1 The classical transfer problem

Since each country has just one factor of production, and production is carried out at constant returns to scale, the production function for commodity $j$ in country $k$ is

\[(S1.1) \quad y_j^k = f_j^k(v_j^k) = \lambda^{-1} f_j^k(\lambda v_j^k) \quad \text{for} \quad \lambda > 0, \ j = k, 3 \]

where $v_j^k$ is the amount of the single factor employed in industry $j$ in country $k$ and $y_j^k$ is the output of commodity $j$ in country $k$. Choosing $\lambda = 1/v_j^k$ in (S1.1) and defining the factor-output coefficient $b_j^k = v_j^k/y_j^k$ we see that

\[(S1.2) \quad b_j^k = \frac{1}{f_j^k(1)} \quad \text{for} \quad j = k, 3. \]

Thus, the factor-output coefficients $b_j^k$ are constant. The resource-allocation constraint therefore becomes

\[(S1.3) \quad v_k^k + v_3^k = b_k^k y_k^k + b_3^k y_3^k = l^k, \]

where $l^k$ is country $k$’s endowment in the unique factor. (S1.3) is the equation of a straight line. Since it is the equation of country $k$’s production-possibility frontier, the Rybczynski (supply) correspondence is the set of pairs $y_k^k, y_3^k$ which maximize the domestic product $p_k^k y_k^k + p_3^k y_3^k$ subject to (S1.3). Now it is easy to see that if $p_k^k/p_3^k > b_k^k/b_3^k$ then country $k$ will specialize in its export good, commodity $k$, and produce none of its nontradable good, commodity $3$. On the other hand, if $p_k^k/p_3^k < b_k^k/b_3^k$ then country $k$ will specialize in its nontradable good, commodity $3$, and produce none of its export good, commodity $k$. Since by hypothesis both of these goods are produced, it therefore follows that

\[(S1.4) \quad \frac{p_k^k}{p_3^k} = \frac{b_k^k}{b_3^k}. \]
This also follows directly from the fact that the minimum-unit cost functions dual to (S1.1) are
\( g_j^k(w) = w b_j^k, \)
whence \( w b_j^k = p_j \) for the produced commodities \( j = k, 3 \). Equation (S1.4) expresses the Ricardian labor theory of value, since if the single factor is labor it states that in a condition of autarky the value of a unit of each commodity is proportional to the amount of labor required to produce it. In this case it also holds under free trade, but only with respect to the two produced commodities.

Now let the aggregate consumer demand function for commodity \( i = 1, 2, 3 \) in country \( k \) be denoted
\[
x_i^k = h_i^k(p_1, p_2, p_3^k, Y^k) \quad (k = 1, 2)
\]
where \( Y^k \) is total disposable income
\[
Y^k = \frac{l^k}{b_k^k} p_k + D^k,
\]
i.e., equal to the sum of country \( k \)'s domestic product
\[
p_k^y y_k^k + p_3^k y_3^k = \frac{p_k^y}{b_k^y} \left( b_k^y y_k^k + b_3^k y_3^k \right) = p_k \frac{l^k}{b_k^k} \quad (k = 1, 2)
\]
(\text{where we have used (S1.4) and (S1.3)}) and the deficit in its balance of trade, \( D^k \); and since by definition of a nontradable we have \( x_3^k = y_3^k \), we have from (S1.6), using (S1.4) and (S1.8),
\[
y_3^k = h_3^k \left( p_1, p_2, \frac{b_3^k}{b_k^3} p_k, \frac{l^k}{b_k^k} p_k + D^k \right) \quad (k = 1, 2).
\]
\text{(a) Now, noting from (S1.3) and (S1.9) that}
\[
y_k^k = \frac{p_k^l}{b_k^l} - \frac{b_k^l}{b_k^3} h_3^k \left( p_1, p_2, \frac{b_3^k}{b_k^3} p_k, \frac{l^k}{b_k^k} p_k + D^k \right),
\]
we may define country \( k \)'s trade-demand functions by
\[
z_j^k = \hat{h}_j^k(p_1, p_2, D^k, t^k) \equiv h_j^k \left( p_1, p_2, \frac{b_j^k}{b_k^j} p_k, \frac{l^k}{b_k^k} p_k + D^k \right) \quad (j \neq k), \text{ and}
\]
\[
z_k^k = \hat{h}_k^k(p_1, p_2, D^k, t^k) \equiv h_k^k \left( p_1, p_2, \frac{b_k^k}{b_k^3} p_k, \frac{l^k}{b_k^k} p_k + D^k \right)
- \frac{1}{b_k^3} \frac{l^k}{b_k^3} h_3^k \left( p_1, p_2, \frac{b_3^k}{b_k^3} p_k, \frac{l^k}{b_k^k} p_k + D^k \right)
\]
respectively.
Differentiating these two functions with respect to the deficit, $D^k$, one readily obtains the formulas given in the question. □

