Separate Appendix for "Optimal City Hierarchy: A Dynamic Programming Approach to Central Place Theory"

Wen-Tai Hsu* Thomas J. Holmes† Frank Morgan‡

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This appendix presents proofs of several propositions and lemmas in the main paper.

Proof of Proposition 1

Recall the statement: *(i) V^*, the infimum function defined in the sequence problem (SP^V), is a solution to the dynamic programming problem (DP^V). (ii) If a function V is a solution to (DP^V) and if*

$$\lim_{i\to\infty}\sum_{k=1}^{K_i} V\left(x_{i,k}\right)=0, \tag{1}$$

then $V=V^*$.

Proof. For all $\mathbf{x}\in\Pi\left(x_1\right)$, let $\mathbf{y}_k$ be k-th part of the sequence starting from the second round of multi-furcation. That is, $\mathbf{y}_k\in\Pi\left(x_{2,k}\right)$, where $\{x_{2,k}\}_{k=1}^{q}\in\Gamma\left(x_1\right)$. Observe that for any $\mathbf{x}\in\Pi\left(x_1\right)$,

$$\begin{aligned} u\left(\mathbf{x}\right) &= F\left(x_1,\mathbf{x}_2\right)+\sum_{i=2}^{\infty}\sum_{k=1}^{K_i}F\left(x_{i,k},\mathbf{x}_{i+1,k}\right) \\ &= F\left(x_1,\mathbf{x}_2\right)+\sum_{k=1}^{q}\left[F\left(x_{2,k},\mathbf{x}_{3,k}\right)+\sum_{i=3}^{\infty}\sum_{k=1}^{K_{i-1}}F\left(x_{i,k},\mathbf{x}_{i+1,k}\right)\right] \\ &= F\left(x_1,\mathbf{y}\right)+\sum_{k=1}^{q}u\left(\mathbf{y}_k\right). \end{aligned} \tag{2}$$

*Department of Economics, National University of Singapore.

†Department of Economics, University of Minnesota, Federal Reserve Bank of Minneapolis, and the National Bureau of Economic Research.

‡Department of Mathematics and Statistics, Williams College.

Part (i): By definition of V^*, for all $\mathbf{x} \in \Pi(x_1)$,

$$V^*(x_1) \leq u(\mathbf{x}), \tag{3}$$

and for any $\epsilon > 0$,

$$V^*(x_1) + \epsilon \geq u(\mathbf{x}) \quad \text{for some } \mathbf{x} \in \Pi(x_1). \tag{4}$$

We say that V satisfies (DP^V) if and only if for all $\mathbf{x}_2 \in \Gamma(x_1)$,

$$V(x_1) \leq F(x_1, \mathbf{x}_2) + \sum_{k=1}^{q} V(x_{2,k}), \tag{5}$$

and for any $\epsilon > 0$ and some $\mathbf{x}_2 \in \Gamma(x_1)$,

$$V(x_1) + \epsilon \geq F(x_1, \mathbf{x}_2) + \sum_{k=1}^{q} V(x_{2,k}). \tag{6}$$

Let $\mathbf{x}_2$ and ϵ be given. Then, by (4), for each k, there exists a $\mathbf{y}_k \in \Pi(x_{2,k})$ such that $V^*(x_{2,k}) + \epsilon \geq u(\mathbf{y}_k)$. It then follows from (2) and (3) that

$$V^*(x_1) \leq u(\mathbf{x}) = F(x, \mathbf{y}) + \sum_{k=1}^{q} u(\mathbf{y}_k) \leq F(x, \mathbf{x}_2) + \sum_{k=1}^{q} V^*(x_{2,k}) + q\epsilon,$$

Since ϵ is arbitrary, (5) follows for $V = V^*$. Next, from (2), (3), and (4), there exists a $\mathbf{x} \in \Pi(x_1)$ such that

$$V^*(x_1) + \epsilon \geq u(\mathbf{x}) = F(x, \mathbf{y}) + \sum_{k=1}^{q} u(\mathbf{y}_k) \geq F(x, \mathbf{x}_2) + \sum_{k=1}^{q} V^*(x_{2,k}),$$

which is actually (6) for $V = V^*$. That (5) and (6) hold establishes Part (i).

