Econ 8181, Topics in Microeconomics: Welfare Economics
Solutions to Homework Problem Set #1

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1 Excise taxes and welfare

(a) Given the utility function (1.2), the demand functions for the two commodities are

$$x_1 = \frac{\theta_1 Y}{p_1} \quad \text{and} \quad x_2 = \frac{\theta_2 Y}{p_2}. \tag{S1.1}$$

From (1.1), the (constant) unit production costs of the two commodities after imposition of the excise tax, and therefore the equilibrium prices, are

$$p_1 = (1+\tau_1)wb_1 \quad \text{and} \quad p_2 = wb_2. \tag{S1.2}$$

At these prices, the consumer demands (S1.1) are

$$x_1 = \frac{\theta_1 Y}{(1+\tau_1)wb_1} \quad \text{and} \quad x_2 = \frac{\theta_2 Y}{wb_2}. \tag{S1.3}$$

It remains to determine disposable income, Y, which is equal to wage income wl plus the government's tax revenues—which are distributed back to the consumers in a lump sum. These tax revenues are

$$\begin{aligned} R(\tau_1, 0) = \tau_1 \cdot wb_1 \cdot x_1 &= \tau_1 wb_1 \frac{\theta_1 Y}{(1+\tau_1)wb_1} \\ &= \frac{\tau_1\theta_1}{1+\tau_1}[wl + R(\tau_1, 0)]. \end{aligned} \tag{S1.4}$$

In case $-1 < \tau_1 < 0$, this is a subsidy, financed by a lump-sum tax $R(\tau_1, 0) < 0$. Solving (S1.4) we obtain

$$R(\tau_1, 0) = \frac{\tau_1\theta_1 wl}{1+\tau_1(1-\theta_1)}. \tag{S1.5}$$

Therefore, disposable income is

$$\begin{aligned} Y = wl + R(\tau_1, 0) &= \left(1 + \frac{\tau_1\theta_1}{1+\tau_1(1-\theta_1)}\right) wl \\ &= \frac{1+\tau_1}{1+\tau_1-\tau_1\theta_1} wl \\ &= \frac{wl}{1 - \frac{\tau_1}{1+\tau_1}\theta_1}. \end{aligned} \tag{S1.6}$$

Setting $x_1 = y_1$, let us verify that the resource-allocation constraint (1.1) holds:

$$\begin{aligned}
b_1x_1 + b_2x_2 &= \left(\frac{\theta_1}{1+\tau_1} + \theta_2\right)\frac{Y}{w} \\
&= \left(\frac{\theta_1}{1+\tau_1} + \theta_2\right)\frac{l}{1 - \frac{\tau_1}{1+\tau_1}\theta_1} \\
&= \frac{\theta_1 + (1+\tau_1)\theta_2}{1+\tau_1-\tau_1\theta_1}\,l \\
&= \frac{1+\tau_1\theta_2}{1+\tau_1(1-\theta_1)}\,l = l.
\end{aligned}$$

Given the utility function (1.2) and the demand functions (S1.1), the indirect utility function is

$$V(p_1, p_2, Y) = \frac{\theta_1^{\theta_1}\theta_2^{\theta_2}}{p_1^{\theta_1}p_2^{\theta_2}}Y. \tag{S1.7}$$

Inserting (S1.2) and (S1.6) into (S1.7) we obtain for the indirect utility as a function of the tax rate τ_1,

$$W(\tau_1, 0) = \frac{\theta_1^{\theta_1}\theta_2^{\theta_2}}{b_1^{\theta_1}b_2^{\theta_2}}\,l\frac{1}{(1+\tau_1)^{\theta_1}(1-\frac{\tau_1}{1+\tau_1}\theta_1)}. \tag{S1.8}$$