(b) World equilibrium requires that

$$h_2^1(p_1, p_2, -T; l^1) + h_2^2(p_1, p_2, T; l^2) = 0.$$  

If $p_1$ is taken to be the numéraire and held fixed, (S1.11) implicitly defines the function

$$p_2 = \tilde{p}_2(T),$$

where the fixed arguments $p_1, l_1$, and $l_2$ are omitted. Substituting (S1.12) into (S1.11) and differentiating, one obtains

$$\left( \frac{\partial h_2^1}{\partial p_2} + \frac{\partial h_2^2}{\partial p_2} \right) \frac{dp_2}{dT} = \frac{\partial h_2^1}{\partial D^1} - \frac{\partial h_2^2}{\partial D^2}. $$

By dynamic stability, the term in parentheses on the left must be negative; it follows that $dp_2/dT > 0$ if and only if

$$\frac{\partial h_2^2}{\partial Y^2} + \frac{b_3^2}{b_2^2} \frac{\partial h_3^2}{\partial Y^2} > \frac{\partial h_2^1}{\partial Y^1}. $$

From (S1.10) this reduces to

$$\frac{\partial h_2^2}{\partial Y^2} + \frac{b_3^2}{b_2^2} \frac{\partial h_3^2}{\partial Y^2} > \frac{\partial h_2^1}{\partial Y^1}. $$

A sufficient condition for this to hold is that

$$\frac{\partial h_2^2}{\partial Y^2} = \frac{\partial h_2^1}{\partial Y^1}$$

and

$$\frac{\partial h_3^2}{\partial Y^2} > 0.$$  

Since preferences have been assumed to be identical and homothetic across the two countries, one might be tempted to believe that this implies $\partial h_2^2/\partial Y^2 = \partial h_2^1/\partial Y^1$; but while the demand functions $h_2^2$ and $h_2^1$ are identical, their arguments are not, since $p_3^2 \neq p_3^1$. Thus we must look for a form of demand function for a tradable that is independent of the price of the nontraded. This is accomplished by the given utility function, $U(x_1^k, x_2^k, x_3^k) = \sum_{i=1}^3 \theta_i \log x_i^k$, which generates the same preferences in both countries, and yields the identical demand functions

$$h_j^k(p_1, p_2, p_3^k, Y^k) = \frac{\theta_j Y^k}{p_j^k} \quad (j = 1, 2, 3; \; k = 1, 2).$$

Since $p_2^k = p_2$ for $k = 1, 2$, we have

$$\frac{\partial h_2^k}{\partial Y^k} = \frac{\theta_2}{p_2} \quad \text{for } k = 1, 2 \quad \text{and} \quad \frac{\partial h_3^k}{\partial Y^k} = \frac{\theta_3}{p_3^k} > 0 \quad (k = 1, 2).$$
Thus (S1.16), and therefore (S1.15), holds. □

We assume that the demand for gold in each country is given by the Cambridge equation

(S1.17) \[ G^k = \kappa Y^k \quad (k = 1, 2), \]

(by assumption the same \(\kappa\) applying to both countries) and that these are equal to the equilibrium stocks of gold in each country, which must satisfy

(S1.18) \[ G^1 + G^2 = G, \]

where \(G\) is the world stock of gold, assumed fixed. Prices in each country must now be expressed relatively to gold.

