1 Incipient and optimal tariffs in a 3-commodity model

Let $T_j = 1 + \tau_j$ denote the tariff factor, and $\tau_j$ the tariff rate, imposed by country 1 on its import of commodity $j$ $(j = 2, 3)$, and let $p_j^1$ denote the price of commodity $j$ on country 1’s markets, while $p_j$ denotes the price on the world market. Denoting country 1’s trade-demand function for commodity $j$ by

$$z_j^1 = h_j^1(p_1^1, p_2^1, p_3^1, D^1; t^1),$$

where $D^1$ is the deficit in its balance of payments on current account and $t^1$ is the vector of its factor endowments, we may define country 1’s tariff-inclusive demand-for-import functions

(S1.1) $$\hat{z}_j^1(p_1, p_2, p_3, T_2, T_3, t^1) \quad (j = 2, 3)$$

implicitly by the two equations

(S1.2)

$$\hat{z}_2^1(\cdot) = \hat{h}_2^1(p_1, T_2p_2, T_3p_3, (T_2 - 1)p_2\hat{z}_2^1(\cdot) + (T_3 - 1)p_3\hat{z}_3^1(\cdot); t^1)$$

$$\hat{z}_3^1(\cdot) = \hat{h}_3^1(p_1, T_2p_2, T_3p_3, (T_2 - 1)p_2\hat{z}_2^1(\cdot) + (T_3 - 1)p_3\hat{z}_3^1(\cdot); t^1).$$

Assuming country 2 to have no trade restrictions, its excess-demand functions for commodities 2 and 3 (its two export goods) are defined by

(S1.3) $$\hat{z}_j^2(p_1, p_2, p_3, t^2) = \hat{h}_2^2(p_1, p_2, p_3, 0; t^2) \quad (j = 2, 3).$$

(a) The functions $p_i = \tilde{p}_i(T_2, T_3)$ $(i = 2, 3)$ are defined implicitly by the following equations of world equilibrium, where the price $p_1^1 = p_1^2 = \tilde{p}_1$ of commodity 1 is taken as numéraire and held fixed:

(S1.4)

$$\hat{z}_2^1(\tilde{p}_1, \tilde{p}_2(\cdot), \tilde{p}_3(\cdot), T_2, T_3, t^1) + \hat{z}_2^2(\tilde{p}_1, \tilde{p}_2(\cdot), \tilde{p}_3(\cdot), t^2) = 0$$

$$\hat{z}_3^1(\tilde{p}_1, \tilde{p}_2(\cdot), \tilde{p}_3(\cdot), T_2, T_3, t^1) + \hat{z}_3^2(\tilde{p}_1, \tilde{p}_2(\cdot), \tilde{p}_3(\cdot), t^2) = 0.$$

From these equations we obtain

(S1.5)

$$\left[ \begin{array}{c} \frac{\partial p_2}{\partial T_2} \\ \frac{\partial p_2}{\partial T_3} \\ \frac{\partial p_3}{\partial T_2} \\ \frac{\partial p_3}{\partial T_3} \end{array} \right] = - \left[ \begin{array}{cccc} \frac{\partial \hat{z}_2^1}{\partial p_2} & \frac{\partial \hat{z}_2^1}{\partial p_3} & \frac{\partial \hat{z}_2^1}{\partial p_4} & \frac{\partial \hat{z}_2^2}{\partial p_2} \\ \frac{\partial \hat{z}_2^1}{\partial p_2} & \frac{\partial \hat{z}_2^1}{\partial p_3} & \frac{\partial \hat{z}_2^1}{\partial p_4} & \frac{\partial \hat{z}_2^2}{\partial p_2} \\ \frac{\partial \hat{z}_3^1}{\partial p_2} & \frac{\partial \hat{z}_3^1}{\partial p_3} & \frac{\partial \hat{z}_3^1}{\partial p_4} & \frac{\partial \hat{z}_3^2}{\partial p_2} \\ \frac{\partial \hat{z}_3^1}{\partial p_2} & \frac{\partial \hat{z}_3^1}{\partial p_3} & \frac{\partial \hat{z}_3^1}{\partial p_4} & \frac{\partial \hat{z}_3^2}{\partial p_2} \end{array} \right]^{-1} \left[ \begin{array}{c} \frac{\partial \tilde{p}_1}{\partial T_2} \\ \frac{\partial \tilde{p}_1}{\partial T_3} \\ \frac{\partial \tilde{p}_2}{\partial T_2} \\ \frac{\partial \tilde{p}_2}{\partial T_3} \end{array} \right]$$
provided the inverse matrix exists. The matrix itself is the Jacobian matrix of the transformation (S1.4), and as assumed by the question, dynamic stability implies that its diagonal elements are negative and that its determinant is positive. (For those interested, a proof is given in the Appendix.) Thus in particular, the inverse matrix exists.

