The Fundamental Theorems of Welfare Economics

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January 31, 2002

1 Preliminary Concepts and Discussion

The so-called "fundamental theorems of welfare economics" state that, under certain conditions, every competitive equilibrium is a Pareto optimum, and conversely, every Pareto optimum is a competitive equilibrium. The proposition was first set forth by Pareto in 1894 [12], and further refined in a series of subsequent writings (cf. [13], [14]—see also Barone [2]); his method of proof used differentiability in an essential way, while the modern approach—pioneered by Arrow [1] in 1951—invoked Minkowski's separation theorem for convex sets, which was being developed at the same time that Pareto wrote (cf. [11]). The proposition was essentially accepted and developed in the 1930s and 40s by Lange [8], [9] and Lerner [10]; however, it did not receive a rigorous proof until 1951, when one was provided in a pioneering article by Arrow [1]. This was followed by a further refinement by Debreu [4]. The two classic sources now relied upon are those of Koopmans [7] and Debreu [5].

The problem will be formulated in a somewhat simplified manner, falling short of the generality provided by Debreu [5], but going beyond the pure-exchange framework employed by Arrow [1]. The general framework and notation are those introduced in Chipman and Moore [3].

Let there be n (final) commodities, consumed by m individuals. We shall assume that production possibilities are completely described by a *production-possibility set* $Y \subseteq E^n_+$ (where E^n_+ denotes the nonnegative orthant of n-dimensional Euclidean space E^n), assumed to be *compact* and *convex* (see Figure 6 below).

Each individual, i, is assumed to have a *preference relation* R_i (defined on the set E^n of commodity bundles), assumed to be *reflexive* ($x_i R_i x_i$), *transitive* ($x_i R_i x_i'$ and $x_i' R_i x_i''$ imply $x_i R_i x_i''$) and *total* (for all x_i, x_i', either $x_i R_i x_i'$ or $x_i' R_i x_i$), as well as *continuous*; the latter condition means that the sets

$$R_i x_i = \{x_i' \mid x_i' R_i x_i\},\ x_i R_i = \{x_i' \mid x_i R_i x_i'\}$$

are closed. Here, x_i is an $1 \times n$ vector in E^n_+, with components x_{ij}, i.e., $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$. It is further assumed that each commodity is capable of being transferred (whether by exchange or otherwise) from one individual to another without cost (i.e., without using up any scarce resources). This is an important assumption, since it implies that leisure is not one of the n commodities, since it is manifestly impossible to transfer leisure directly from one person to another.[1]

We denote by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix}$$

the $m \times n$ *allocation matrix*, or *allocation*. It is an element of E_+^{mn}. The column sums of X will be denoted

$$\sigma(X) = \sum_{i=1}^{m} x_i = (1, 1, \dots, 1)X.$$

If $y = \sigma(X)$, we say that X is an allocation of y, and if $\sigma(X) \in Y$ we say that X is an allocation of Y. We define the *set of allocations* of Y by

$$A(Y) = \{X \in E_+^{mn} \mid \sigma(X) \in Y\}.$$

In the special case of pure exchange, in which Y consists of a single aggregate bundle $\omega = (\omega_1, \omega_2, \dots, \omega_n)$, $A(Y)$ corresponds (in the case $n = 2$) to the well-known "Edgeworth box".

We define the relation P_i of *strict preference* by $x_i P_i x_i'$ if and only if $x_i R_i x_i'$ and not $x_i' R_i x_i$. Given an m-tuple

$$R = (R_1, R_2, \dots, R_m)$$

of preference relations, we say that the allocation X is (weakly) *Pareto superior* to the allocation X' if each row x_i of X is preferred or indifferent to the corresponding row x_i' of X'; in symbols,

$$XRX' \quad \text{if and only if} \quad x_i R_i x_i' \quad \text{for } i = 1, 2, \dots, m.$$

Note that this definition involves an implicit assumption of *individualism*, i.e., each individual is concerned only with the bundle of commodities he or she consumes.

We say that X is *strictly Pareto superior* to X', written XPX', if X is Pareto superior to X' but X' is not Pareto superior to X, or equivalently,

$$\begin{array}{lll} XPX' & \text{if and only if} & x_i R_i x_i' \ \text{ for all } i = 1, 2, \dots, m, \\ & \text{and} & x_k P_k x_k' \ \text{ for at least one } k. \end{array}$$

[1] Someone who desires more leisure may cut his or her hours of work, and this person's spouse may cut down on leisure by working more hours; if both are paid at the same rate (but not otherwise), these activities will result in an *indirect* transfer of a certain number of hours of leisure from one to the other. Another illustration would be that of two men, Smith and Jones, who previously mowed their own lawns, until Smith decided to hire Jones to mow his lawn; then leisure is indirectly transferred from Jones to Smith, but the amount of leisure lost by Jones is not equal to the amount gained by Smith unless they both earn the same wage in their regular jobs and in lawn-mowing.

We define an *economy* as an ordered pair (Y, R) where Y is the set of aggregate production- (and therefore consumption-) possibilities and R is the m-tuple of individual preference orderings R_i, each defined on the commodity space E^n_+.

An allocation X of Y will be called a *Pareto-optimal allocation of* Y *relative to* R, or a *Pareto-optimal allocation for the economy* (Y, R), if there is no other allocation X' of Y which is strictly Pareto superior to X. The *set of Pareto-optimal allocations for* (Y, R) is denoted

$$O(Y, R) = \{X \in A(Y) \mid X'PX \;\text{ for no }\; X' \in A(Y)\}.$$

The statement "X is a Pareto-optimal allocation for the economy (Y, R)" is therefore written: "$X \in O(Y, R)$."

We now define what we mean by *competitive equilibrium.* Let p be an n-tuple of n prices, not all zero. Let the *budget set* for an individual i be defined by

$$B(p, I_i) = \{x_i \in E^n_+ \mid p \cdot x_i \leqq I_i\}$$

where I_i represents the income of the individual i, and $p \cdot x_i$ denotes the inner product $p \cdot x = \sum_{j=1}^{n} p_j x_{ij}$. From this definition, it is clear that $B(p, I_i)$ is *convex*. The pair (X, p) will be called a *competitive equilibrium for the economy* (Y, R) if the following two conditions are satisfied:

(i) $x_i R_i x'_i$ for all $x'_i \in B(p, p \cdot x_i)$, $i = 1, 2, \ldots, m$.

(ii) $p \cdot y \geqq p \cdot y'$ for all $y' \in Y$, where $y = \sum_{i=1}^{m} x_i = \sigma(X)$.