Clearly, we may no longer retain the normalization \(p_1 = \tilde{p}_1\). From (S1.9) and (S1.17) we have

(S1.19) \[ G^1 = \kappa \left( \frac{p_1}{b_1} l^1 p_1 - T \right) \quad \text{and} \quad G^2 = \kappa \left( \frac{p_2}{b_2} l^2 p_2 + T \right). \]

Summing these equations and using (S1.18) we obtain

(S1.20) \[ \kappa \left( \frac{p_1}{b_1} l^1 + \frac{p_2}{b_2} l^2 \right) = G. \]

This is the new normalization of the prices, replacing \(p_1 = \tilde{p}_1\).

World equilibrium is now defined by the system

(S1.21) \[ \dot{h}^1_2(p_1, p_2, -T; l^1) + \dot{h}^2_2(p_1, p_2, T; l^2) = 0 \]
\[ \kappa \left( \frac{p_1}{b_1} l^1 + \frac{p_2}{b_2} l^2 \right) = G. \]

These equations implicitly define the functions

(S1.22) \[ p_k = \tilde{p}_k(T; l^1, l^2) \quad (k = 1, 2). \]

Substituting (S1.22) into (S1.21) and differentiating, we obtain

(S1.23) \[ \begin{bmatrix} \frac{\partial \dot{h}^1_2}{\partial p_1} + \frac{\partial \dot{h}^2_2}{\partial p_1} \frac{\partial \dot{h}^1_2}{\partial p_2} + \frac{\partial \dot{h}^2_2}{\partial p_2} \\ \kappa \frac{l^1}{b_1} \end{bmatrix} \begin{bmatrix} \frac{\partial p_1}{\partial T} \\ \frac{\partial p_2}{\partial T} \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{h}^1_2}{\partial D^1} - \frac{\partial \dot{h}^2_2}{\partial D^2} \\ 0 \end{bmatrix}. \]

The Jacobian matrix on the left in (S1.23) has as its determinant

(S1.24) \[ \Delta = \left( \frac{\partial \dot{h}^1_2}{\partial p_1} + \frac{\partial \dot{h}^2_2}{\partial p_1} \right) \kappa \frac{l^2}{b_2} - \left( \frac{\partial \dot{h}^1_2}{\partial p_2} + \frac{\partial \dot{h}^2_2}{\partial p_2} \right) \kappa \frac{l^1}{b_1}. \]
We may assume as dynamic stability conditions the inequalities

\[
\frac{\partial \hat{h}_1^1}{\partial p_1} + \frac{\partial \hat{h}_2^2}{\partial p_2} > 0 \quad \text{and} \quad \frac{\partial \hat{h}_1^1}{\partial p_2} + \frac{\partial \hat{h}_2^2}{\partial p_1} < 0.
\]

Then \( \Delta > 0 \). Accordingly, the partial derivatives \( \partial \hat{p}_k / \partial T \) for \( k = 1, 2 \) are given by

\[
\begin{bmatrix}
\frac{\partial \hat{p}_1}{\partial T} \\
\frac{\partial \hat{p}_2}{\partial T}
\end{bmatrix}
= \frac{1}{\Delta} \begin{bmatrix}
\frac{\kappa l^2}{b_2^1} - \frac{\partial \hat{h}_1^1}{\partial p_2} - \frac{\partial \hat{h}_2^2}{\partial p_2} \\
-\frac{\kappa l^1}{b_1^1} + \frac{\partial \hat{h}_1^1}{\partial p_1} + \frac{\partial \hat{h}_2^2}{\partial p_1}
\end{bmatrix}
= \frac{\kappa}{\Delta} \left( \frac{\partial \hat{h}_2^2}{\partial D^2} - \frac{\partial \hat{h}_1^1}{\partial D^1} \right) \begin{bmatrix}
\frac{l^2}{b_2^1} - \frac{l^1}{b_1^1}
\end{bmatrix}.
\]

Assuming (S1.14) to hold, it follows that \( \partial \hat{p}_1 / \partial T < 0 \) and \( \partial \hat{p}_2 / \partial T > 0 \), i.e., country 1’s export price declines absolutely and country 2’s rises absolutely.

