1 National-income comparisons with fixed production

Assume that there is an economy with two commodities and two individuals whose preference relations $R_i$ for these two commodities are represented by the utility functions

\[ U_1(x_{11}, x_{12}) = x_{11} \quad \text{and} \quad U_2(x_{21}, x_{22}) = x_{21} + x_{22} \]

respectively, and that there are fixed amounts

\[ y^1 = (y^1_1, y^1_2) = (9, 3) \quad \text{and} \quad y^2 = (y^2_1, y^2_2) = (7, 7) \]

of the two commodities to be allocated between the two individuals in periods 1 and 2, where $y^j_t$ denotes the fixed supply of commodity $j$ in period $t$. The symbol $R$ denotes the pair $(R_1, R_2)$, where $x^1_iR_ix^2_i$ stands for “$x^1_i$ is preferred or indifferent to $x^2_i$ by individual $i$.”

(a) Show that the pair $(X^1, p^1)$, where

\[ X^1 = \begin{bmatrix} x^1_{11} & x^1_{12} \\ x^1_{21} & x^1_{22} \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad p^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \]

is

(i) a competitive equilibrium for $(y^1, R)$, and

(ii) a Pareto-optimal allocation of $y^1$ for $R$.

(b) Show that the pair $(X^2, p^2)$, where

\[ X^2 = \begin{bmatrix} x^2_{11} & x^2_{12} \\ x^2_{21} & x^2_{22} \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 1 & 7 \end{bmatrix} \quad \text{and} \quad p^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \]

is

(i) a competitive equilibrium for $(y^2, R)$, and

(ii) a Pareto-optimal allocation of $y^2$ for $R$. 

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(c) Show that although
\[ p_1^2 y_1^2 + p_2^2 y_2^2 > p_1^1 y_1^1 + p_2^1 y_2^1 \]
and
\[ p_1^1 y_1^1 + p_2^1 y_2^1 > p_1^2 y_1^2 + p_2^2 y_2^2, \]
i.e., national income in period 2 than in period 1 when measured in either period-2 or period-1 prices, nevertheless there exists no allocation \( \bar{X}^2 \) of \( y^2 \) that is weakly Pareto superior to the observed allocation \( X^1 \) of \( y^1 \) (that is, preferred or indifferent by both individuals to the observed allocation in period 1).

HINT: Draw the Edgeworth boxes corresponding to the two equilibria, and show that the Scitovsky set of \( X^1 \) (the set of all aggregate bundles
\[
(x_1, x_2) = (x_{11} + x_{21}, x_{12} + x_{22})
\]
such that \( (x_{i1}, x_{i2}) \mathord{R}^i (x_{i1}^1, x_{i2}^1) \)) lies above and to the right of the Edgeworth box subtended by \( y^2 \).

2 The cost-of-living index

An individual consuming two commodities has constant preferences represented by the utility function
\[ U(x_1, x_2) = x_1 x_2. \]
In period 1 this individual has an income of \( Y^1 = 10,000 \) and faces prices of \( p_1^1 = 10 \) and \( p_2^1 = 5 \) for commodities 1 and 2 respectively; in period 2, an income of \( Y^2 = 16,000 \) and prices of \( p_1^2 = 8 \) and \( p_2^2 = 25 \) for commodities 1 and 2 respectively. All income is spent on these two commodities.

(a) Find the quantities \( x^1 = (x_1^1, x_2^1) \) and \( x^2 = (x_1^2, x_2^2) \) consumed in the two periods (the superscripts denote the period of time; they are not exponents). Is the individual better or worse off in period 2 than in period 1?

(b) Find the hypothetical income \( \bar{Y}^2 \) that would make the individual just as well off in period 2 with prices \( p_1^2 = 8 \) and \( p_2^2 = 25 \) as in period 1 with income \( Y^1 = 10,000 \) and prices \( p_1^1 = 10 \) and \( p_2^1 = 5 \). Is \( \bar{Y}^2 \) greater or smaller than \( Y^2 \)?

(c) Find the hypothetical bundle \( \bar{x}^2 = (\bar{x}_1^2, \bar{x}_2^2) \) that would be consumed at period-2 prices with the hypothetical income \( \bar{Y}^2 \). Show that
\[ p_1^2 x_1^1 + p_2^2 x_2^1 > p_1^2 \bar{x}_1^2 + p_2^2 \bar{x}_2^2, \]
and display this in a graph with parallel budget lines for prices \( p^2 = (p_1^2, p_2^2) \) going through the points \( x^1 = (x_1^1, x_2^1) \) and \( \bar{x}^2 = (\bar{x}_1^2, \bar{x}_2^2) \).
(d) Define the “true” cost-of-living index as \( C = \bar{Y}^2 / Y^1 \). Use your answer to (c) to show that the Laspeyres price index

\[
L = \frac{p_2^2 x_1^1 + p_2^2 x_2^1}{p_1^1 x_1^1 + p_2^1 x_2^1}
\]

overestimates this “true” index of the cost of living, i.e., \( L > C \). By what percentage?