It is convenient to provide a notation for the set of possible competitive equilibria (X, p) for an economy (Y, R). Accordingly, we define

$$C(Y, R) = \{(X, p) \mid (X, p) \text{ is a competitive equilibrium for } (Y, R)\}.$$

Two comments may be made concerning the above definition. (1) Condition (i) assumes that the individual is free to choose among all bundles in $B(p, p \cdot x_i)$, and that any such choice is feasible. Actually, not all such bundles are compatible with survival. It has become customary, following Koopmans and Debreu, to define preferences for individual i only over a "consumption set" or "survival set" $S_i \subseteq E^n_+$. However, while it is of course reasonable to require that an *equilibrium bundle* belong to such a set, there is no reason why a consumer, insufficiently informed (or even adequately informed) about nutrition, should not choose a bundle outside of his or her survival set, even if a bundle is available in the intersection $B(p, I_i) \cap S_i \neq \varnothing$. Examples of such behavior can be documented; an illustration is given in Figure 1.[2] Thus, we can require condition (i) to be replaced by

(i*) $x_i \in S_i$, and $x_i R_i x'_i$ for all $x_i \in B(p, p \cdot x_i) \cap S_i$, $i = 1, 2, \ldots, m$,

[2] The Aztecs hit upon a diet of corn and beans, providing for their minimum protein requirements. It is recounted, however, that some neighboring communities consumed corn but not beans, and others beans but not corn, and perished from malnutrition. In very recent times it has been reported that many Central Americans have died of starvation after refusing to eat unfamiliar but nutritious kinds of corn and sorghum.

and likewise we can alter the definition of the attainable set of allocations to

$$A^*(Y) = \{X \in E_+^{mn} \mid \sigma(X) \in Y, x_i \in S_i \text{ for } i = 1, 2, \ldots, m\}$$

or, equivalently,

$$A^*(Y) = \left\{ X \in E_+^{mn} \;\middle|\; \sigma(X) \in Y \cap \sum_{i=1}^{m} S_i \right\}.$$

Likewise, we can alter the definition of the set of Pareto optima to

$$O^*(Y, R) = \left\{ X \in A^*(Y) \;\middle|\; X'PX \text{ for no } X' \in A^*(Y) \right\}.$$

In what follows it can be verified that all the results go through if $A(Y)$ and $O(Y, R)$ are replaced by $A^*(Y)$ and $O^*(Y, R)$ and at the same time $C(Y, R)$ is replaced by

$$C^*(Y, R) = \left\{ (X, p) \;\middle|\; x_i \text{ and } y = \sum_{i=1}^{m} x_i \text{ satisfy (i}^*\text{) and (ii)} \right\}.$$

To maintain simplicity, however, we shall deal instead with the case in which we formally take $S_i = E_+^n$.

(2) The second comment that may be made concerning the above definition of competitive equilibrium is that condition (ii):

(a) involves a hidden assumption concerning individuals' preferences, and

(b) leaves undescribed and only implicit an important aspect of competitive equilibrium.

One way to think of condition (ii) is to assume that each commodity, j, is produced by means of a production function with primary factors of production as arguments, and that each such factor is in fixed total supply and perfectly mobile among industries. This would mean that there is a fixed demand for leisure, and that labor is indifferent as between alternative occupations; this is the hidden assumption (a). To be sure, it is a limiting assumption, which should be relaxed in a more realistic analysis. However, inclusion of this consideration at this stage would complicate the analysis and possibly, therefore, detract from a clear understanding of the logic of the fundamental theorems.

As for (b) we may remark that if the maximization of the value of output $p \cdot y$ subject to $y \in Y$ is carried out, Lagrangean multipliers will appear which correspond to the rentals of factors of production; given the pattern of ownership of resources, this will determine individual's incomes which will be set equal to their expenditures. Conversely, if each firm minimizes costs in the face of given market prices of products and factor services, the value of output will be maximized at those prices. Thus, a more complete description could be given than is indicated by condition (ii); but it would be supplementary to the theorems that follow and not affect their validity, so long as we are prepared to assume that labor and other factor services do not enter individual's preferences.

2 Proof of the fundamental theorems

A preference relation R_i will be called *locally non-satiating* if, for all $x_i \in E^n_+$ and any neighborhood $N(x_i)$ of x_i (with respect to the relative topology of E^n_+),[3] there exists a bundle $x'_i \in N(x_i)$ such that $x'_i P_i x_i$.

A bundle x_i is said to *maximize the preference relation* R_i on a set $C \subseteq E^n_+$ if $x_i R_i x'_i$ for all $x'_i \in C$.

Lemma 1. If R_i is locally non-satiating and x^0_i maximizes R_i on $B(p^0, I^0_i)$, then $p^0 \cdot x^0_i = I^0_i$.

Proof. The set

$$\text{int } B(p^0, I^0_i) = \left\{ x \in E^n_+ \;\middle|\; p^0 \cdot x < I^0_i \right\}$$

is open in the relative topology of E^n_+. Therefore, by local non-satiation, for each $x_i \in \text{int } B(p^0, I^0_i)$ there is a neighborhood $N(x_i) \subseteq \text{int } B(p^0, I^0_i)$ such that $x'_i P_i x_i$ for some $x'_i \in N(x_i)$. Consequently R_i has no maximum in int $B(p^0, I^0_i)$. It follows that $p^0 \cdot x^0_i = I^0_i$. ■

The following two lemmas provide conditions under which, respectively, preference maximization implies cost minimization and cost minimization implies preference maximization. Cf. Debreu [5, pp. 68–71].

Lemma 2.1. Let R_i be locally non-satiating. If x^0_i maximizes R_i on $B(p^0, I^0_i)$ then x^0_i minimizes $p^0 \cdot x_i$ on $R_i x^0_i$, i.e., $p^0 \cdot x^0_i \leqq p^0 \cdot x_i$ for all $x_i \in R_i x^0_i$.

Proof. If the conclusion does not hold then there exists an $x^1_i \in R_i x^0_i$ which is cheaper than x^0_i, i.e., such that $p^0 \cdot x^1_i < p^0 \cdot x^0_i$ (see Figure 2 where, in violation of local non-satiation, there is a "thick" indifference curve shown by the shaded area including its lower boundary). By local non-satiation, there exists an x_i sufficiently close to x^1_i such that $p^0 \cdot x_i < p^0 \cdot x^0_i$ and $x_i P_i x^1_i R_i x^0_i$, hence $x_i P_i x^0_i$ by transitivity. This contradicts the assumption that x^0_i maximizes R_i on $B(p^0, I^0_i)$. ■

Lemma 2.2. Let I^0_i satisfy

$$I^0_i > \inf_{x_i \in E^n_+} p^0 \cdot x_i, \tag{1}$$

and assume that $B(p^0, I^0_i) \neq \varnothing$. [If all components of p^0 are nonnegative, (1) becomes equivalent to the condition $I^0_i > 0$.] Then if x^0_i minimizes $p^0 \cdot x_i$ on $R_i x^0_i$, x^0_i maximizes R_i on $B(p^0, I^0_i)$.

