1 Flexible exchange rate

Krugman’s dynamic system (his equation (10)) is:

\[ \begin{align*}
\dot{m} &= [g - \pi(m/F)]m \\
\dot{F} &= Y - G - C(Y - T, m + F).
\end{align*} \tag{1.1} \]

He takes a Taylor approximation of (1.1) around the equilibrium values \( \bar{m}, \bar{F} \), i.e., the values of \( m, F \) for which \( \dot{m}, \dot{F} = 0 \):

\[ \begin{bmatrix}
\dot{u}_1 \\
\dot{u}_2
\end{bmatrix} =
\begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}, \quad \text{or} \quad \dot{u} = Bu,
\] \tag{1.2}

where

\[ u_1 = m - \bar{m}, \quad u_2 = F - \bar{F} \]

and

\[ \begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix} =
\begin{bmatrix}
-\bar{m} \pi_1 (\bar{m} / \bar{F}) & (\bar{m} / \bar{F})^2 \pi_1 (\bar{m} / \bar{F}) \\
-C_2 & -C_2
\end{bmatrix}. \tag{1.3} \]

Whenever the eigenvalues of \( B \) are distinct (as is the case here), there exists a nonsingular matrix \( V \), whose columns are the eigenvectors, \( \nu^i \), of \( B \), such that \( BV = \Lambda \), where \( \Lambda \) is a diagonal matrix whose diagonal elements are the eigenvalues, \( \lambda_j \), of \( B \).\footnote{See, for example, Perlis (1952, p. 172). In this \( 2 \times 2 \) case, suppose that for some scalars \( c_1, c_2 \) we have \( c_1 \nu^1 + c_2 \nu^2 = 0 \). Premultiplying this equation by \( B \) we get, since \( B \nu^i = \nu^i \lambda_i \), \( c_1 \nu^1 \lambda_1 + c_2 \nu^2 \lambda_2 = 0 \). These two equations may be be written as the single matrix equation \( (c_1, c_2) \begin{bmatrix} 1 & \lambda_1 \\
1 & \lambda_2 \end{bmatrix} = (0,0) \). (Continued on next page.)} Hence,

\[ V^{-1}BV = \Lambda. \tag{1.4} \]
Defining the transformation

\[(1.5) \quad u = V u^*, \quad \text{i.e.,} \quad u^* = V^{-1} u,\]

we have from (1.5), (1.2), (1.5) again, and (1.4),

\[(1.6) \quad \ddot{u}^* = V^{-1} \ddot{u} = V^{-1} B u = V^{-1} B V u^* = \Lambda u^*,\]

hence

\[(1.7) \quad \dot{u}_j^* = \lambda_j u_j^* \quad (j = 1, 2).\]

Writing this as \(du_j^*/dt = \lambda_j dt\) and integrating, we obtain

\[\log u_j^* = \lambda_j t + \text{constant}.\]

Thus,

\[(1.8) \quad u_j^* = c_j e^{\lambda_j t} \quad (j = 1, 2),\]

where the \(c_j\)s are constants. Substituting (1.8) in (1.5) we obtain the solution of (1.2):

\[(1.9) \quad u = v^1 u_1^* + v^2 u_2^* = v^1 c_1 e^{\lambda_1 t} + v^2 c_2 e^{\lambda_2 t},\]

where \(v^j\) denotes the \(j\)th column of \(V\). We can write this out as

\[(1.10) \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} c_1 e^{\lambda_1 t} + \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} c_2 e^{\lambda_2 t} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ 0 \\ c_2 \\ 0 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix}\]

which corresponds to Krugman’s equation (A2) where

\[
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}.
\]

**Lemma.** Let a \(2 \times 2\) matrix \(B\) have eigenvectors \(v^j\) and corresponding distinct eigenvalues \(\lambda_j\), where \(Bv^j = v^j \lambda_j\), i.e.,

\[
\begin{bmatrix}
 b_{11} & b_{12} \\
 b_{21} & b_{22}
\end{bmatrix} \begin{bmatrix} v_{1j} \\ v_{2j} \end{bmatrix} = \begin{bmatrix} v_{1j} \\ v_{2j} \end{bmatrix} \lambda_j \quad (j = 1, 2),
\]

where \(\lambda_1 \neq \lambda_2\). Then \(v_{11} \neq 0\), and \(v_{12} = 0\) if and only if \(b_{12} = 0\).