Part (ii): Let V be a solution to (DP^V), then (5) and (6) hold. It suffices to show that this implies (3) and (4) hold. Observe that (5) implies that for all $\mathbf{x} \in \Pi(x_1)$,

$$
\begin{aligned}
V(x_1) &\leq F(x_1, \mathbf{x}_2) + \sum_{k=1}^{q} V(x_{2,k}) \\
&\leq F(x_1, \mathbf{x}_2) + \sum_{k=1}^{q} \left[F(x_{2,k}, \mathbf{x}_{3,k}) + \sum_{k=1}^{q} V(x_{3,k}) \right] \\
&\vdots \\
&\leq F(x_1, \mathbf{x}_2) + \sum_{i=2}^{n} \sum_{k=1}^{K_i} F(x_{i,k}, \mathbf{x}_{i+1,k}) + \sum_{k=1}^{K_{n+1}} V(x_{n+1,k}) \,. \qquad (7)
\end{aligned}
$$

Taking n to infinity in (7), (1) implies (3) holds for V. Next, fix $\epsilon > 0$ and choose $\{\delta_i\}_{i=2}^{\infty}$ such that $\sum_{i=2}^{\infty} \delta_i = \epsilon/2$. Since (6) holds, we can choose $\mathbf{x}_{i+1} \in \Gamma_i(\mathbf{x}_i)$, such that for all $i \geq 2$ and $k = 1, 2, ..., K_i$,

$$
V(x_{i,j}) \geq F(x_{i,j}, \mathbf{x}_{i+1,j}) + \sum_{k_j} V\left(x_{i+1,k_j}\right) - \frac{\delta_i}{K_i}
$$

Then, the sequence so picked is in $\Pi(x_1)$. By (6) and induction using the above inequality, we have

$$
\begin{aligned}
V(x_1) + \frac{\epsilon}{2} &\geq F(x_1, \mathbf{x}_2) + \sum_{k=1}^{q} V(x_{2,k}) \\
&\geq F(x_1, \mathbf{x}_2) + \sum_{k=1}^{K_2} \left[F(x_{2,j}, \mathbf{x}_{3,j}) + \sum_{k_j} V\left(x_{3,k_j}\right) \right] - \delta_2 \\
&\vdots \\
&\geq u_n(\mathbf{x}) + \sum_{k=1}^{K_{n+1}} V(x_{n+1,k}) - \sum_{i=2}^{n} \delta_i.
\end{aligned}
$$

Taking n to infinity and noting (1), we have $V(x_1) + \epsilon \geq u(\mathbf{x})$, which is (4) with V in place of V^*. That (3) and (4) hold for V implies that $V = V^*$. ■

Proof of Proposition 2

First, recall the statement: *(i) If a feasible sequence $x^* \in \Pi(x_1)$ attains the infimum in (SP^V), then it satisfies*

$$
V^*(x_{i,j}^*) = F\left(x_{i,j}^*, \mathbf{x}_{i+1,j}^*\right) + \sum_{k_j} V^*\left(x_{i+1,k_j}^*\right). \qquad (8)
$$

(ii) If a feasible sequence $x^* \in \Pi(x_1)$ satisfies (8), and if it satisfies (1) with V^* in place of V, then it attains the infimum in (SP^V).

Proof. *Part (i).* By (2), since $\mathbf{x}^*$ attains the minimum, for $\mathbf{x} \in \Pi(x_1)$,

$$\begin{aligned} V^*(x_1) &= u(\mathbf{x}^*) = F(x_1, \mathbf{y}^*) + \sum_{k=1}^{q} u(\mathbf{y}_k^*) . \\ &\leq u(\mathbf{x}) = F(x_1, \mathbf{y}) + \sum_{k=1}^{q} u(\mathbf{y}_k) . \end{aligned} \tag{9}$$

In particular, the inequality holds for any $\mathbf{x} \in \Pi(x_1)$ such that $\mathbf{x}_2 = \mathbf{x}_2^*$. Then, for each j, for all $\mathbf{y}_j \in \Pi\left(x_{2,j}^*\right)$,

$$F(x_1, \mathbf{x}_2) + \sum_{k=1}^{q} u(\mathbf{y}_k^*) \leq F(x_1, \mathbf{x}_2) + u(\mathbf{y}_j) + \sum_{k \neq j} u(\mathbf{y}_k^*) .$$

Hence, for each j, $u\left(\mathbf{y}_j^*\right) = V^*\left(x_{2,j}^*\right)$. Substituting these into (9) gives (8) for $i = 1$, $k = 1$. Continuing by induction establishes (8) for i and k.