Proof. From (1) and the non-emptiness of $B(p^0, I^0_i)$ there exists an $x^1_i \in$ int $B(p^0, I^0_i)$. Now for any $x_i \in \text{int } B(p^0, I^0_i)$ we have $x^0_i P_i x_i$; for if not, then since $x_i R_i x^0_i$ and $p^0 \cdot x_i < I^0_i$, x^0_i would not minimize $p^0 \cdot x_i$ on $R_i x^0_i$. Consider now any $x^2_i \in B(p^0, I^0_i)$ satisfying $p^0 \cdot x^2_i = I^0_i$ (see Figure 3). Since $B(p^0, I^0_i)$ is convex, for any t in the interval $0 \leqq t < 1$ we have $\bar{x}_i = (1-t)x^1_i + t x^2_i \in B(p^0, I^0_i)$; and clearly, $p^0 \cdot \bar{x}_i < I^0_i$, hence by the above argument $x^0_i P_i \bar{x}_i$. Since t can be made arbitrarily

[3] Some technical points: A "topology" is a collection of "open sets"; in the Euclidean topology, the open sets in E^n are unions of "balls" of radius r, $B(x, r) = \{x' \mid \sum_{i=1}^n (x'_i - x_i)^2 < r\}$. The open sets in the *relative topology* of E^n_+ are the intersections of these sets in E^n with the nonnegative orthant E^n_+. Thus they include points on the boundary of E^n_+.

close to 1, this shows that x_i^2 belongs to the closure of the set

$$x_i^0 P_i = \left\{ x_i \in E_+^n \;\middle|\; x_i^0 P_i x_i \right\}.$$

But since R_i is continuous (in particular, the set $x_i^0 R_i$ is closed) we have $\overline{x_i^0 P_i} \subseteq x_i^0 R_i$ hence x_i^2 is a member of the set $x_i^0 R_i$, or in other words $x_i^0 R_i x_i^2$. Since x_i^2 was arbitrary, it follows that $x_i^0 R_i x_i$ for all $x_i \in B(p^0, I_i^0)$ such that $p^0 \cdot x_i = I_i^0$, and thus that $x_i^0 R_i x_i$ for all $B(p^0, I_i^0)$. ■

Figure 4 illustrates the indispensability of assumption (1) of Lemma 2.2. In particular, it shows that if assumption (1) does not hold, then it is no longer the case in the proof of Lemma 2.2 that $x_i^0 R_i x_i^2$. In fact, while x_i^0 minimizes $p^0 \cdot x_i$ on $R_i x_i^0$ at prices $p^0 = (0, p_2^0)$, it does not maximize R_i on $B(p^0, 0)$, since clearly no such maximum exists.

Lemmas 2.1 and 2.2 provide one of the two pairs of keys needed to prove the Fundamental Theorems. The other pair consists of the following two simple but important lemmas. First, if $S_1, S_2, \ldots, S_m$ is any collection of sets $S_i \subseteq E^n$, their sum is defined as

$$\sum_{i=1}^{m} S_i = \left\{ x \in E^n \;\middle|\; (\exists x_i \in S_i)\; x = \sum_{i=1}^{m} x_i \right\}.$$

Likewise, the (algebraic) difference $S - T$ between two sets S and T is defined as

$$S - T = \left\{ z \in E^n \;\middle|\; (\exists x \in S, y \in T)\; z = x - y \right\}.$$

The following lemmas will be found in Koopmans [7, pp. 12–13].

Lemma 3.1. Let x_i^0 maximize $p \cdot x_i^0$ on S_i for each $i = 1, 2, \ldots, m$, and define $x^0 = \sum_{i=1}^m x_i^0$. Then x^0 maximizes $p \cdot x$ on $S = \sum_{i=1}^m S_i$.

Proof. Let $x^1 \in S$; then $x^1 = \sum_{i=1}^m x_i^1$ for $x_i^1 \in S_i$. It follows that

$$p \cdot x^1 = \sum_{i=1}^{m} p \cdot x_i^1 \leqq \sum_{i=1}^{m} p \cdot x_i^0 = p \cdot x^0. \quad ■$$

Lemma 3.2. Let $x^0 = \sum_{i=1}^m x_i^0$ maximize $p \cdot x$ on $S = \sum_{i=1}^m S_i$, where $x_i^0 \in S_i$. Then for each $i = 1, 2, \ldots, m$, x_i^0 maximizes $p \cdot x_i$ on S_i.

Proof. Suppose the conclusion is false. Then for some $i = k$ there exists $x_k^1 \in S_k$ such that $p \cdot x_k^1 > p \cdot x_k^0$. Defining

$$x^1 = x_k^1 + \sum_{i \neq k} x_i^0,$$

then we have certainly $x^1 \in S$, and furthermore

$$p \cdot x^1 = p \cdot x_k^1 + \sum_{i \neq k} p \cdot x_i^0 > \sum_{i=1}^{m} p \cdot x_i^0 = p \cdot x^0,$$

contradicting the hypothesis that x^0 maximizes $p \cdot x$ on S. ■

We shall have occasion to consider a particular sum of sets, which we shall call the *Scitovsky set of* X, defined as

$$\mathrm{Sci}(X) = \sum_{i=1}^{m} R_i x_i. \tag{2}$$

This consists of all aggregate bundles $x' = \sum_{i=1}^{m} x'$ such that $x'_i R_i x_i$ for each $i = 1, 2, \ldots, m$; i.e., of all aggregate bundles x' such that, for some allocation X' of this aggregate, every person is at least as well off as with the allocation X of x. The boundary of $\mathrm{Sci}(X)$ is known (in the case $n = 2$) as the *Scitovsky indifference curve* corresponding to the allocation X, first introduced by Scitovsky [15] in his analysis of the theory of tariffs. A geometric method of construction of $\mathrm{Sci}(X)$ for the case $m = n = 2$ and $X = X^0$ may be illustrated by Figure 5: by rigidly "sliding" the northeast O_2 axis of the Edgeworth box, together with individual 2's indifference curve (measured from O_2), along individual 1's indifference curve (measured from O_1), so as to maintain tangency between the two curves, the displaced origin O_2 traces out the locus of points along which both individuals' utility remain at the same level as at the original allocation X^0; this is the Scitovsky indifference curve. The diagram also illustrates the parallogram law showing that a point on the Scitovsky indifference curve is the sum of the component points on the two individuals' indifference curves (now both measured from O_1) when the slopes of the two indifference curves are the same.

We are now ready to prove the first (and easier) of the two Fundamental Theorems (cf. Koopmans [7, pp. 48–9]).

Theorem 1. Let (X^0, p^0) be a competitive equilibrium for (Y, R), where each R_i is locally non-satiating. Then X^0 is a Pareto-optimal allocation for (Y, R).