**Proof.** First we show that \(v_{11} \neq 0\). Suppose by way of contradiction that \(v_{11} = 0\). Then \(\lambda_1\) and \(\nu^1\) must satisfy

\[
\begin{bmatrix}
 0 \\ 0
\end{bmatrix} = \begin{bmatrix}
 \lambda_1 - b_{11} & -b_{12} \\
 -b_{21} & \lambda_1 - b_{22}
\end{bmatrix} \begin{bmatrix} 0 \\ v_{21} \end{bmatrix} = \begin{bmatrix} -b_{12} \\ v_{21} \end{bmatrix} \begin{bmatrix} \lambda_1 - b_{22} \\ \lambda_1 - b_{22} \end{bmatrix} v_{21}.
\]

If \(\lambda_1 \neq \lambda_2\) then the ("Vandermonde") matrix on the left is nonsingular, hence postmultiplying both sides of the equation by its inverse we conclude that each \(c_j v^j = 0\). But each \(v^j \neq 0\), hence each \(c_j = 0\); therefore \(v^1\) and \(v^2\) are linearly independent.
Now \( v_{21} \neq 0 \), since \( v^1 \neq 0 \) by definition of an eigenvector. So the above equation could hold only if \( b_{12} = 0 \) and \( \lambda_1 = b_{22} \). But if \( b_{12} = 0 \) then \( \lambda_1 = b_{11} \) and \( \lambda_2 = b_{22} \), so the equality could hold only if \( b_{11} = b_{22} \), contrary to assumption.

Now we show that \( v_{12} = 0 \) if and only if \( b_{12} = 0 \). First, suppose as above that \( b_{12} = 0 \); then \( \lambda_1 = b_{11} \) and \( \lambda_2 = b_{22} \), and \( V \) and \( \Lambda \) must satisfy

\[
\begin{bmatrix}
  b_{11} & 0 \\
  b_{21} & b_{22}
\end{bmatrix}
\begin{bmatrix}
  v_{11} & v_{12} \\
  v_{21} & v_{22}
\end{bmatrix}
= \begin{bmatrix}
  v_{11} & v_{12} \\
  v_{21} & v_{22}
\end{bmatrix}
\begin{bmatrix}
  b_{11} & 0 \\
  0 & b_{22}
\end{bmatrix},
\]

or

\[
\begin{bmatrix}
  b_{11}v_{11} & b_{11}v_{12} \\
  b_{21}v_{11} + b_{22}v_{21} & b_{21}v_{12} + b_{22}v_{22}
\end{bmatrix}
= \begin{bmatrix}
  v_{11}b_{11} & v_{12}b_{22} \\
  v_{21}b_{11} & v_{22}b_{22}
\end{bmatrix}.
\]

Concentrating first on the left columns of these two matrices we see that the equality implies (since \( b_{11} \neq b_{22} \))

\[
v_{21} = \frac{b_{21}}{b_{11} - b_{22}} v_{11},
\]

which implies \( v_{11} \neq 0 \) (since otherwise \( v^1 = 0 \)). Now from the equality of the right columns we have

\[
\begin{bmatrix}
  b_{11} - b_{22} \\
  b_{21}
\end{bmatrix}
\begin{bmatrix}
  v_{12} \\
  v_{22}
\end{bmatrix}
= 0.
\]

Since \( b_{11} \neq b_{22} \), this implies \( v_{12} = 0 \). This shows that \( b_{12} = 0 \) implies \( v_{12} = 0 \).

It remains to show the converse. Let \( v_{12} = 0 \). Then \( \lambda_2 \) must satisfy

\[
\begin{bmatrix}
  \lambda_2 - b_{11} & -b_{12} \\
  -b_{21} & \lambda_2 - b_{22}
\end{bmatrix}
\begin{bmatrix}
  0 \\
  v_{22}
\end{bmatrix}
= \begin{bmatrix}
  -b_{12} \\
  \lambda_2 - b_{22}
\end{bmatrix}
\begin{bmatrix}
  v_{12} \\
  0
\end{bmatrix}.
\]

But since \( v_{12} = 0 \) by hypothesis, and \( v^2 \neq 0 \) by the definition of an eigenvector, therefore \( v_{22} \neq 0 \). It follows that \( b_{12} = 0 \) and \( \lambda_2 = b_{22} \). □