Part (ii). Let $\mathbf{x}^* \in \Pi(x_1)$ be a sequence that satisfies (8) and let V^* be the minimum cost function defined by (SP^V). By (8) for $i = 1$, we have

$$V^*(x_1) = F(x_1, \mathbf{y}^*) + \sum_{k=1}^{q} V^*(\mathbf{y}_k^*) .$$

Substituting V^* on the right-hand side using (8) and continuing by an induction similar to that shown in (7), we have

$$V^*(x_1) = F(x_1, \mathbf{x}_2^*) + \sum_{i=2}^{n} \sum_{k=1}^{K_i} F\left(x_{i,k}^*, \mathbf{x}_{i+1,k}^*\right) + \sum_{k=1}^{K_{n+1}} V^*\left(x_{n+1,k}^*\right) .$$

Taking n to infinity and using (1) with $V = V^*$, we see that $u(\mathbf{x}^*) = V^*(x_1)$. By definition of V^*, $u(\mathbf{x}) \geq V^*(x_1)$ for all $\mathbf{x} \in \Pi(x_1)$. Hence, this sequence $\mathbf{x}^*$ that satisfies (1) and (8) attains the infimum in (SP^V). ■

Proof of Proposition 4

For this proposition, first recall three assumptions.

A1. Γ is compact-valued.

A2. For each $x_1 \in X$ and for each feasible sequence $\mathbf{x} \in \Pi(x_1)$, $\lim_{i\to\infty} x_{i,k} = 0$.

A3. $F : X \times X^q \to \mathbb{R}_+$ is a continuous function with nonnegative values, and $F(\mathbf{0}) = 0$.

Then, recall the statement: *Suppose that A1, A2, and A3 hold. Also suppose that there exists a continuous and strictly increasing function $M : X \to R_+$ with $M(\mathbf{0}) = 0$ such that if any continuous real-valued function v satisfies $0 \le v(x) \le M(x)$, then $0 \le Tv(x) \le M(x)$. Denote the set of continuous functions v satisfying $0 \le v(x) \le M(x)$ as $V(X)$. Then, for any $v \in V(X)$, the sequence $\{T^n v\}$ converges to V^*.*

Proof. Let v be an arbitrary element in $\mathcal{V}(X)$. For $n = 2$,

$$\begin{aligned} T^2 v(x) &= \min_{\mathbf{x}_2 \in \Gamma(x)} F(x, \mathbf{x}_2) + \sum_{k=1}^{q} Tv(x_{2,k}) \\ &= \min_{\mathbf{x}_2 \in \Gamma(x)} \left\{ F(x, \mathbf{x}_2) + \sum_{j=1}^{q} \min_{\mathbf{x}_{3,j} \in \Gamma\left(x_{2,j}\right)} \left[F(x_{2,j}, \mathbf{x}_{3,j}) + \sum_{k_j} v\left(x_{3,k_j}\right) \right] \right\}. \end{aligned}$$

Recursively,

$$\begin{aligned} & T^n v(x) \\ = \; & \min_{\mathbf{x}_2 \in \Gamma(x)} F(x, \mathbf{x}_2) + \sum_{k=1}^{q} T^{n-1} v(x_{2,k}) \\ = \; & \min_{\mathbf{x}_2 \in \Gamma(x)} \left\{ F(x, \mathbf{x}_2) + \sum_{j=1}^{q} \min_{\substack{\mathbf{x}_{3,j} \in \Gamma(x_{2,j}), \\ x \in X \text{ given.}}} \left[F(x_{2,j}, \mathbf{x}_{3,j}) + \sum_{k_j} T^{n-2} v\left(x_{3,k_j}\right) \right] \right\} \\ & \vdots \\ = \; & \min_{\mathbf{x}_2 \in \Gamma(x)} \left\{ F(x, \mathbf{x}_2) + \sum_{j=1}^{q} \min_{\substack{\mathbf{x}_{3,j} \in \Gamma(x_{2,j}), \\ x \in X \text{ given.}}} \left[F(x_{2,j}, \mathbf{x}_{3,j}) + \sum_{k_j} \min_{\mathbf{x}_{4,kj} \in \Gamma\left(x_{3,k_j}\right)} \{\cdots\} \right] \right\}. \quad (10) \end{aligned}$$