Proof. Since $(X^0, p^0) \in C(Y, R)$ we have, by definition,

(i) $x_i^0 R_i x_i$ for all $x_i \in B(p^0, p^0 \cdot x_i^0)$, $i = 1, 2, \ldots, m$;

(ii) $p^0 \cdot y^0 \geqq p^0 \cdot y$ for all $y \in Y$, where $y^0 = \sigma(X^0)$.

From (i) and local non-satiation we have, by Lemma 2.1,

(i′) $p^0 \cdot x_i^0 \leqq p^0 \cdot x_i$ for all $x_i \in R_i x_i^0, i = 1, 2 \ldots, m$.

Now suppose that $X^0 \notin O(Y, R)$, i.e. $X^1 P X^0$ for some $X^1 \in A(Y)$. Then $x_i^1 R_i x_i^0$ for all i and $x_k^1 P_k x_k^0$ for some k. It follows that $p^0 \cdot x_k^1 > p^0 \cdot x_k^0$, since if $p^0 \cdot x_k^1 \leqq p^0 \cdot x_k^0$ (i.e., $x_k^0 \in B(p^0, p^0 \cdot x_k^0)$), we would have $x_k^0 R_k x_k^1$ from (i′). Consequently, defining

$$y^1 = x_k^1 + \sum_{i \neq k} x_i^0$$

we have

$$p^0 \cdot y^1 = p^0 \cdot x_k^1 + \sum_{i \neq k} p^0 \cdot x_i^0 > \sum_{i=1} p^0 \cdot x_i^0 = p^0 \cdot y^0,$$

contradicting (ii). ■

The converse of Theorem 1 is not true without a qualification; and unfortunately, this qualification is quite awkward to state.

Theorem 2. Let X^0 be a Pareto-optimal allocation for (Y, R), where each R_i is convex, i.e., for any i and any x_i, the set $R_i x_i$ is convex. Then there exists a price vector $p^0 \neq 0$ such that

(i′) $p \cdot x_i^0 \leqq p \cdot x_i$ for all $x_i \in R_i x_i^0,\ i = 1, 2 \ldots, m.$

(ii) $p^0 \cdot y^0 \geqq p^0 \cdot y$ for all $y \in Y$, where $y^0 = \sigma(X^0)$.

(Such a situation is called a "valuation equilibrium" by Debreu [4].) Furthermore, if p^0 is such that

$$p^0 \cdot x_i^0 > \inf_{x \in E_+^n} p^0 \cdot x \text{ for each } i = 1, 2, \ldots, m, \tag{3}$$

then (i′) implies

(i) $x_i^0 R_i x_i$ for all $x_i \in B(p^0,\, p^0 \cdot x_i^0),\ i = 1, 2, \ldots, m;$

i.e., (X^0, p^0) is a competitive equilibrium for (Y, R).

Proof. For any $k \in \{1, 2, \ldots, m\}$ we have, by the continuity of each R_i,

$$\text{int Sci}(X^0) = P_k x_k^0 + \sum_{i \neq k} R_i x_i^0.$$

Let X^1 be any allocation such that $x_k^1 P_k x_k^0$ and $x_i^1 R_i x_i^0$ for $i \neq k$, i.e., such that

$$\sigma(X^1) \in \text{int Sci}(X^0).$$

Then $X^1 P X^0$, hence $X^1 \notin A(Y)$—since $X^0 \in O(Y, R)$—and thus $\sigma(X^1) \notin Y$. Therefore,

$$\text{int Sci}(X^0) \cap Y = \varnothing.$$

Now the sets int $\text{Sci}(X^0)$ and Y are both convex, hence by Minkowski's separating hyperplane theorem (cf. Debreu [5, p. 25]), there exists an n-tuple of prices $p^0 \neq 0$, and a constant c, such that

(i′′′′) $p^0 \cdot x > c$ for all $x \in \text{int Sci}(X^0)$;

(ii′′′) $p^0 \cdot y \leqq c$ for all $y \in Y$

(see Figure 6). From the continuity of preferences, (i′′′′) implies

(i′′′) $p^0 \cdot x \geqq c$ for all $x \in \text{Sci}(X^0)$.

Now, $y^0 = \sigma(X^0) \in \text{Sci}(X^0)$, hence by (i′′′) we have

(a) $p^0 \cdot y^0 \geqq c.$

Likewise, since $X^0 \in (Y, R)$ it follows that $y^0 = \sigma(X^0) \in Y$, hence by (ii′′′) we have

(b) $p^0 \cdot y^0 \leqq c.$

It follows from (a) and (b) that $p^0 \cdot y^0 = c$, hence inequalities (i‴) and (ii‴) may be written

(i″) $p^0 \cdot x \geqq p^0 \cdot y^0$ for all $x \in \mathrm{Sci}(X^0)$;

(ii) $p^0 \cdot y \leqq p^0 \cdot y^0$ for all $y \in Y$.

By definition (2) and Lemma 3.2, (ii″) implies

(i′) $p^0 \cdot x_i \geqq p^0 \cdot x_i^0$ for all $x_i \in R_i x_i^0$, $i = 1, 2, \ldots, m$.

Now, using assumption (3) it follows by Lemma 2.2 that (i′) implies

(i) $x_i R_i x_i^0$ for all $x_i \in B(p^0, p^0 \cdot x_i^0)$, $i = 1, 2, \ldots, m$.

(i) and (ii) constitute the definition of competitive equilibrium. ■

It may be noted that there is nothing in Theorem 2 to require that prices be positive. Figure 6 illustrates the possibility of *non-free-disposability*—of commodity 1 in this case. If commodity 1 is considered noxious to consumers, as is the case, say, with garbage, then they will be willing to pay to have it disposed of. However, this is not quite the same as being induced to purchase it by a negative price. In general, negative prices cannot be implemented in a competitive economy, if only because it is in no one's interest to produce something at a nonnegative cost and pay people to purchase it. Negative prices can be ruled out either by introducing an assumption of free disposability, or by postulating, in place of the assumption of local non-satiation, the much stronger assumption of *monotonicity*, namely that each individual will prefer a larger quantity of any commodity to a smaller one.

Figure 7 illustrates the indispensability of condition (3) of Theorem 2, in the form $p^0 \cdot x^0 > 0$ (for the case $p^0 \geqq 0$). It is an example of a case in which the Edgeworth box shrinks to a line segment, all available allocations are Pareto optimal, but no competitive equilibrium exists. In effect in this example there is only one commodity, and the concept of price is meaningless unless there is more than one commodity. Thus, no competitive equilibrium is possible in a one-commodity economy; the situation is reduced to Hobbes's state of nature [6], which is necessarily a state of war.