It is not hard to show (but this is not part of the question) that the price of country 1’s nontradable falls absolutely and that of country 2’s rises absolutely, hence a Laspeyres index of country 1’s prices falls and a Laspeyres index of country 2’s prices rises.

To show that gold must move from country 1 to country 2, upon substituting (S1.26) in (S1.19) to define the functions

\[
\hat{G}^1(T) = \kappa \left( \frac{l^1}{b_1^1} \hat{p}_1(T) - T \right) \quad \text{and} \quad \hat{G}^2(T) = \kappa \left( \frac{l^2}{b_2^2} \hat{p}_2(T) + T \right),
\]

we see that

\[
\frac{\partial \hat{G}^1}{\partial T} = \kappa \left( \frac{l^1}{b_1^1} \frac{\partial \hat{p}_1}{\partial T} - 1 \right) < 0 \quad \text{and} \quad \frac{\partial \hat{G}^2}{\partial T} = \kappa \left( \frac{l^2}{b_2^2} \frac{\partial \hat{p}_2}{\partial T} + 1 \right) > 0. \quad \Box
\]

(d) Let us denote the pre-transfer consumption of commodity \( i \) in country \( k \) (when \( T = 0 \)) by \( \bar{x}_i^k \). Then from the assumption that the monetary authorities in the two countries stabilize their respective price levels, we have

\[
\bar{x}_1^k \hat{p}_1^k + \bar{x}_2^k \hat{p}_2^k + \bar{x}_3^k \hat{p}_3^k = \hat{p}_k^k \quad \text{for} \ k = 1, 2.
\]

However, from (S1.4) we have

\[
\hat{p}_3^k = \frac{l^1}{b_2^1} \hat{p}_2^k \quad (k = 1, 2).
\]

Substituting (S1.30) in (S1.29) we obtain

\[
\bar{x}_j^k \hat{p}_j^k + (\bar{x}_k^k + \bar{x}_3^k \frac{l^1}{b_2^1}) \hat{p}_k^k = \hat{p}_j^k \quad (j \neq k, k = 1, 2).
\]
Moreover, to take account of varying exchange rates we posit

\[ p_i^2 = e p_i^1 \quad \text{for } i = 1, 2. \]

Thus, \( e \) is the price of country 1’s currency in units of country 2’s; a rise in \( e \) is an appreciation of country 1’s currency. We want to show that a transfer from country 1 to country 2 will result in a fall in \( e \).

Putting together (S.31) and (S.32) we obtain the two equations

\[
\begin{align*}
\left( \bar{x}_1^1 + \bar{x}_1^1 \frac{b_1^1}{b_1^1} \right) p_1^1 + \bar{x}_2^1 p_2^1 &= \bar{p}_1^1 \\
\bar{x}_2^2 e p_1^1 + \left( \bar{x}_2^2 + \bar{x}_3^2 \frac{b_2^2}{b_2^2} \right) e p_2^1 &= \bar{p}_2^1.
\end{align*}
\]

These constitute the normalization replacing that of \( p_1^1 = p_2^1 = \bar{p}_1 \).