Thus, $T^n v(x)$ can be understood as a *hierarchy* of minimization problems in which each $\min$ operator has q *immediate* $\min$ operators embedded in it. Hence, $T^n v(x)$ has $1 + q + ... + q^{n-1} = (q^n - 1)/(q - 1)$ $\min$ operators in total (each $\{\cdots\}$ contains $(q^n - q^3)/(q - 1)$ $\min$ operators). Although this problem is different from either (SP^V) or (DP^V), the optimal sequence $\{\mathbf{x}_i\}_{i=1}^{n+1}$ solving this problem and giving the value of $T^n v(x)$ obviously satisfies the Γ constraint recursively. Hence, taking n to infinity, $\mathbf{x} \equiv \{\mathbf{x}_i\}_{i=1}^{\infty} \in \Pi(x)$. It is easily verified that by induction,

$$T^n v(x) \leq \min_{\{\mathbf{x}_i\}_{i=1}^{n+1}} \left[F(x, \mathbf{x}_2) + \sum_{i=2}^{n-1} \sum_{k=1}^{K_i} F(x_{i,k}, \mathbf{x}_{i+1,k}) + \sum_{k=1}^{K_n} Tv(x_{n,k}) \right]. \tag{11}$$

Let $\{\tilde{\mathbf{x}}_i\}_{i=1}^{n+1}$ be the sequence that solves $T^n v(x)$ according to (10) and $\{\hat{\mathbf{x}}_i\}_{i=1}^{n+2}$ the sequence that solves $T^{n+1}v$. Using (11) and noting that $0 \leq v \leq M$ and $0 \leq Tv \leq M$, we have

$$\begin{aligned}
\left(T^{n+1}v - T^n v\right)(x) &\leq F(x, \tilde{\mathbf{x}}_2) + \sum_{i=2}^{n} \sum_{k=1}^{K_i} F(\tilde{x}_{i,k}, \tilde{\mathbf{x}}_{i+1,k}) + \sum_{k=1}^{K_{n+1}} Tv(\tilde{x}_{n+1,k}) \\
&\quad - \left\{ F(x, \tilde{\mathbf{x}}_2) + \sum_{i=2}^{n-1} \sum_{k=1}^{K_i} F(\tilde{x}_{i,k}, \tilde{\mathbf{x}}_{i+1,k}) + \sum_{k=1}^{K_n} Tv(\tilde{x}_{n,k}) \right\} \\
&= \sum_{k=1}^{K_n} F(\tilde{x}_{n,k}, \tilde{\mathbf{x}}_{n+1,k}) + \sum_{k=1}^{K_{n+1}} Tv(\tilde{x}_{n+1,k}) - \sum_{k=1}^{K_n} Tv(\tilde{x}_{n,k}) \\
&\leq \sum_{k=1}^{K_n} F(\tilde{x}_{n,k}, \tilde{\mathbf{x}}_{n+1,k}) + \sum_{k=1}^{K_{n+1}} M(\tilde{x}_{n+1,k})
\end{aligned} \tag{12}$$

Similarly, noting that $F \geq 0$, we have

$$\begin{aligned}
\left(T^n v - T^{n+1}v\right)(x) &\leq F(x, \hat{\mathbf{x}}_2) + \sum_{i=2}^{n-1} \sum_{k=1}^{K_i} F(\hat{x}_{i,k}, \hat{\mathbf{x}}_{i+1,k}) + \sum_{k=1}^{K_n} Tv(\hat{x}_{n,k}) \\
&\quad - \left[F(x, \hat{\mathbf{x}}_2) + \sum_{i=2}^{n} \sum_{k=1}^{K_i} F(\hat{x}_{i,k}, \hat{\mathbf{x}}_{i+1,k}) + \sum_{k=1}^{K_{n+1}} Tv(\hat{x}_{n+1,k}) \right] \\
&= \sum_{k=1}^{K_n} Tv(\hat{x}_{n,k}) - \sum_{k=1}^{K_n} F(\hat{x}_{n,k}, \hat{\mathbf{x}}_{n+1,k}) - \sum_{k=1}^{K_{n+1}} Tv(\hat{x}_{n+1,k}) \\
&\leq \sum_{k=1}^{K_n} M(\hat{x}_{n,k}).
\end{aligned} \tag{13}$$