(b) Noticing that that the expression in the denominator of (S1.8) may be written, as a function of τ_1 and $\theta_1 = 1-\theta_2$, as

$$F(\tau_1, \theta_1) \equiv (1+\tau_1)^{\theta_1}\left(1 - \frac{\tau_1}{1+\tau_1}\theta_1\right) = \frac{1+\tau_1\theta_2}{(1+\tau_1)^{\theta_2}}, \tag{S1.9}$$

we may define our measure of welfare loss, denoting explicitly its dependence on $\theta_1 = 1-\theta_2$, by

$$L(\tau_1, \theta_1) = W(0,0) - W(\tau_1, 0) = Kl\left(1 - \frac{1}{F(\tau_1, \theta_1)}\right) = Kl\left(1 - \frac{(1+\tau_1)^{\theta_2}}{1+\tau_1\theta_2}\right) \tag{S1.10}$$

where

$$K = \frac{\theta_1^{\theta_1}\theta_2^{\theta_2}}{b_1^{\theta_1}b_2^{\theta_2}}. \tag{S1.11}$$

We find that

$$\frac{\partial L}{\partial \tau_1} = Kl\frac{\theta_1\theta_2\tau_1}{(1+\tau_1)^{\theta_1}(1+\tau_1\theta_2)^2}. \tag{S1.12}$$

Thus, $L(\tau_1, \theta_1)$ is stationary with respect to τ_1 at $\tau_1 = 0$. To compute the second derivative we first note that

$$\begin{aligned}
\frac{\partial}{\partial \tau_1}[(1+\tau_1)^{\theta_1}(1+\tau_1\theta_2)^2] &= \theta_1(1+\tau_1)^{\theta_1-1}(1+\tau_1\theta_2)^2 + (1+\tau_1)^{\theta_1}2(1+\tau_1\theta_2)\theta_2 \\
&= (1+\tau_1)^{\theta_1-1}(1+\tau_1\theta_2)[\theta_1(1+\tau_1\theta_2) + 2\theta_2(1+\tau_1)] \\
&= (1+\tau_1)^{\theta_1-1}(1+\tau_1\theta_2)[2-\theta_1+\theta_1\theta_2\tau_1+2\theta_2\tau_1].
\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\partial^2 L}{\partial \tau_1^2} &= Kl\cdot\theta_1\theta_2\frac{(1+\tau_1)^{\theta_1-1}(1+\tau_1\theta_2)[(1+\tau_1)(1+\tau_1\theta_2)-(2-\theta_1+\theta_1\theta_2\tau_1+2\theta_2\tau_1)\tau_1]}{(1+\tau_1)^{2\theta_1}(1+\tau_1\theta_2)^4}\\ &= Kl\cdot\theta_1\theta_2\frac{1+(\theta_2-\theta_1\theta_2-2\theta_2)\tau_1^2}{(1+\tau_1)^{\theta_1+1}(1+\tau_1\theta_2)^3}\\ &= Kl\cdot\theta_1\theta_2\frac{1-(1-\theta_1^2)\tau_1^2}{(1+\tau_1)^{\theta_1+1}(1+\tau_1\theta_2)^3}.\end{aligned}$$

(S1.13)

This is positive when $\tau_1 = 0$, so that $L(\tau_1, \theta_1)$ is minimized with respect to τ_1 at $\tau_1 = 0$.

(c) Under the erroneous assumption that the demand for commodity 1 is unaffected by the return of the tax revenues, it will be equal to

$$x_1 = \frac{\theta_1 wl}{(1+\tau_1)wb_1} = \frac{\theta_1 l}{(1+\tau_1)b_1} \tag{S1.14}$$

after the tax, rather than

$$x_1 = \frac{\theta_1 l}{(1+\tau_1)b_1}\frac{1}{1-\frac{\tau_1}{1+\tau_1}\theta_1}\ , \tag{S1.15}$$

from (S1.3) and (S1.6). The change in consumer's surplus[1] will then be

$$\begin{aligned}\Delta(\mathrm{CS}) = -\int_{wb_1}^{(1+\tau_1)wb_1}\frac{\theta_1 wl}{p_1}dp_1 &= -\theta_1 wl\int_{wb_1}^{(1+\tau_1)wb_1}\frac{1}{p_1}dp_1\\ &= -\theta_1 wl\cdot\log p_1\Big|_{wb_1}^{(1+\tau_1)wb_1}\\ &= -\theta_1 wl\Big(\log[(1+\tau_1)wb_1]-\log(wb_1)\Big)\\ &= -\theta_1 wl\,\log(1+\tau_1).\end{aligned} \tag{S1.16}$$