Figure 8 illustrates the indispensability of the convexity of preferences. Here, the point P is a Pareto optimum, since individual 1 (with the serpentine indifference curves) cannot become better off without individual 2 becoming worse off. But the point P cannot be a competitive equilibrium, since at the price line through P this individual will prefer the point Q to P. It cannot be objected that Q is unattainable; for by the definition of competitive equilibrium the only information available to the consumer (and in this case the information happens to be incorrect) is that anything is available that is in the individual's budget set. Thus, the point P cannot be sustained as a competitive equilibrium. But this would still be the case even if the Edgeworth box were to be enlarged so as to include the point Q.

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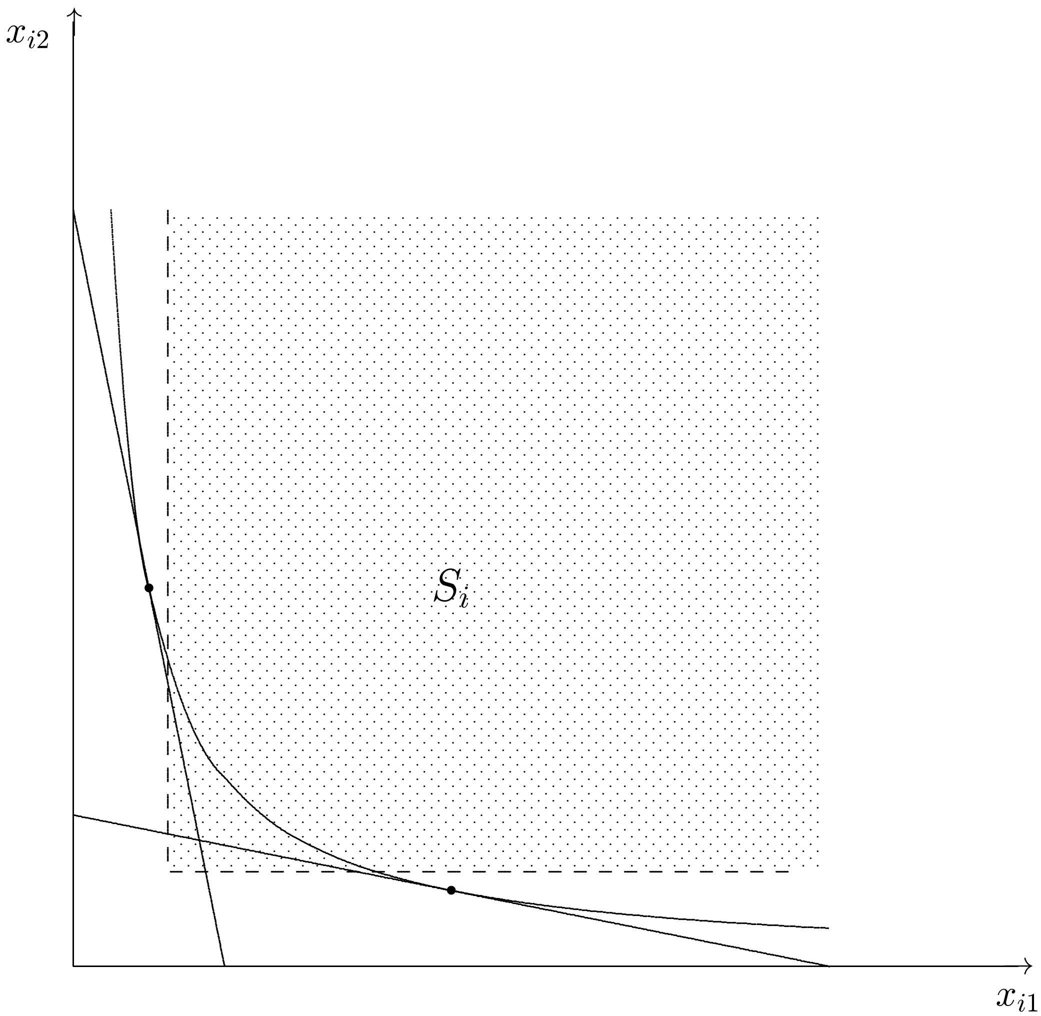