By (12) and (13), we have

$$\left| \left(T^{n+1}v - T^n v\right)(x) \right| \leq \sum_{k=1}^{K_n} F(\tilde{x}_{n,k}, \tilde{\mathbf{x}}_{n+1,k}) + \sum_{k=1}^{K_{n+1}} M(\tilde{x}_{n+1,k}) + \sum_{k=1}^{K_n} M(\hat{x}_{n,k}),$$

which goes to 0 as n goes to infinity because $M(\mathbf{0}) = 0$, $F(\mathbf{0}) = 0$, and $\lim_{i \to \infty} x_{i,k} = 0$ for all k. This implies that $\{T^n v\}$ is Cauchy, and hence it converges. Now, note that (11) can be written as

$$T^n v(x) \leq \min_{\mathbf{x}\in\Pi(x)} \left[u_{n-1}(\mathbf{x}) + \sum_{k=1}^{K_n} Tv(x_{n,k}) \right].$$

That $0 \leq Tv \leq M$, $\lim_{i\to\infty} x_{i,k} = 0$ for all k, and $M(\mathbf{0}) = 0$ implies that (1) holds for Tv in place of V, and hence

$$\lim_{n\to\infty} T^n v(x) \leq \min_{\mathbf{x}\in\Pi(x)} u(\mathbf{x}) = V^*(x).$$

Now, denote the optimal sequence that solves this $T^n v(x)$ problem as $\{\tilde{\mathbf{x}}_i\}_{i=1}^{n+1}$. The existence of such a sequence is guaranteed by Proposition 3 because each of the minimization problems in the expanded problem implied by $T^n v(x)$ has a minimizer. Obviously, $\tilde{\mathbf{x}} \equiv \{\tilde{\mathbf{x}}_i\}_{i=1}^{\infty} \in \Pi(x_1)$. Taking n to infinity and by definition of V^*,

$$\begin{aligned}\lim_{n\to\infty} T^n v(x) &= \lim_{n\to\infty} \left\{ u_{n-1}(\tilde{\mathbf{x}}) + \sum_{k=1}^{K_n} Tv(\tilde{x}_{n,k}) \right\} \\ &= u(\tilde{\mathbf{x}}) \geq V^*(x).\end{aligned}$$

Therefore, $\lim_{n\to\infty} T^n v(x) = V^*(x)$. ■

Proof of Lemma 3

The proof of part (i) of this lemma is included in the paper, but we duplicate that part here, as it is needed for the proof of part (ii). First, recall that the optimal magnitude of z' when z' is interior is determined by

$$s^1(\ell,\alpha) = \frac{t\ell^2}{2}\alpha(1-\alpha) = \phi(z'). \tag{14}$$

Recall the statement: *For any two positive real numbers ℓ_1 and z_1, let $X = [0,\ell_1] \times [0,z_1]$, and let $D(X)$ denote the set of all real-valued continuous functions $d : X \to R_+$ such that*

$$0 \leq d(\ell,z) \leq A(\ell,z). \tag{15}$$

Let $T : D(X) \to D(X)$ be given by, for each $d \in D(X)$,

$$Td(\ell,z) \equiv \inf_{\alpha\in(0,1),z'\in[0,z]} A(\ell,z) - A(\ell,z') + d(\alpha\ell,z') + d((1-\alpha)\ell,z'). \tag{16}$$

Then,

(i) $0 \leq Td(\ell,z) \leq A(\ell,z)$.

(ii) *Given* $(\ell, z) \in X$, *there exists an* $\epsilon > 0$ *such that every optimal choice of* $\alpha \in [\epsilon, 1-\epsilon]$ *for all* d. *That is, the optimal choice of* α *cannot be arbitrarily close to* 0 *or* 1.