The equation of world equilibrium is now

\[ (S.34) \quad \hat{h}_1^1(p_1^1, p_2^1, -T; l^1) + \hat{h}_2^2(e p_1^1, e p_2^1, T; l^2) = 0. \]

This equation, together with the two normalization equations (S.33), determines the three unknowns \( p_1^1, p_2^1, \) and \( e \) as functions of \( T \) as well as \( l^1 \) and \( l^2 \). Breves over these three variables will indicate these functions. For simplicity we assume that the exchange rate \( e \) is initially \( = 1 \). From (S.34) and (S.33) we obtain the Jacobian system (evaluated at the initial pre-transfer equilibrium)

\[ (S.35) \quad \begin{bmatrix}
\frac{\partial \hat{h}_1^1}{\partial p_1^1} + \frac{\partial \hat{h}_2^2}{\partial p_2^1} & \frac{\partial \hat{h}_1^1}{\partial p_2^1} + \frac{\partial \hat{h}_2^2}{\partial p_2^1} & 0 \\
\bar{x}_1^2 + \bar{x}_2^2 & 0 & \bar{x}_1^2 p_1^1 + \left( \bar{x}_2^2 + \bar{x}_3^2 \frac{b_2^2}{b_2^2} \right) p_2^1 \\
\bar{x}_1^1 + \bar{x}_2^1 \frac{b_1^1}{b_1^1} & \bar{x}_2^2 + \bar{x}_3^2 \frac{b_2^2}{b_2^2} & \bar{x}_1^2 p_1^1 + \left( \bar{x}_2^2 + \bar{x}_3^2 \frac{b_2^2}{b_2^2} \right) p_2^1
\end{bmatrix}
\begin{bmatrix}
\frac{\partial p_1^1}{\partial T} \\
\frac{\partial p_2^1}{\partial T} \\
\frac{\partial e}{\partial T}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial \hat{h}_1^1}{\partial D^1} - \frac{\partial \hat{h}_2^2}{\partial D^2} \\
0 \\
0
\end{bmatrix},
\]

where in the top-right element of the Jacobian we have made use of the homogeneity of degree zero of country 2’s trade-demand function \( \hat{h}_2^2(p_1^2, p_2^2, D^2; l^2) \) in \( p_1^2, p_2^2, D^2, \) and the fact that trade is balanced in the initial equilibrium, so that

\[
\frac{\partial \hat{h}_2^2}{\partial p_1^2} p_1^2 + \frac{\partial \hat{h}_2^2}{\partial p_2^2} p_2^2 = -\frac{\partial \hat{h}_2^2}{\partial D^2} D^2 = 0.
\]

The system of equations (S.35) is block-triangular, hence it may be solved recursively, as follows. From the top-left block of (S.35), namely

\[ (S.36) \quad \begin{bmatrix}
\frac{\partial \hat{h}_1^1}{\partial p_1^1} + \frac{\partial \hat{h}_2^2}{\partial p_2^1} & \frac{\partial \hat{h}_1^1}{\partial p_2^1} + \frac{\partial \hat{h}_2^2}{\partial p_2^1} \\
\bar{x}_1^2 + \bar{x}_2^2 \frac{b_2^2}{b_2^2} & \bar{x}_1^2 \end{bmatrix}
\begin{bmatrix}
\frac{\partial p_1^1}{\partial T} \\
\frac{\partial p_2^1}{\partial T}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial \hat{h}_1^1}{\partial D^1} - \frac{\partial \hat{h}_2^2}{\partial D^2} \\
0
\end{bmatrix},
\]
we note from (S1.25) that the determinant $\Delta_1$ of the $2 \times 2$ matrix is positive, and

\[
\begin{bmatrix}
\frac{\partial p_1^1}{\partial T} \\
\frac{\partial p_2^1}{\partial T}
\end{bmatrix} = \frac{1}{\Delta_1} \begin{bmatrix}
x_2^1 - x_1^1 \frac{b_1^1}{b_2^1} \\
-\frac{x_1^2}{\bar{x}_1^1} - \frac{x_2^2}{\bar{x}_2^1} \frac{b_1^1}{b_2^1}
\end{bmatrix} \begin{bmatrix}
-\frac{\partial h_1^1}{\partial p_1^1} - \frac{\partial h_2^1}{\partial p_2^1} \\
\frac{\partial h_1^1}{\partial p_1^1} + \frac{\partial h_2^1}{\partial p_2^1}
\end{bmatrix} \begin{bmatrix}
\frac{\partial h_1^2}{\partial D^1} - \frac{\partial h_2^2}{\partial D^2} \\
0
\end{bmatrix}
\]