Likewise, the erroneous calculation of the government revenue is, in place of (S1.5) (setting $Y = wl$ instead of $Y = wl + R(\tau_1, 0)$ in (S1.4)),

$$R_1(\tau_1, 0) = \tau_1\cdot wb_1\cdot\frac{\theta_1 wl}{(1+\tau_1)wb_1} = \frac{\tau_1}{1+\tau_1}\theta_1 wl. \tag{S1.17}$$

Therefore the erroneous calculation of the deadweight loss is

$$L_1(\tau_1,\theta_1) = -\Delta(\mathrm{CS}) - R_1(\tau_1, 0) = wl\cdot\theta_1\left(\log(1+\tau_1) - \frac{\tau_1}{1+\tau_1}\right) \tag{S1.18}$$

in place of (S1.10). Clearly this is zero for $\tau_1 = 0$. We verify that

$$\frac{\partial L_1}{\partial \tau_1} = wl\cdot\theta_1 = wl\cdot\theta_1\left(\frac{1}{1+\tau_1} - \frac{1}{(1+\tau_1)^2}\right) = wl\cdot\frac{\theta_1\tau_1}{(1+\tau_1)^2}\ , \tag{S1.19}$$

[1] The consumer's surplus itself, if the lower limit of integration is taken to be $p_1 = 0$, is undefined; only changes in consumer's surplus can be defined.

which vanishes if and only if $\tau_1 = 0$. The second derivative is found to be

$$\frac{\partial^2 L_1}{\partial \tau_1^2} = wl \cdot \theta_1\theta_1 \frac{1-\tau_1^2}{(1+\tau_1)^4}\,, \tag{S1.20}$$

which is positive at $\tau_1 = 0$. Hence the deadweight loss is minimized when $\tau_1 = 0$.[2]

(d) Now we need to use a money-metric indirect utility function that makes the correct and incorrect measures of welfare loss commensurate. The logical candidate for this is the Hurwicz-Uzawa income-compensation function[3]

$$\mu(\bar{p}; p, Y) = \min\{\bar{Y} > 0 \,:\, (\bar{p}, \bar{Y}) R^*(p, Y)\}, \tag{S1.21}$$

where "R^*" means "is indirectly preferred or indifferent to", and in this application $p = (p_1, p_2)$. In our particular case of homothetic preferences we may express (S1.7) as

$$V(p, Y) = \frac{Y}{C(p)} \quad \text{where } C(p) = \frac{p_1^{\theta_1} p_2^{\theta_2}}{\theta_1^{\theta_1}\theta_2^{\theta_2}}\,, \tag{S1.22}$$

$C(p)$ being a Jevonian geometric cost-of-living index. From (S1.22) and (S1.21) we see that necessarily

$$\mu(\bar{p}; p, Y) = \frac{C(\bar{p})}{C(p)} Y = \frac{\bar{p}_1^{\theta_1}\bar{p}_2^{\theta_2}}{p_1^{\theta_1}p_2^{\theta_2}} Y = \bar{Y}. \tag{S1.23}$$