Figure 1

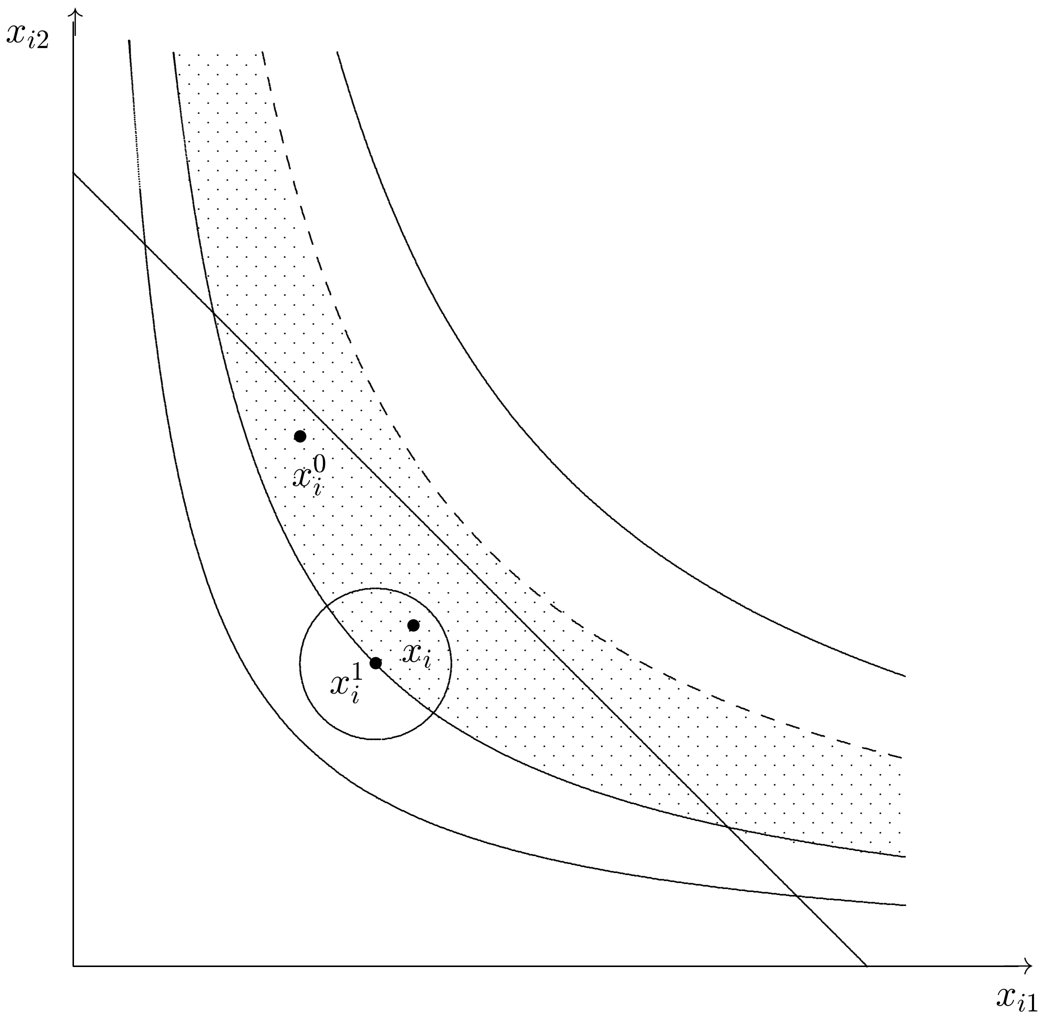

Figure 2

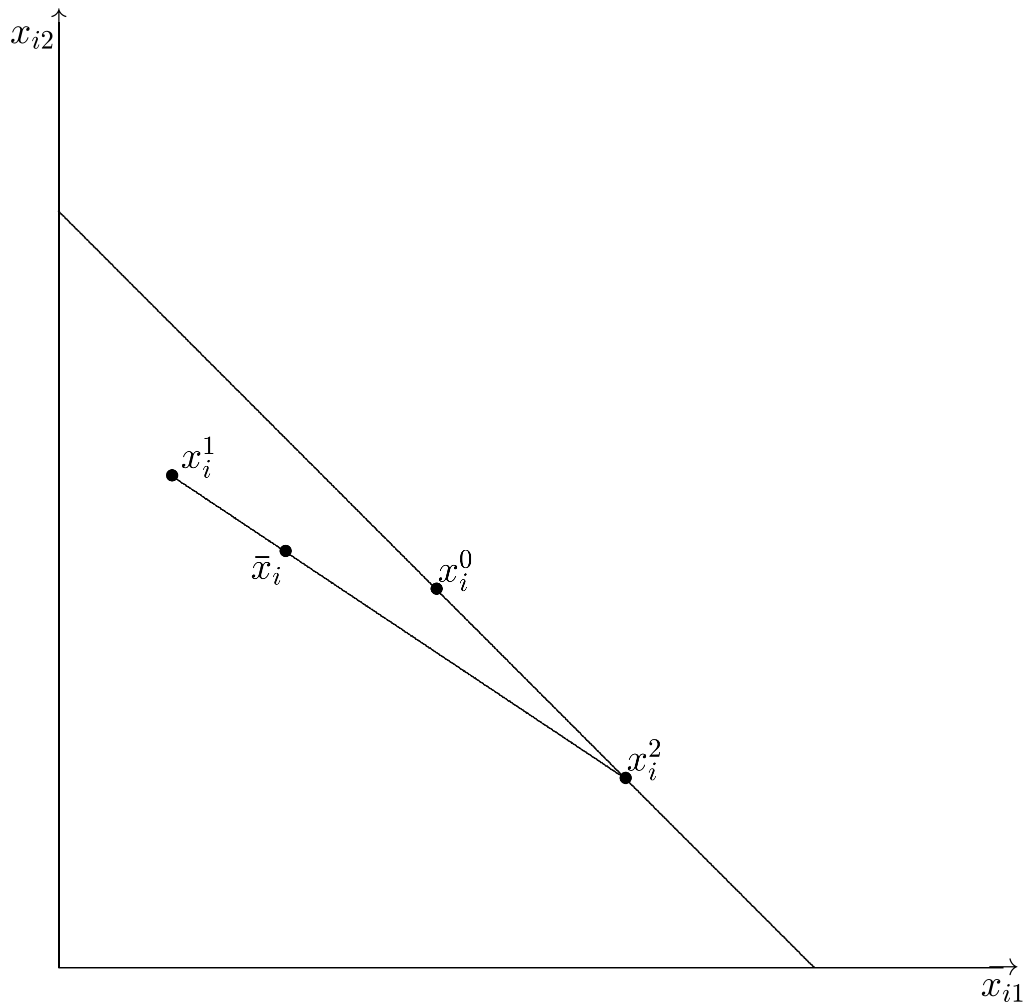

Figure 3