Let us now obtain expressions for the $\partial z_i^1 / \partial T_j$ for $i, j = 2, 3$. Differentiating equations (S1.2) with respect to $T_2$ and $T_3$ we obtain

$$
\begin{bmatrix}
\frac{\partial z_2^1}{\partial T_2} & \frac{\partial z_3^1}{\partial T_2} \\
\frac{\partial z_2^1}{\partial T_3} & \frac{\partial z_3^1}{\partial T_3}
\end{bmatrix} = 
\begin{bmatrix}
1 - \frac{T_2 - 1}{T_2} p_2 \frac{\partial h_2^1}{\partial D_1^1} & - \frac{T_3 - 1}{T_3} p_3 \frac{\partial h_2^1}{\partial D_1^1} \\
- \frac{T_2 - 1}{T_2} p_2 \frac{\partial h_3^1}{\partial D_1^1} & 1 - \frac{T_3 - 1}{T_3} p_3 \frac{\partial h_3^1}{\partial D_1^1}
\end{bmatrix}^{-1}
\begin{bmatrix}
\frac{\partial h_2^1}{\partial T_2} \\
\frac{\partial h_3^1}{\partial T_2}
\end{bmatrix}.
$$

(S1.6)

where the

$$
\frac{\partial h^1_i}{\partial T_j} = \frac{\partial h^1_i}{\partial D^1_j} \frac{\partial D^1_j}{\partial T_i}
$$

are country 1’s trade-Slutsky terms, provided the inverse matrix exists. That it does exist follows by computation of the determinant, which is

$$
\Delta = \left| 1 - \frac{T_2 - 1}{T_2} p_2 \frac{\partial h_2^1}{\partial D^1_2} - \frac{T_3 - 1}{T_3} p_3 \frac{\partial h_2^1}{\partial D^1_3} \\
- \frac{T_2 - 1}{T_2} p_2 \frac{\partial h_3^1}{\partial D^1_2} 1 - \frac{T_3 - 1}{T_3} p_3 \frac{\partial h_3^1}{\partial D^1_3} \right|
= 1 - \frac{T_2 - 1}{T_2} p_2 \frac{\partial h_2^1}{\partial D^1_2} - \frac{T_3 - 1}{T_3} p_3 \frac{\partial h_3^1}{\partial D^1_3}.
$$

(S1.7)

Since $(1 - 1/T_i) \in [0, 1)$ for $T_i \in [1, \infty)$, and since if we assume that

$$
p_1 \frac{\partial h^1_i}{\partial D^1} > 0 \text{ and } p_1 \frac{\partial h^1_i}{\partial D^1} \geq 0 \text{ for } i = 2, 3,
$$

it follows that

$$
p_1 \frac{\partial h^1_i}{\partial D^1} < 1 \text{ for } i = 2, 3,
$$

therefore the determinant (S1.7) is necessarily positive. The inverse matrix in (S1.6) is then

$$
\frac{1}{\Delta} \begin{bmatrix}
1 - \frac{T_3 - 1}{T_3} p_3 \frac{\partial h_2^1}{\partial D^1_2} & \frac{T_3 - 1}{T_3} p_3 \frac{\partial h_2^1}{\partial D^1_3} \\
- \frac{T_2 - 1}{T_2} p_2 \frac{\partial h_3^1}{\partial D^1_2} 1 - \frac{T_3 - 1}{T_3} p_3 \frac{\partial h_3^1}{\partial D^1_3}
\end{bmatrix},
$$

which has all its elements nonnegative.