The advantage of this lemma is that if \( b_{12} \neq 0 \), one may normalize the eigenvectors to \( v_{11} = v_{12} = 1 \) and \( v_{21} = v_1 \) and \( v_{22} = v_2 \).\(^2\) The quantities \( v_1 \) and \( v_2 \) are known as “distribution coefficients.” They may be solved for in the same way as the eigenvalues \( \lambda_1 \) and \( \lambda_2 \). Since an eigenvalue \( \lambda \) and the corresponding eigenvector \( v \) satisfy

\[
[I - \lambda B] v = 0 \quad (v \neq 0),
\]

the eigenvalues are obtained from

\[
(1.11) \quad \begin{vmatrix}
  \lambda - b_{11} & -b_{12} \\
  -b_{21} & \lambda - b_{22}
\end{vmatrix}
= \lambda^2 - (b_{11} + b_{22})\lambda + (b_{11}b_{22} - b_{21}b_{12}) = 0,
\]

which has the well-known solution

\[
(1.12) \quad \lambda = \frac{b_{11} + b_{22} \pm \sqrt{(b_{11} - b_{22})^2 + 4b_{21}b_{12}}}{2}.
\]

---

\(^2\)This is the procedure followed in many textbooks, e.g., Andronov et al. (1966, p. 257), who state that this general form is “well known” and follows under the assumption that “both roots have real parts different from zero and that there are no multiple roots.” However, the above Lemma is true without the first of these conditions; and untrue without the condition \( b_{12} = 0 \).
To solve for the $v$s we write

$$
\begin{pmatrix}
\lambda - b_{11} & -b_{12} \\
-b_{21} & \lambda - b_{22}
\end{pmatrix}
\begin{pmatrix}
v \\
v
\end{pmatrix}
= \begin{pmatrix}
\lambda - b_{11} - b_{12}v \\
-b_{21} + (\lambda - b_{22})v
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
$$

Eliminating $\lambda$ from the pair of equations on the right we obtain

$$b_{21} = (\lambda - b_{22})v = (b_{12}v + b_{11} - b_{22})v,$$

or

$$b_{12}v^2 + (b_{11} - b_{22})v - b_{21} = 0.$$

This has the solution

$$v = \frac{b_{22} - b_{11} \pm \sqrt{(b_{11} - b_{22})^2 + 4b_{21}b_{12}}}{2b_{12}}.$$

Comparing (1.14) with (1.12) we see—as can also be verified directly from (1.13)—that $\lambda_j$ and $v_j$ are related by

$$v_j = \frac{\lambda_j - b_{11}}{b_{12}} = \frac{b_{21}}{\lambda_j - b_{22}}.$$

Applying these results to Krugman’s $B$ given by (1.3) above, and denoting $\bar{\pi}_1 = \pi_1(\bar{m}/\bar{F})$, we obtain, taking $\lambda_1$ to be the smaller of the two eigenvalues,

$$\begin{align*}
\lambda_1 &= -\frac{1}{2}[C_2 + (\bar{m}/\bar{F})\bar{\pi}] + \frac{1}{2}\sqrt{(C_2 - (\bar{m}/\bar{F})\bar{\pi})^2 - 4C_2(\bar{m}/\bar{F})^2\bar{\pi}_1}; \\
\lambda_2 &= -\frac{1}{2}[C_2 + (\bar{m}/\bar{F})\bar{\pi}] + \frac{1}{2}\sqrt{(C_2 - (\bar{m}/\bar{F})\bar{\pi})^2 - 4C_2(\bar{m}/\bar{F})^2\bar{\pi}_1}.
\end{align*}$$

(Notice that for the first term in the discriminant Krugman (1979, p. 324) wrote a $+$ sign where it should be a $-$ sign. Krugman’s formula for $\lambda_2$ also contains the misprint $\pi_2$ for $\pi_1$.) The nature of the eigenvalues may be determined as follows. First one may observe, writing (1.11) as