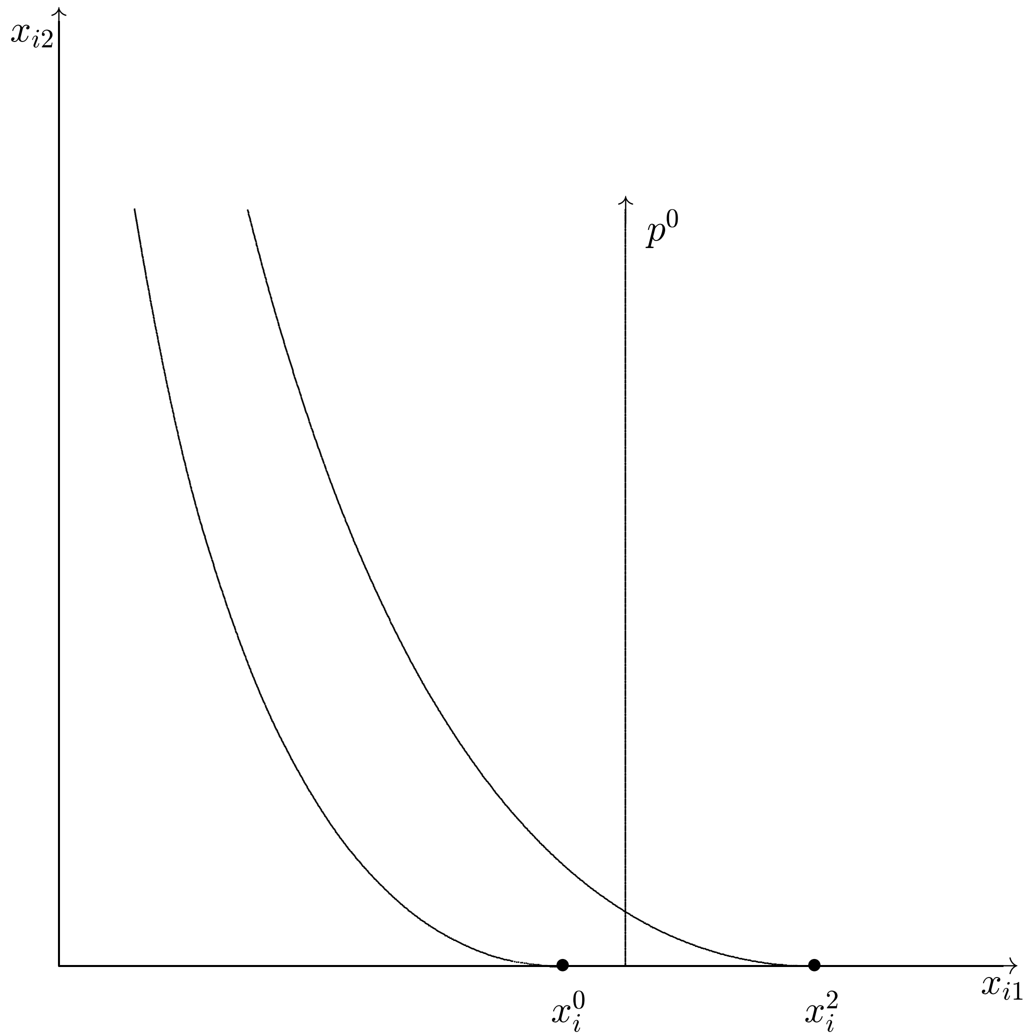

Figure 4

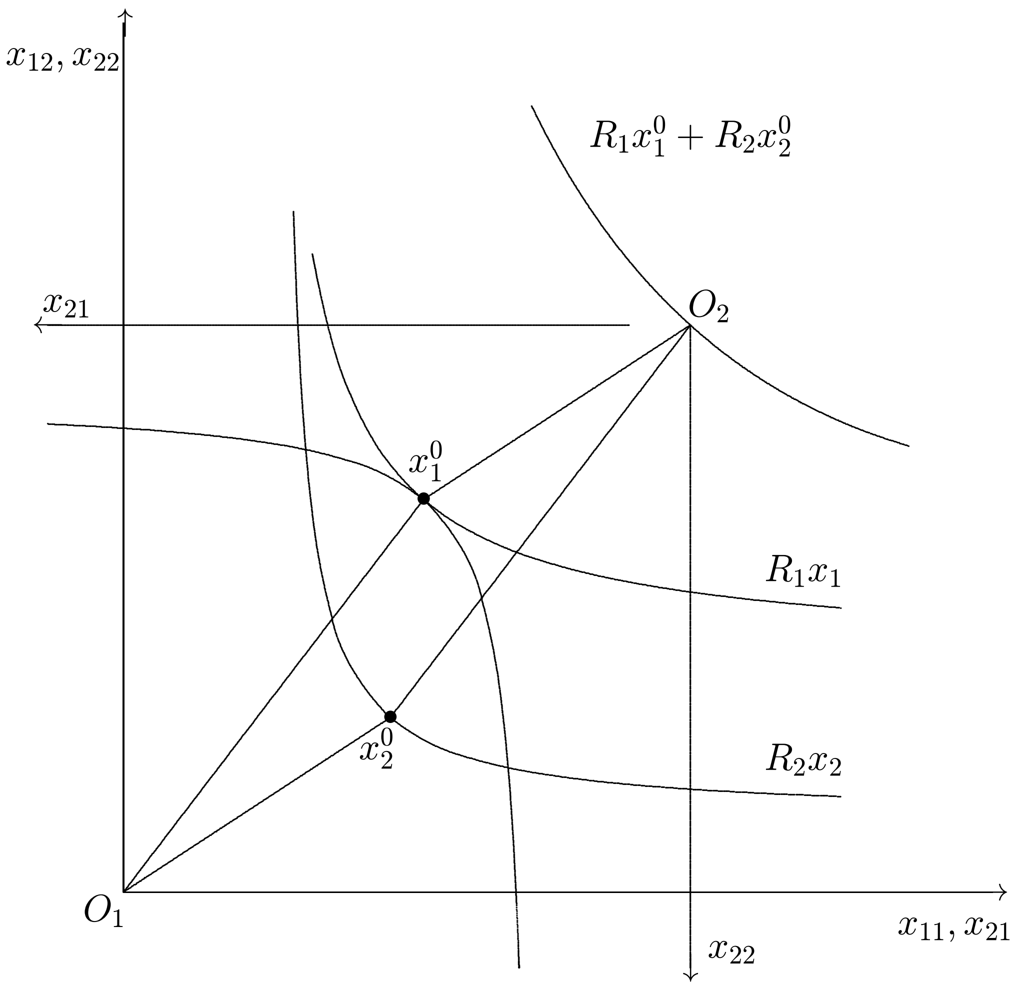

Figure 5

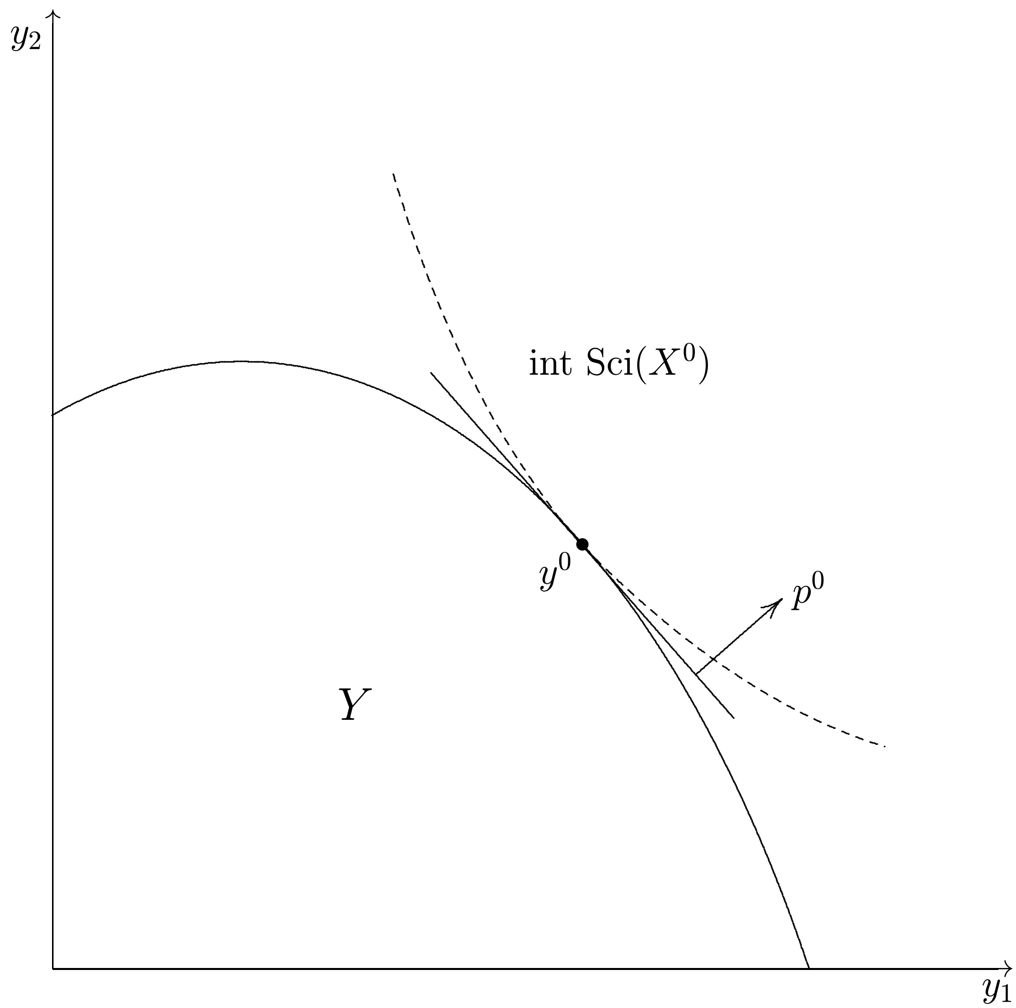