Now it is natural to take the base prices as $\bar{p}_j = wb_j$ and the current prices as $p_j = (1+\tau_j)wb_j$, where in our application $\tau_2 = 0$. Then further substituting (S1.6) in (S1.23) our modified welfare function becomes

$$\begin{aligned} W_0(\tau_1, 0) \;&\equiv\; \mu\Big(wb_1, wb_2\,;\, (1+\tau_1)wb_1, wb_2, wl/(1-\tfrac{\tau_1}{1+\tau_1})\Big) \\ &=\; wl\, \frac{1}{(1+\tau_1)^{\theta_1}(1-\frac{\tau_1}{1+\tau_1}\theta_1)}. \end{aligned} \tag{S1.24}$$

This simply replaces the factor Kl in (S1.8) (where K is defined by (S1.11)) by the initial national income wl. Analogously to (S1.10) we define our modified measure of welfare loss by

$$L_0(\tau_1, \theta_1) = W_0(0,0) - W_0(\tau_1, 0) = wl\left(1 - \frac{(1+\tau_1)^{\theta_2}}{1+\tau_1\theta_2}\right)\,, \tag{S1.25}$$

[2]This of course assumes that the consumer's-surplus analysis is extended to negative tax rates (subsidies). In terms of the definition (S1.17) this means that the entire gross gain in consumer's surplus is more than offset by the unit subsidy $\tau_1 wb_1$ multiplied by the new, larger, computed consumption level (S1.14). In the usual Marshallian diagram, this lump-sum or income tax would be indicated by a rectangle *enclosing* the trapezoid of the gross gain in consumer's surplus, showing the subsidy to entail a welfare loss. Apparently the Marshallian method has never been used in this way (certainly not by Marshall (1890) himself). A mechanical use of the Marshallian diagram in which the unit subsidy is multiplied by the *old* consumption level would lead to the absurd conclusion that an excise subsidy is always beneficial. However, the extension to negative tax rates presented here is needed mathematically to define the various welfare-loss measures as differentiable functions of the tax rate over the interval $(-1, \infty)$.

[3]Cf. Hurwicz and Uzawa (1971, p. 119, 125–130), McKenzie (1957), Chipman and Moore (1980, p. 935).

which again simply replaces K in (S1.10) by w. The replacement of the factor Kl by wl makes it possible to make a direct comparison of the welfare loss measured by indirect utility with that measured by the consumer's surplus. Both measures are equal to the initial national income wl when $\tau_1 = 0$. However, (S1.18) cannot be expressed as an indirect utility (or income-compensation) function, for the simple reason that the consumer's surplus procedure is inconsistent with the consumer's demand function; it includes the returned tax revenues in the consumer's money measure of utility, but it assumes that none of it is spent on commodity 1, contrary to (S1.1). It wouldn't help to assume that the government revenues are retained by government officials; they cannot be consumed by these officials until they are transformed into goods, but then we are back to the same problem.

Let us now consider the difference between the consumer's-surplus measure $L_1(\tau_1, \theta_1)$ of deadweight loss and the money-metric-utility measure of welfare loss $L_0(\tau_1, \theta_1)$,

$$D(\tau_1, \theta_1) = L_1(\tau_1, \theta_1) - L_0(\tau_1, \theta_1) = wl\left(\theta_1 \log(1+\tau_1) - \frac{\theta_1\tau_1}{1+\tau_1} - 1 + \frac{(1+\tau_1)^{\theta_2}}{1+\tau_1\theta_2}\right). \tag{S1.26}$$

We shall now show that this difference is positive for $\tau_1 \neq 0$ in the interval $-1 < \tau_1 < \infty$, i.e., that the consumer's-surplus measure *overestimates the true welfare loss.*