Putting together (S1.5) and (S1.6) we then have

$$
\begin{bmatrix}
\frac{\partial \bar{p}_2}{\partial T_2} & \frac{\partial \bar{p}_2}{\partial T_3} \\
\frac{\partial \bar{p}_3}{\partial T_2} & \frac{\partial \bar{p}_3}{\partial T_3}
\end{bmatrix} =
$$
\[
\begin{align*}
- \begin{bmatrix}
\frac{\partial z_1^1}{\partial p_1} & \frac{\partial z_2^1}{\partial p_2} & \frac{\partial z_1^2}{\partial p_3} & \frac{\partial z_2^2}{\partial p_3} \\
\frac{\partial z_3^1}{\partial p_1} & \frac{\partial z_3^2}{\partial p_2} & \frac{\partial z_3^1}{\partial p_3} & \frac{\partial z_3^2}{\partial p_3}
\end{bmatrix}^{-1}
\begin{bmatrix}
1 & -\frac{T_2-1}{T_2} \frac{\partial h_1^1}{\partial D^1} & -\frac{T_3-1}{T_3} \frac{\partial h_1^1}{\partial D^1} \\
-\frac{T_2-1}{T_2} \frac{\partial h_1^2}{\partial D^1} & 1 & -\frac{T_3-1}{T_3} \frac{\partial h_1^2}{\partial D^1}
\end{bmatrix}^{-1}
\begin{bmatrix}
\frac{p_2^1}{T_2} & 0 \\
0 & \frac{p_3^1}{T_3}
\end{bmatrix}.
\end{align*}
\]

Provided the off-diagonal elements \(\frac{\partial z_i^j}{\partial p_j} + \frac{\partial z_j^i}{\partial p_j}\) are positive for \(i \neq j\)—an assumption unfortunately omitted from the question!—the first inverse matrix has all its elements nonpositive; and the second inverse matrix has all its elements nonnegative. Thus, the \(\frac{\partial \tilde{p}_i}{\partial T_j}\) are proportional to nonnegative weighted averages of the trade-Slutsky terms \(\tilde{s}_{2j}^1, \tilde{s}_{3j}^1\) for \(j = 2, 3\).

When \(T_j = 1\) for \(j = 2, 3\), they are of course proportional to the corresponding trade-Slutsky terms. \(\Box\)

(b) The indirect trade-utility function of country 1 is defined by

\[
\hat{V}^1(p_1^1, p_2^1, p_3^1, D^1; l^1) = \hat{U}^1(h_1^1(p_1^1, p_2^1, p_3^1, D^1; l^1)),
\]

where \(\hat{h}_1^1(\cdot) = (\hat{h}_1^1(\cdot), \hat{h}_2^1(\cdot), \hat{h}_3^1(\cdot))\). Accordingly, we may define country 1’s potential welfare as a function of the two tariff factors by

\[
W_1(T_2, T_3) = \hat{V}^1(\bar{p}_1, \bar{p}_2(\cdot), \bar{p}_3(\cdot), (T_2-1)\bar{p}_2(\cdot)z_2^1(\cdot) + (T_3-1)\bar{p}_3(\cdot)z_3^1(\cdot); l^1),
\]

where \(\bar{p}_i(\cdot)\) denotes \(\bar{p}_i(T_2, T_3)\) and likewise \(\bar{z}_i^1(\cdot)\) denotes

\[
z_1^1(T_2, T_3) = \bar{z}_1^1(p_1, \bar{p}_2(T_2, T_3), \bar{p}_3(T_2, T_3), T_2, T_3, l^1) \quad (i = 2, 3).
\]