Figure 6

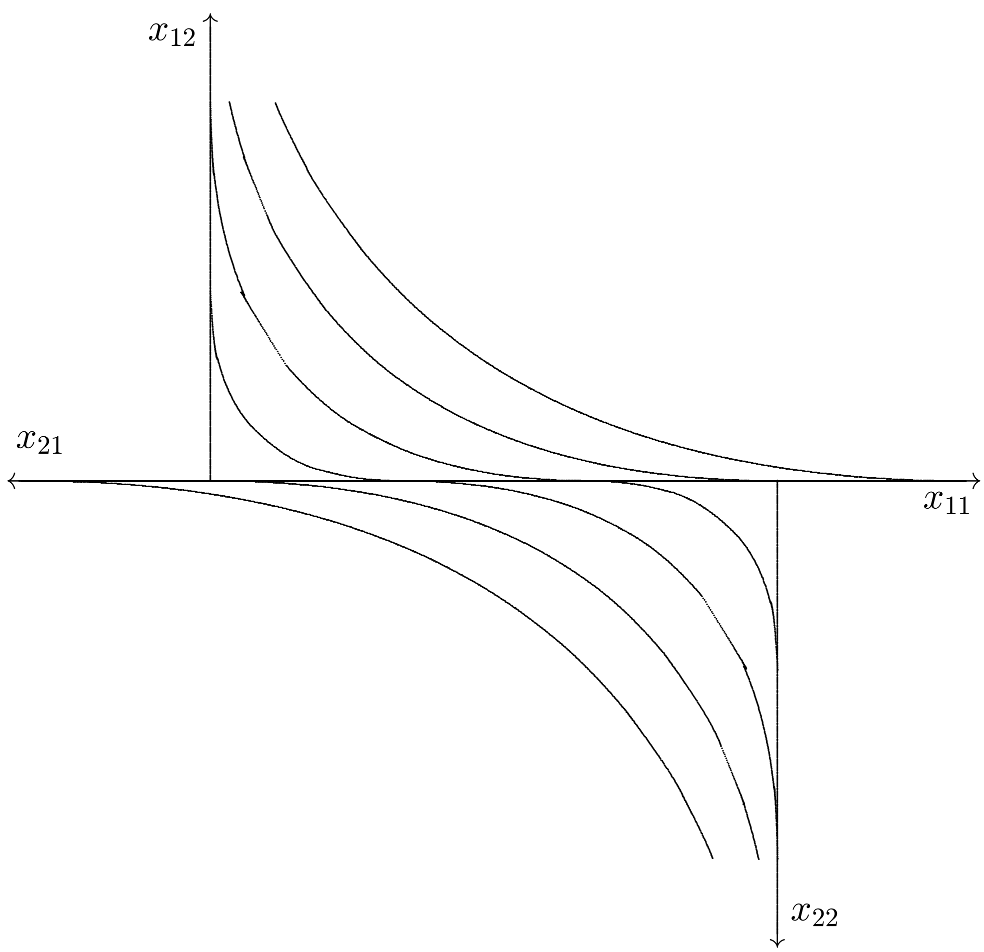

Figure 7

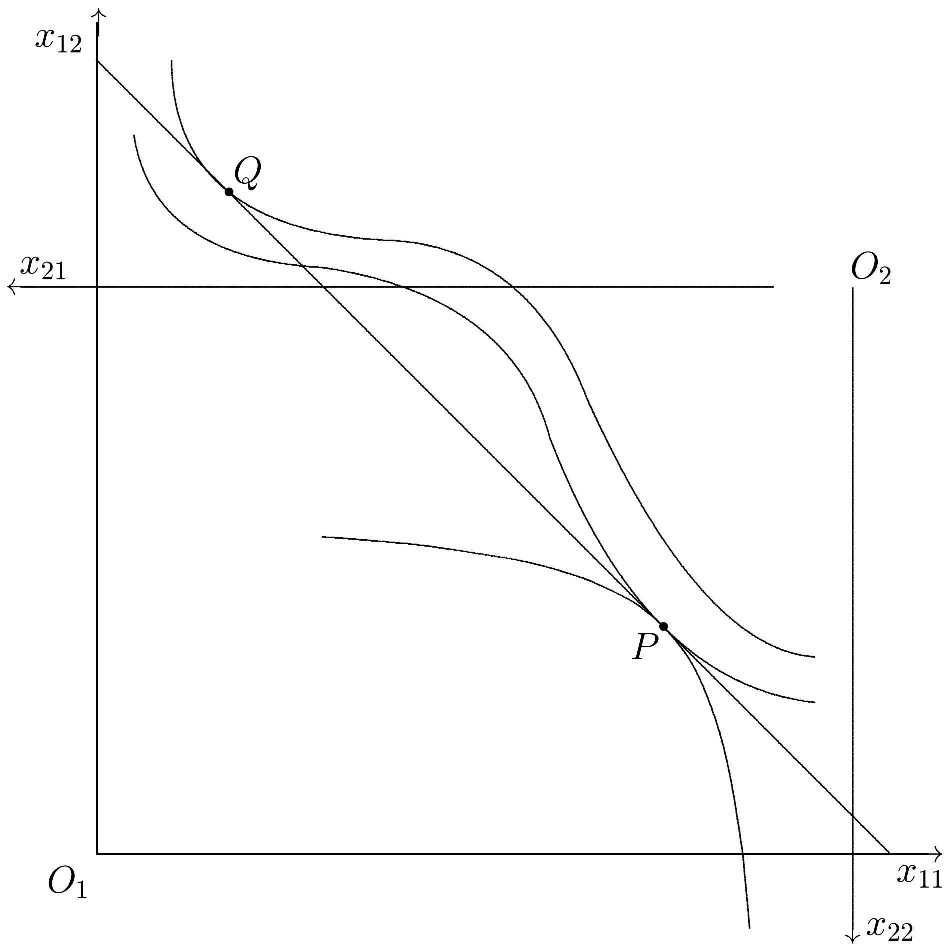

Figure 8