Proceeding as before, we first compute, using (S1.19) and (S1.12) (but replacing Kl by wl in the latter),

$$\frac{\partial D}{\partial \tau_1} = \frac{\partial L_1}{\partial \tau_1} - \frac{\partial L_0}{\partial \tau_1} = wl\left(\frac{\theta_1\tau_1}{(1+\tau_1)^2} - \frac{\theta_1\theta_2\tau_1}{(1+\tau_1)^{\theta_1}(1+\tau_1\theta_2)^2}\right), \tag{S1.27}$$

which vanishes at $\tau_1 = 0$ because each term does so. Next we compute, using (S1.20) and (S1.13) (but again, with wl replacing Kl in the latter),

$$\frac{\partial^2 D}{\partial \tau_1^2} = wl\left(\frac{\theta_1(1-\tau_1^2)}{(1+\tau_1)^4} - \frac{[1-(1-\theta_1^2)\tau_1^2]\theta_1\theta_2}{(1+\tau_1)^{\theta_1+1}(1+\tau_1\theta_2)^3}\right) = wl\theta_1^2 > 0 \quad \text{at } \tau_1 = 0. \tag{S1.28}$$

Thus, $D(\tau_1, \theta_1)$ reaches an isolated minimum value of 0 at $\tau_1 = 0$, and is elsewhere positive throughout the interval $(-1, \infty)$.

References

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2 Public goods and misreprentation of preferences

(a) Setting marginal rates of substitution equal to the respective "personalized prices" t_i, we have

$$\frac{\partial U_i/\partial y}{\partial U_i/\partial x_i} = \frac{\partial U_i}{\partial y} = \frac{\sqrt{a}}{2\sqrt{y}} = t_i, \quad \text{hence} \quad y_i = \frac{a}{4t_i^2} \quad (i = 1, 2),$$

where y_i denotes individual i's demand for the public good. For these two demands to be equal, clearly the t_i must be equal; and since Pareto optimality requires $t_1 + t_2 = 1$ (from the production transformation relation), we have $t_i = \frac{1}{2}$, hence $y_i = y = a$. Let us denote this by $\hat{y}(a) = a$. From the budget constraint we also have $\hat{x}_i(a) = (1 - a)/2$.

(b) From (a) it follows immediately that the amount of the public good produced and consumed will be $\hat{y}(b) = b < a = \hat{y}(a)$. Thus, each individual will consume less of the public good. However, since the amounts of the private good consumed will be $\hat{x}_i(b) = (1 - b)/2$, and $b < a$, it follows that

$$\hat{x}_i(b) = \frac{1-b}{2} > \frac{1-a}{2} = \hat{x}_i(a),$$

that is, each individual will consume more of the private good (which is clear from the nature of the production transformation relation and the fact that each individual demands the same amount of the private good). Thus, to see whether this increase in consumption of the private good is counterbalanced by the decrease in consumption of the public good, we must consult the individuals' "true" utility functions $U_i(x_i, y; a) = x_i + \sqrt{ay}$. Specifically, we need to show that

$$U_i\Big(\hat{x}_i(a), \hat{y}(a); a\Big) = U_i\left(\frac{1-a}{2}, a; a\right) = \frac{1-a}{2} + a$$

is greater than

$$U_i\Big(\hat{x}_i(b), \hat{y}(b); a\Big) = U_i\left(\frac{1-b}{2}, b; a\right) = \frac{1-b}{2} + \sqrt{ab}.$$

This is shown as follows:

$$\begin{aligned}
U_i\Big(\hat{x}_i(a), \hat{y}(a); a\Big) - U_i\Big(\hat{x}_i(b), \hat{y}(b); a\Big) &= \frac{1-a}{2} + a - \frac{1-b}{2} - \sqrt{ab} \\
&= \sqrt{a}\Big(\sqrt{a} - \sqrt{b}\Big) - \frac{a-b}{2} \\
&= \sqrt{a}\Big(\sqrt{a} - \sqrt{b}\Big) - \tfrac{1}{2}\Big(\sqrt{a} - \sqrt{b}\Big)\Big(\sqrt{a} + \sqrt{b}\Big) \\
&= \Big(\sqrt{a} - \sqrt{b}\Big)\left(\sqrt{a} - \frac{\sqrt{a}}{2} - \frac{\sqrt{b}}{2}\right) \\
&= \frac{\Big(\sqrt{a} - \sqrt{b}\Big)^2}{2} \\
&> 0.
\end{aligned}$$