(S1.37)

Thus, as long as (S1.14) holds, $\partial p_1^1/\partial T < 0$ and $\partial p_2^1/\partial T > 0$. Now from the third equation of (S1.35) we have

\[
\frac{\partial \bar{e}}{\partial T} = \frac{\bar{x}_1^2 \frac{\partial p_1^1}{\partial T} + (\bar{x}_2^2 + \bar{x}_2^2) \frac{\partial p_1^1}{\partial T}}{\bar{x}_2^2 p_1^1 + (\bar{x}_2^2 + \bar{x}_2^2) p_2^1},
\]

where the $p_i^1$s and their derivatives are evaluated at $T = 0$. We wish to show that $\partial \bar{e}/\partial T < 0$, i.e., that the transfer from country 1 to country 2 results in a depreciation in country 1’s (an appreciation in country 2’s) currency. The denominator of (S1.38) is clearly positive; thus, we wish to show that the numerator is positive. Since $\Delta_1 > 0$, and (S1.14) holds, in order to show that the numerator of (S1.38) is positive, it is enough (in view of (S1.37)) to show that the inner product of the vectors

\[
(x_1^2, x_2^2 + \frac{x_2^2}{\bar{x}_2^1} b_2^2) \cdot (-x_1^2, x_1^2 + \frac{b_1^1}{\bar{x}_1^1} b_2^1)
\]

(S1.39)

is positive.

Now we need to make use of the assumption about consumer preferences. In the present case (S1.8) becomes $Y^k = l^k p_k^1 / b_k^1$, and in the initial equilibrium we have (using (S1.4))

\[
\bar{x}_i^k = \frac{\theta_i Y^k}{p_i^k} = \frac{\theta_i l^k}{b_k^1 p_i^k} \quad \text{and} \quad \bar{x}_3^k b_3^k b_2^1 = \frac{\theta_3 l^k}{b_k^1} \quad (i = 1, 2, 3; \ k = 1, 2).
\]

(S1.40)

Consequently the vectors in (S1.39) become (again using (S1.4))

\[
\begin{pmatrix}
\theta_1 l^2 p_2^2 / b_2^1, \theta_2 l^2 / b_2^2 + \theta_3 l^2 / b_2^1 & \\
-\theta_2 l^1 p_1^1 / b_1^1, \theta_1 l^1 / b_1^1 + \theta_3 l^1 / b_1^1 \end{pmatrix}.
\]

(S1.41)

We note that

\[
\begin{pmatrix}
\bar{x}_1^2 \bar{x}_2^2 \\
-\bar{x}_1^2 \bar{x}_2^2 + \bar{x}_1^2 \bar{x}_2^1
\end{pmatrix} = -\bar{x}_1^2 \bar{x}_2^1 + \bar{x}_2^2 \bar{x}_1^1 = -\frac{\theta_1 \theta_2 l^2 l^1}{b_2^1 b_1^1} + \frac{\theta_3 \theta_1 l^2 l^1}{b_2^1 b_1^1} = 0.
\]

The inner product (S1.39) then reduces to

\[
\begin{vmatrix}
\frac{\theta_1 l^1 l^2}{b_2^1 b_1^1} \\
\frac{\theta_1 l^1 l^2}{b_2^1 b_1^1}
\end{vmatrix} = \frac{\theta_3 l^1 l^2}{b_2^1 b_1^1} (\theta_1 + \theta_2 + \theta_3) = \frac{\theta_3 l^1 l^1}{b_2^1 b_1^1} > 0.
\]

This proves the desired result. \(\Box\)

7
References