Proof. That $Td \geq 0$ is trivial. Utilize (15) to write

$$\begin{aligned} Td(\ell, z) &\leq \inf_{\alpha\in(0,1), z'\in[0,z]} A(\ell, z) - A(\ell, z') + A(\alpha\ell, z') + A((1-\alpha)\ell, z') \\ &= \inf_{\alpha\in(0,1), z'\in[0,z]} A(\ell, z) + \Phi(z') - \frac{z' t\ell^2}{2}\alpha(1-\alpha) \end{aligned} \tag{17}$$

By (14),

$$\Phi(z') - \frac{z' t\ell^2}{2}\alpha(1-\alpha) = \int_0^{z'} \left[\phi(y) - \frac{t\ell^2}{2}\alpha(1-\alpha)\right] dy < 0.$$

Hence, Td satisfies (15). To prove part (ii), we only need to show the lower bound part, as the upper bound part follows by symmetry. Suppose the contrary is true. That is, suppose for each $\epsilon > 0$, there exists a d such that the optimal $\alpha < \epsilon$. Then, since ϵ can be arbitrarily small, the optimal α for some d is also arbitrarily small, and hence by (14), the optimal z' is also arbitrarily small. By (17), such a choice of α and z' implies that the value of the objective is arbitrarily close to $A(\ell, z)$, which is the value when no z'-city is built. Note that the net savings from having a z'-city are given by

$$S(\ell, \alpha, z') \equiv \int_0^{z'} \left[\frac{t\ell^2}{2}\alpha(1-\alpha) - \phi(y)\right] dy. \tag{18}$$

By setting $\alpha = 1/2$ in (18), the net savings is $\int_0^{z'} \left[\frac{t\ell^2}{8} - \phi(y)\right] dy$. By (14) and the constraint $z' \leq z$, $z' = \min\left\{\phi^{-1}\left(t\ell^2/8\right), z\right\}$. Hence, the net savings is a positive constant independent of d. Thus, an arbitrarily small α cannot be optimal, and there exists an $\epsilon > 0$ such that optimal choice of $\alpha \in [\epsilon, 1-\epsilon]$ for all d. ■

Part of the Proof of Proposition 5

What is left to prove for this proposition is that $\alpha = 1/2$ is the unique solution to

$$\max_{\alpha\in(0,1)} \frac{[\alpha(1-\alpha)]^{\frac{b+1}{b}}}{1 - \alpha^{\frac{2(b+1)}{b}} - (1-\alpha)^{\frac{2(b+1)}{b}}}.$$

Let the objective function be denoted as $R(\alpha)$. Observe that for all $\alpha \neq 1/2$,

$$R(\alpha) - R\left(\frac{1}{2}\right) < 0$$

if and only if

$$K(\alpha) \equiv 2^{2+\frac{2}{b}} [\alpha(1-\alpha)]^{1+\frac{1}{b}} + \left[\alpha^{1+\frac{1}{b}} - (1-\alpha)^{1+\frac{1}{b}}\right]^2 < 1.$$

It suffices to show that for all $\alpha \in (0, 1/2)$, $K(\alpha) < 1$. Recall that $b > 0$ and observe that

$$K'(\alpha) = \frac{b+1}{b} \left\{ 2^{2+\frac{2}{b}} [\alpha(1-\alpha)]^{\frac{1}{b}} (1-2\alpha) + 2\left[\alpha^{1+\frac{1}{b}} - (1-\alpha)^{1+\frac{1}{b}}\right] \left[\alpha^{\frac{1}{b}} + (1-\alpha)^{\frac{1}{b}}\right] \right\}.$$

Since $K(0) = 1$ and $K'(0) = -2(b+1)/b$, $K(\alpha) < 1$ for at least a right neighborhood of 0. Also, since $K(1/2) = 1$, if $K(\alpha) > 1$ for some $\alpha \in (0, 1/2)$, then there are at least two solutions to $K'(\alpha) = 0$ for $\alpha \in (0, 1/2)$. Thus, if $K'(\alpha) = 0$ entails a unique solution for $\alpha \in (0, 1/2)$, then the result follows from the contradiction.

