), it ends the period with \( q = (1 + i)k \) units of capital and this is the output of the firm.

Capital depreciates at rate \( \delta \) from period to period. So if a firm ends period \( t \) with \( q_k \) units of capital, it begins period \( t + 1 \) with \( k_{t+1} = (1 - \delta) \) units of capital.

The discount factor is \( \delta \).

Demand conditions are the same in each period and are summarized by a demand curve, \( Q^D = D(p) \). Let the inverse demand curve be given by \( p = P(Q) \), where \( Q \) is the industry output.

There are two production sectors in the economy, a dominant firm sector and a fringe sector. Let \( k^d_t \) and \( k^f_t \) denote the capital levels in the two sectors at the beginning of period \( t \). Assume that firms in the fringe sector behave competitively. These firms take the sequence of prices as given when making investment decisions. Assume that the dominant firm behaves strategically. It recognizes the impact of its investment decisions on the price. Assume that the dominant firm and the fringe sector simultaneously select their investment rates.

2. Perfect Competition

Consider a special case of this model where there is no dominant firm so the industry is competitive (i.e. assume \( k^d_t = 0 \) for all \( t \)). This section determines the stationary competitive equilibrium for this economy.

Suppose agents take as given that the price is constant in each period at \( p \). Let \( v \) be the
discounted value of owning one unit of capital at the beginning of the period. This must solve

(1) \[ v = \max_i p(1 + i) - H(i) + \beta(1 - \delta)(1 + i)v \]

The FONC of this problem is

(2) \[ p - H'(i) + \beta(1 - \delta)v = 0 \]

Let \( \tilde{v}(p) \) and \( \tilde{i}(p) \) solve (1) and (2). To solve for these functions, note that (1) implies

\[ v = \frac{p(1 + i) - H(i)}{1 - \beta(1 - \delta)(1 + i)}. \]

Substituting this into (2) yields the following equation that gives investment \( i \) as an implicit function of \( p \),

(3) \[ p + \frac{\beta(1 - \delta)p(1 + i) - \beta(1 - \delta)H(i)}{1 - \beta(1 - \delta)(1 + i)} = H'(i) \]

Note that a solution might not exist if the price is large enough and if \( \beta \) is large. So assume that \( p \) is small enough so that a solution exists. It is straightforward to show that \( \tilde{v}(p) \) and \( \tilde{i}(p) \) are both strictly increasing in \( p \).

Consider the investment rate \( i^o \) such that the size of a fringe firm stays constant:

\[ (1 - \delta)(1 + i^o) = 1 \]

or

\[ i^o = \frac{1}{1 - \delta} - 1 \]

Let \( p^o \) be the price so that \( i^o = \tilde{i}(p^o) \). Letting \( i = i^o \) and substituting this into (3) yields the following equation for the stationary competitive price \( p^o \),

\[ p^o + \frac{\beta p^o - \beta(1 - \delta)H(i^o)}{1 - \beta} = H'(i^o) \]

Solving for \( p^o \) yields

\[ p^o = (1 - \beta)H'(i^o) + \beta(1 - \delta)H(i^o) \]
3. Pure Monopoly

Suppose that in the initial period $k_0^f = 0$ so there is no fringe sector. The problem reduces to one of pure monopoly. This section solves for the stationary equilibrium in this case.

Let $w^M(k)$ be the value function for the monopolist in this situation beginning a period with $k$ units of capital.

$$w^M(k) = \max_i P(k(1+i))k(1+i) - kH(i) + \beta w^M((1-\delta)k(1+i))$$

The FONC is

$$p + P'q - h + \beta(1-\delta)\frac{dw^M}{dk} = 0$$

It is possible to use the envelope theorem to verify that

$$\frac{dw^M}{dk} = (1+i)H' - H$$

Now plug this into the first-order condition and evaluate at the steady state investment level $i^* = \frac{1}{1+A} - 1$. This yields

$$p + P'Q - H'(i^*) + \beta(1-\delta)\left[(1+i^*)H'(i^*) - H(i^*)\right] = 0$$

$$p + P'Q - H'(i^*) + \beta H'(i^*) - \beta(1-\delta)H(i^*) = 0$$

which can be rewritten as

$$p + P'Q = (1-\beta)H'(i^*) + \beta(1-\delta)H(i^*).$$

Notice that the right-hand side is the stationary marginal cost in the competitive case. So this is the familiar marginal revenue equals marginal cost condition for monopoly. The above equation determines the stationary monopoly output level $Q^M$ and the stationary monopoly price $p^M$. Note $p^M > p^*.$

4. Equilibrium in the Dominant-Firm Model

Now consider the general model where there is both a dominant firm and a fringe. I consider Markov-perfect equilibrium. In a Markov-perfect equilibrium, policy functions depend upon the
payoff-relevant state which is the beginning-of-period capital levels for the dominant firm and the fringe \((k^d_t, k^f_t)\).

Let \(\tilde{d}(k^d, k^f)\) and \(\tilde{f}(k^d, k^f)\) be the policy functions of the dominant firm and the fringe. Let \(\tilde{u}(k^d, k^f)\) be the value function of the dominant firm in the Markov-perfect equilibrium. The value to a fringe firm is proportional to the amount of capital a fringe firm holds. Let \(\tilde{v}(k^d, k^f)\) be the value to a fringe firm that begins the period owing one unit of capital. Define the dominant firm’s quantity to be \(\tilde{q}(k^d, k^f) = k^d(1 + \tilde{d}(k^d, k^f))\) and analogously define \(\tilde{q}(k^d, k^f)\). Finally, let \(\tilde{p}(k^d, k^f)\) be the equilibrium price given the state \((k^d, k^f)\).

In a Markov-perfect equilibrium, these functions must satisfy the following conditions:

1. The value function for the dominant firm solves

   \[
   \tilde{v}(k^d, k^f) = \max_{q^d \geq k^d} P\left(q^d + \tilde{q}(k^d, k^f)\right) q^d - k^d H\left(\frac{q^d}{k^d} - 1\right) + \beta \tilde{v}\left((1 - \delta)q^d, (1 - \delta)\tilde{q}(k^d, k^f)\right)
   \]

2. The value \(\tilde{v}(k^d, k^f)\) of one unit of capital to a fringe firm solves

   \[
   \tilde{v}(k^d, k^f) = \max_{i \geq 0} \tilde{p}(k^d, k^f)(1 + i) - H(i) + \beta (1 - \delta)(1 + i)\tilde{v}\left((1 - \delta)q^d, (1 - \delta)\tilde{q}(k^d, k^f)\right)
   \]

3. There is market clearing,

   \[
   \tilde{p}(k^d, k^f) = P(\tilde{q}(k^d, k^f) + \tilde{q}(k^d, k^f))
   \]