3.1 Reduced-mean-square error estimation

An investigator wishes to estimate the parameter $\mu$ in the model

(Q3.1.1) $y_t = \mu + \varepsilon_t; \quad E\{\varepsilon_t\} = 0; \quad E\{\varepsilon_t^2\} = \sigma^2; \quad E\{\varepsilon_t \varepsilon_{t'}\} = 0$ for $t \neq t'$.

Instead of the sample mean

(Q3.1.2) $\bar{y} = \frac{\sum_{t=1}^{n} y_t}{n}$

this investigator proposes to use the estimator

(Q3.1.3) $\hat{\mu} = \frac{\sum_{t=1}^{n} y_t}{n + a}$,

where $a > 0$.

(a) Show that this procedure can be justified by a mean-square-error criterion if it is known a priori that

(Q3.1.4) $\frac{\mu^2}{\sigma^2} \leq \frac{2}{a}$.

(b) Suppose the model (Q3.1.1) is modified so that $\mu$ is a random variable distributed independently of the $\varepsilon_t$'s. Specify values for the prior mean $\bar{\mu}$ of $\mu$ and the prior variance $\tau^2$ of $\mu$ for which $\hat{\mu}$ (as defined by (Q3.1.3)) is the wide-sense posterior mean of $\mu$ given $y$. 

3.2 Specification error

In the linear model

\[(Q3.2.1) \quad y = X\beta + \varepsilon, \quad \text{E}\{\varepsilon\} = 0, \quad \text{E}\{\varepsilon\varepsilon^t\} = \sigma^2\Omega,\]

assume that $X$ is $n \times k$ of rank $k$ and that $\Omega$ is symmetric positive definite. Suppose that for some fixed $r \times k$ matrix $\Psi$ of rank $r$, $\beta$ is estimated subject to the erroneous homogeneous linear restriction $\Psi \beta = 0$ by the formula

$$\hat{\beta} = (I - \Psi^t\Psi)\tilde{\beta},$$

where

$$\Psi^t = (X'\Omega^{-1}X)^{-1}\Psi(X'\Omega^{-1}X)^{-1}\Psi'^{-1}$$

and

$$\tilde{\beta} = X^t y, \quad X^t = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}.$$ 

(a) Show that

\[(Q3.2.2) \quad \text{Var}\{\tilde{\beta}\} \geq \text{Var}\{\hat{\beta}\}.\]

(b) Suppose that it is desired to estimate a linear function $L\beta$, where $L$ is a given $l \times k$ matrix ($1 \leq l < k$). Show that for this $L$,

\[(Q3.2.3) \quad \text{Var}\{L\tilde{\beta}\} = \text{Var}\{L\hat{\beta}\}\]

if and only if

\[(Q3.2.4) \quad \text{E}\{L\hat{\beta}\} = L\beta \quad \text{for all } \beta,\]

i.e., if and only if $L\hat{\beta}$ is an unbiased estimator of $L\beta$.

(c) Apply the results of (a) and (b) to the following special case:

$$X = [X_1, X_2], \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},$$

where $X_i$ is $n \times k_i$, $k = k_1 + k_2$, and likewise $\beta_i$ is a $k_i \times 1$ vector; further,

$$\Psi = [0_{k_2 \times k_1}, I_{k_2}] \quad \text{and} \quad L = [I_{k_1}, 0_{k_1 \times k_2}],$$

where $l = k_1$. Show that (Q3.2.4) holds if and only if $X_1'\Omega^{-1}X_2 = 0$. Interpret this result.
3.3 Bias in estimation of sampling variances

An investigator is employing the following model:

\[(Q3.3.1) \quad y = X\beta + \varepsilon = [\iota, Z]\beta + \varepsilon \quad \text{where} \quad \mathbb{E}\{\varepsilon\} = 0 \quad \text{and} \quad \mathbb{E}\{\varepsilon\varepsilon'\} = \sigma^2\Omega.\]

Here, \(\iota\) denotes a column of \(n\) ones, and \(Z\) an \(n \times (k - 1)\) matrix such that the rank of \(X = [\iota, Z]\) is equal to \(k\), and

\[(Q3.3.2) \quad \Omega = (1 - \rho)I + \rho \mu\mu', \quad \text{where} \quad -\frac{1}{n - 1} < \rho < 1.\]

This investigator estimates \(\beta\) by ordinary least squares, \(b = (X'X)^{-1}X'y\), and \(\sigma^2\) by

\[(Q3.3.3) \quad s^2 = \frac{\sum_{i=1}^{n} \varepsilon_i^2}{n - k},\]

where \(e = y - Xb\).