That $K'(\alpha) = 0$ is equivalent to

$$2^{2+\frac{2}{b}} [\alpha(1-\alpha)]^{\frac{1}{b}} (1-2\alpha) = 2\left[(1-\alpha)^{1+\frac{1}{b}} - \alpha^{1+\frac{1}{b}}\right] \left[\alpha^{\frac{1}{b}} + (1-\alpha)^{\frac{1}{b}}\right],$$

which is equivalent to

$$\frac{(1-\alpha)^{\frac{b+2}{b}} - \alpha^{\frac{b+2}{b}}}{(1-2\alpha)[\alpha(1-\alpha)]^{\frac{1}{b}}} = 2^{\frac{b+2}{b}} - 1.$$

Denote the left-hand side of the above equation as $D(\alpha)$. Observe that $\lim_{\alpha \to 0} D(\alpha) = \infty$ and $\lim_{\alpha \to 1/2} D(\alpha) = (b+2)/b < 2^{\frac{b+2}{b}} - 1$. Thus, if $D'(\alpha) < 0$ for $\alpha \in (0, 1/2)$, then there is one and only one solution to $K'(\alpha) = 0$. Now,

$$D'(\alpha) = -\frac{[(1+b)(1-\alpha) - \alpha]\alpha^{1+\frac{2}{b}} - [(1+b)\alpha - (1-\alpha)](1-\alpha)^{1+\frac{2}{b}}}{(1-2\alpha)^2 b[\alpha(1-\alpha)]^{1+\frac{1}{b}}}$$

So, $D'(\alpha) < 0$ if and only if

$$G(\alpha) \equiv [(1+b)(1-\alpha) - \alpha]\alpha^{1+\frac{2}{b}} - [(1+b)\alpha - (1-\alpha)](1-\alpha)^{1+\frac{2}{b}} > 0.$$

Observe that $G(1/2) = 0$ and for $\alpha \in (0, 1/2)$,

$$G'(\alpha) = -\frac{(1+b)(2+b)}{b}(1-2\alpha)\left[(1-\alpha)^{2/b} - \alpha^{2/b}\right] < 0.$$

Thus, $G(\alpha) > 0$ for all $\alpha \in (0, 1/2)$.

Proof of Proposition 6

Recall that the optimal distance between two neighboring z_1-cities is determined by

$$\ell_1^* = \arg \min_{\ell_1 \in \left(0, \bar{\ell}(z_1)\right)} C^*(\ell_1, z_1). \tag{19}$$

Recall the statement of this proposition: *Suppose that the central place property holds in an optimal solution. Then, there exists an $\ell_1^* \in \left(0, \bar{\ell}(z_1)\right)$ that solves (19). That is, an optimal distance between two neighboring z_1-cities exists. Moreover, $\ell_1^* \in [\bar{\ell}(z_1)/2, \bar{\ell}(z_1))$.*

Proof. Note that we can write $C^*\left(\bar{\ell}(z_1), z_1\right) = C^*\left(\bar{\ell}(z_1)/2, z_1\right)$ because the allocation under $\left(\bar{\ell}(z_1)/2, z_1\right)$ and $\left(\bar{\ell}(z_1), z_1\right)$ are identical, which can also be easily verified by examining $C^*\left(\bar{\ell}(z_1), z_1\right)$ and $C^*\left(\bar{\ell}(z_1)/2, z_1\right)$ using the definition in (SP). Also note that for any given z_1, $\lim_{\ell_1 \to 0} C^*(\ell_1, z_1) = \infty$. To see this, we first note that the city hierarchy on the interval of ℓ_1 approaches nil because $z_2^* \to 0$ as $\ell_1 \to 0$. Then, $C^*(\ell_1, z_1) \to A(\ell_1, z_1) \to \infty$, as $\ell_1 \to 0$. The fact that $\lim_{\ell_1 \to 0} C^*(\ell_1, z_1) = \infty$ rules out the neighborhood of $(0, \delta)$ as possible minimizers for some $\delta > 0$. To search for optimal ℓ_1, we can include $\bar{\ell}(z_1)$ as candidate because if it attains the minimum, then we set $\ell_1^* = \bar{\ell}(z_1)/2$. Hence, (19) can be viewed as a minimization of a continuous function C^* on a compact set $\left[\delta, \bar{\ell}(z_1)\right]$. Hence, a minimizer exists.