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + (\lambda_1\lambda_2) = 0,$$

and noting from (1.11) and (1.17) that

$$\begin{align*}
\text{tr } B &= b_{11} + b_{22} = \lambda_1 + \lambda_2 \\
\det B &= b_{11}b_{22} - b_{21}b_{12} = \lambda_1\lambda_2,
\end{align*}$$

that $\det B \geq 0$ in two possible cases: (i) $\lambda_1$ and $\lambda_2$ are complex conjugates $\lambda_1 = \mu - i\nu$ and $\lambda_2 = \mu + i\nu$ (where $i = \sqrt{-1}$ and $\nu \neq 0$), hence $\lambda_1\lambda_2 = \mu^2 + \nu^2 > 0$; and (ii) $\lambda_1$ and $\lambda_2$ are real of like sign (including zero). It follows that $\det B < 0$ if and only if $\lambda_1$ and $\lambda_2$ are real of opposite sign, in which case the origin of the system of differential equations is a saddlepoint. We verify that this is the case with Krugman’s $B$, since

$$|B| = \begin{vmatrix}
-(\bar{m}/\bar{F})\bar{\pi}_1 & (\bar{m}/\bar{F})^2\bar{\pi}_1 \\
-C_2 & -C_2
\end{vmatrix} = C_2\bar{\pi}_1 \bar{m} \bar{F} \left( 1 + \frac{\bar{m}}{\bar{F}} \right) < 0,$$

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since \( \pi_1 < 0 \) (from Krugman’s equation (9)) and \( C_2 > 0 \) (a positive wealth effect on consumption). This by itself proves that the discriminant in (1.16) is positive and that the larger \( \lambda_2 \) is positive and the smaller \( \lambda_1 \) negative.

Now let us consider the distribution coefficients. From (1.14) and (1.3) we have, corresponding to the eigenvalues of (1.16),

\[
\begin{align*}
v_1 &= \frac{(\bar{m}/\bar{F})\pi_1 - C_2 - \sqrt{[C_2 - (\bar{m}/\bar{F})\pi_1]^2 - 4C_2(\bar{m}/\bar{F})^2\pi_1}}{2(\bar{m}/\bar{F})^2\pi_1}, \\
v_2 &= \frac{(\bar{m}/\bar{F})\pi_1 - C_2 + \sqrt{[C_2 - (\bar{m}/\bar{F})\pi_1]^2 - 4C_2(\bar{m}/\bar{F})^2\pi_1}}{2(\bar{m}/\bar{F})^2\pi_1}.
\end{align*}
\]

Since \( \pi_1 < 0 \), the discriminant is certainly positive, and the numerator of \( v_1 \) is negative; since the denominator of \( v_1 \) is also negative, \( v_1 > 0 \).

From (1.10), the stable branch of the solution is given by

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \begin{bmatrix}
  1 \\
  v_1
\end{bmatrix} e^{\lambda_1 t},
\]

hence the stable separatrix is defined by

\[
u_2 = v_1 u_1, \quad \text{or} \quad F - \bar{F} = v_1 (m - \bar{m}).
\]

This may, using (1.3) and the first equation of (1.15), be written as

\[
m - \bar{m} = \frac{1}{v_1} (F - \bar{F}) = \frac{(\bar{m}/\bar{F})^2\pi_1}{\lambda_1 + (\bar{m}/\bar{F})\pi_1} (F - \bar{F}).
\]

(Krugman’s formula contains the misprint \( \pi_1 \) as the first term in the denominator of the above expression in place of \( \lambda_1 \)—cf. Krugman (1979, p. 325).)

Krugman’s Figure 2 (p. 317) shows the slope \( 1/v_1 \) to be flatter than that defined by the locus \( \bar{m} = \bar{u}_1 = 0 \), which is (from (1.2)) \( b_{11} u_1 + b_{12} u_2 = 0 \), i.e., that

\[
\frac{1}{v_1} < \frac{du_1}{du_2} = -\frac{b_{12}}{b_{11}} = \frac{\bar{m}}{\bar{F}}.
\]

This follows from (1.20), since (1.21) is equivalent to

\[
\frac{(\bar{m}/\bar{F})\pi_1}{\lambda_1 + (\bar{m}/\bar{F})\pi_1} < 1,
\]

which follows from the fact that \( \bar{m}/\bar{F} > 0 \), \( \bar{\pi}_1 < 0 \), and \( \lambda_1 < 0 \).