(a) Find the expected value of \(s^2\).

(b) Defining the bias in estimation of sampling variances by

\[(Q3.3.4) \quad B(\rho) = \mathbb{E}\{s^2\}(X'X)^{-1} - \text{Var}\{b\},\]

determine the conditions under which the sampling variances of the individual regression coefficients \(b_j\) (the diagonal elements of \(s^2(X'X)^{-1}\)) are biased upward or downward as estimates of the true variances \(\text{Var}\{b_j\}\), for each \(j = 1, 2, \ldots, k\).

3.4 Efficiency of the Cochrane-Orcutt estimator of linear trend relative to least squares

In the model of linear trend

\[(Q3.4.1) \quad y_t = \alpha + \beta t + \varepsilon_t, \quad \mathbb{E}\{\varepsilon_t\} = 0, \quad \mathbb{E}\{\varepsilon_t\varepsilon_{t'}\} = \sigma^2 \frac{\rho|t-t'|}{1 - \rho^2}, \quad (t, t' = 1, 2, \ldots, n),\]

an investigator decides to replace the variables \(y_t\) by \(y^*_t = y_t - \rho y_{t-1}\), \(t\) by \(t^* = t - \rho(t - 1)\), and \(\varepsilon_t\) by \(\varepsilon^*_t = \varepsilon_t - \rho \varepsilon_{t-1}\), and estimate \(\beta\) by applying ordinary least squares to the transformed system

\[(Q3.4.2) \quad y^*_t = \alpha^* + \beta t^* + \varepsilon^*_t \quad (t = 2, 3, \ldots, n).\]

Show that the efficiency of the resulting estimator \(\hat{\beta}\) of \(\beta\), relative to the estimator \(b\) of \(\beta\) obtained by applying ordinary least squares to (Q3.4.1), i.e., \(\text{Var}\{b\}/\text{Var}\{\hat{\beta}\}\), goes to zero as \(\rho \to 1\), for each sample size \(n \geq 3\). [You may use any relevant formulas from the Notes.]
3.5 Estimation of growth rates

An investigator is studying the growth of real national income $Y_t$ over a period of $n$ consecutive years in terms of the model

(Q3.5.1) \[ y_t = \alpha + \beta t + \varepsilon_t \quad (t = 1, 2, \ldots, n) \]

where $y_t = \log Y_t$, and $E\{\varepsilon_t\} = 0$, $E\{\varepsilon_t^2\} = \sigma^2$, and $E\{\varepsilon_t \varepsilon_{t'}\} = 0$ for $t \neq t'$. Denote by $b$ the ordinary least-squares estimator of the growth rate, $\beta$, and define by

(Q3.5.2) \[ \hat{\beta} = \frac{\sum_{t=1}^{n-1} \Delta y_t}{n-1} \quad \text{(where } \Delta y_t = y_{t+1} - y_t) \]

the first-difference estimator of the growth rate.

(a) Find the means and variances of $b$ and $\hat{\beta}$. (HINT: you may wish to use the series formulas

\[ \sum_{t=1}^n t = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{t=1}^n t^2 = \frac{n(n+1)(2n+1)}{6}. \]

(b) Define the efficiency of the first-difference estimator relative to the least-squares estimator by

(Q3.5.3) \[ E(n) = \frac{\text{Var}\{b\}}{\text{Var}\{\hat{\beta}\}}, \]

show that:

(i) $E(2) = E(3) = 1$ and $E(n)$ is decreasing in $n$ for $n \geq 4$;
(ii) $\lim_{n \to \infty} E(n) = 0$. 

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