Next, we show that $\ell_1^* \geq \bar{\ell}(z_1)/2$. Suppose that $\ell_1^* < \bar{\ell}(z_1)/2$. Then, by definition of ℓ_1^*, $C(\ell_1^*, z_1) \leq C\left(\bar{\ell}(z_1)/2, z_1\right)$. Consider $\ell_1 = 2\ell_1^*$. Since ℓ_1^* is the cost-minimizing distance between two neighboring z_1^*-cities, at $\ell_1 = 2\ell_1^*$, it is optimal to set $z_2^* = z_1$ so that the effective distance between two neighboring z_1-cities is exactly ℓ_1^* and the per capita cost is minimized. However, we also have $\ell_1 = 2\ell_1^* < \bar{\ell}(z_1)$, which means that at $\ell_1 = 2\ell_1^*$, it is optimal to set $z_2^* < z_1$. The result follows from the contradiction. ■

Proof of Proposition 7

Recall that for $i \geq 2$, z_i^e and z_i^* are given by

$$\phi(z_i^e) = \frac{t(\ell_i^e)^2}{2}, \tag{20}$$

$$\phi(z_i^*) = \frac{t(\ell_i^*)^2}{2}. \tag{21}$$

Recall the statement of this proposition: *With the central place property, the following holds.*

1. *If $\ell_1^e = \ell_1^*$, then entry for each good is identical in both the equilibrium and the optimal*

solution.

2. *If $\ell_1^e > \ell_1^*$, then*

    (a) $z_{i+1}^* < z_{i+1}^e \leq z_i^*$ *for all* $i \geq 1$. *The* $[0, z_1]$ *continuum can be partitioned into sets of the form* $(z_{i+1}^e, z_i^*]$ *and* $(z_{i+1}^*, z_{i+1}^e]$, *for all* $i \in N$.

    (b) *For all* $y \in (z_{i+1}^*, z_{i+1}^e]$, *equilibrium entry is weakly more than the optimal one.*

    (c) *For all* $y \in (z_{i+1}^e, z_i^*]$, *equilibrium entry is less than the optimal one.*

3. *If $\ell_1^e < \ell_1^*$, then the result in (b) holds with the superscripts of * and e exchanged.*

Proof. The first point is trivial by comparing (20) and (21). The proof for Point 3 is the same as that for Point 2 with the superscripts of * and e exchanged. Suppose that $\ell_1^e > \ell_1^*$. For (a) of Point 2, noting (20), (21), and that $\ell_1^e < 2\bar{\ell}(z_1) \leq 2\ell_1^*$ (Corollary 3), we have

$$\phi^{-1}\left(\frac{t\left(\ell_1^*\right)^2}{2^{2i+1}}\right) < \phi^{-1}\left(\frac{t\left(\ell_1^e\right)^2}{2^{2i+1}}\right) \leq \phi^{-1}\left(\frac{t\left(\ell_1^*\right)^2}{2^{2i-1}}\right),$$

that is, $z_{i+1}^* < z_{i+1}^e \leq z_i^*$. For (b) in Point 2, let the distance between any two neighboring locations of y be denoted as ℓ_y. For all $y \in (z_{i+1}^*, z_{i+1}^e]$, the following inequality holds.

$$\ell_y^e = \ell_{i+1}^e = \frac{\ell_1^e}{2^i} \leq \frac{\ell_1^*}{2^{i-1}} = \ell_i^* = \ell_y^*,$$

which implies that the equilibrium entry is more than optimal if $\ell_1^e < 2\ell_1^*$, and is equal to the optimal entry if $\ell_1^e = 2\ell_1^*$. Similarly, for (c) in Point 2, consider $y \in (z_{i+1}^e, z_i^*]$.

$$\ell_y^e = \ell_i^e = \frac{\ell_1^e}{2^{i-1}} > \frac{\ell_1^*}{2^{i-1}} = \ell_i^* = \ell_y^*,$$

which implies that the equilibrium entry is less than optimal. ■