In a similar way we see that the unstable separatrix is given by

\[
u_2 = v_2 u_1, \quad \text{or} \quad F - \bar{F} = v_2 (m - \bar{m}).
\]

The numerator of \( v_2 \) is positive if and only if

\[
\sqrt{[C_2 - (\bar{m}/\bar{F})\pi_1]^2 - 4C_2(\bar{m}/\bar{F})^2\pi_1} > C_2 - (\bar{m}/\bar{F})\pi_1.
\]
The right side is positive, since $\bar{\pi}_1 < 0$, hence we can square both sides to obtain the inequality

$$[C_2 - (\bar{m}/\bar{F})\bar{\pi}]^2 - 4C_2(\bar{m}/\bar{F})^2\bar{\pi}_1 > [C_2 - (\bar{m}/\bar{F})\bar{\pi}_1]^2,$$

which holds since the second term on the left is negative (because $\bar{\pi}_1 < 0$). Since the denominator of $v_2$ is negative, it follows that $v_2 < 0$; hence the unstable separatrix has a negative slope.

2 Fixed exchange rate

The system of differential equations in this section, while not explicitly formulated in Krugman’s paper, appears to consist of two variables, wealth ($W$) and reserves ($R$). However, the stock of reserves plays only a passive role (it cannot become negative). The system consists of the two differential equations

\begin{align}
\dot{W} &= Y - T - C(Y - T, W);
\dot{R} &= L[Y - T - C(Y - T, W)] - (G - T).
\end{align}

(2.1)

The first equation states that the growth of wealth is equal to the saving rate. The second is based on the assumption that “the government can pay for its deficit $G - T$ either by issuing new domestic money or by drawing on its reserves of foreign money $R$”, which he writes

$$\dot{M}/P + \dot{R} = G - T = g \cdot (M/P),$$

(equation (15), p. 318), it having been assumed (on p. 315) that “the government adjusts its expenditure so as to keep the deficit a constant fraction of the [real] money supply.” Note, however, that if the government finances its deficit by drawing down its reserves, the plus sign in the above equation should be a minus sign, i.e., the correct equation is

\begin{align}
\dot{M}/P - \dot{R} &= G - T = g \cdot (M/P).
\end{align}

(2.2)

With this correction, and using the assumption that a proportion $L$ of saving goes into increased domestic money holdings $\dot{M}/P$, the second equation of (2.1) results.

The system (2.1) has the peculiarity that $R$ does not appear as a state variable. If we were to try to linearize it, we would obtain, with $W$ as the only state variable,

$$B = \begin{bmatrix}
-LC_2 & 0 \\
-C_2 & 0
\end{bmatrix}$$

with eigenvalues $\lambda_1 = -LC_2 < 0$ and $\lambda_2 = 0$. Krugman appears to treat the government deficit $G - T$ as a parameter. Since the condition $b_{12} \neq 0$ of the above Lemma does not hold, we need to try a different normalization. From $BV = VA$ we get

$$\begin{bmatrix}
-LC_2 & 0 \\
-C_2 & 0
\end{bmatrix} \begin{bmatrix}
v_{11} & v_{12} \\
v_{21} & v_{22}
\end{bmatrix} = \begin{bmatrix}
v_{11} & v_{12} \\
v_{21} & v_{22}
\end{bmatrix} \begin{bmatrix}
-LC_2 & 0 \\
0 & 0
\end{bmatrix},$$

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or
\[
\begin{bmatrix}
-\lambda_2 v_{11} & -\lambda_2 v_{12} \\
-\lambda_2 v_{11} & -\lambda_2 v_{12}
\end{bmatrix}
= \begin{bmatrix}
-\lambda_2 v_{11} & 0 \\
-\lambda_2 v_{21} & 0
\end{bmatrix}.
\]

Equating the first columns we see that \( v_{11} = L v_{21} \), and equating the second columns we see that, in accordance with the Lemma, \( v_{12} = 0 \). Accordingly we may choose
\[
V = \begin{bmatrix} L & 0 \\ 1 & 1 \end{bmatrix}.
\]

The solution of the linearized system is then
\[
\begin{bmatrix} W - \bar{W} \\ R - \bar{R} \end{bmatrix} = \begin{bmatrix} L \\ 1 \end{bmatrix} c_1 e^{-\lambda_2 t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} c_2,
\]
which converges to the constant term \( [c_1 \bar{c}_2] \). If \( G - T > 0 \), presumably \( c_2 < 0 \), showing that reserves keep falling, until eventually they become zero. If \( G - T = 0 \) then \( \bar{R} = L \bar{W} \), in which case \( c_2 = 0 \).

References