Differentiating (S1.9) with respect to \(T_2\) we obtain

\[
\frac{\partial W_1}{\partial T_2} = \frac{\partial \hat{V}^1}{\partial p_1} \left[\frac{\partial \bar{p}_2}{\partial T_2} + T_2 \frac{\partial \bar{p}_2}{\partial T_2}\right] + \frac{\partial \hat{V}^1}{\partial p_2} \left[T_3 \frac{\partial \bar{p}_3}{\partial T_2} + \frac{\partial \hat{V}^1}{\partial D^1}\right] \times
\]

\[
\left[\frac{p_2^1}{T_2} + (T_2-1)z_2^1 \frac{\partial p_2}{\partial T_2} + (T_2-1)z_2^1 \frac{\partial \bar{p}_2}{\partial T_2} + (T_3-1)z_3^1 \frac{\partial p_3}{\partial T_2} + (T_3-1)z_3^1 \frac{\partial \bar{p}_3}{\partial T_2}\right].
\]

Now using Antonelli’s partial differential equation

\[
\frac{\partial \hat{V}^1}{\partial p_j^1} = -\frac{\partial \hat{V}^1}{\partial D^1} \frac{\partial \hat{h}_j^1}{\partial T_2},
\]

this becomes, after cancelling like terms,

\[
\frac{\partial W_1}{\partial T_2} = \frac{\partial \hat{V}^1}{\partial D^1} \left[\frac{z_1^1}{T_2} \frac{\partial \bar{p}_2}{\partial T_2} + \frac{z_1^1}{T_2} \frac{\partial \bar{p}_2}{\partial T_2} + (T_2-1)p_2 \frac{\partial \bar{p}_2}{\partial T_2} + (T_3-1)p_3 \frac{\partial \bar{p}_3}{\partial T_2}\right].
\]

An exactly similar equation is obtained upon differentiating with respect to \(T_3\); thus (S1.12) holds for \(\partial/\partial T_j\) \((j = 2, 3)\) in place of just \(\partial/\partial T_2\).
Starting from free trade, the last two terms of (S.12) vanish. Since $\partial \hat{V}^1/\partial D^1 > 0$, an increase in $T_j$ raises potential welfare if and only if it lowers a Laspeyres world price index of imports, $\hat{z}_1^1 \hat{p}_2(T_2, T_3) + \hat{z}_3^1 \hat{p}_3(T_2, T_3)$, where the quantity weights are the volumes of country 1’s imports in the initial free-trade situation.

(c) Country 1’s potential-welfare function can also be expressed in terms of its direct trade-utility function, composed with the negatives of the foreign excess-demand functions:

$$W^1(T_2, T_3) = \hat{U}^1\left(-\hat{z}_1^2(\hat{p}_1, \hat{p}_2(\cdot), \hat{p}_3(\cdot), l^2), -\hat{z}_2^2(\hat{p}_1, \hat{p}_2(\cdot), \hat{p}_3(\cdot), l^2), -\hat{z}_3^2(\hat{p}_1, \hat{p}_2(\cdot), \hat{p}_3(\cdot), l^2)\right).$$

(S.13)

Differentiating with respect to $T_j$ and using the first-order conditions for maximum trade-utility subject to the balance-of-trade constraint,

$$\frac{\partial \hat{U}^1}{\partial \hat{z}_i^1} = \lambda p_i^1 \quad \text{(where} \quad \lambda = \frac{\partial \hat{V}^1}{\partial D^1},$$

we have

$$\frac{\partial W^1}{\partial T_j} = -\sum_{i=1}^{3} \frac{\partial \hat{U}^1}{\partial \hat{z}_i^1} \left[ \sum_{k=2}^{3} \frac{\partial \hat{z}_k^2}{\partial \hat{p}_k} \frac{\partial \hat{p}_k}{\partial T_j} \right]$$

$$= -\lambda \sum_{k=2}^{3} \left[ \sum_{i=1}^{3} p_i^1 \frac{\partial \hat{z}_i^2}{\partial \hat{p}_k} \right] \frac{\partial \hat{p}_k}{\partial T_j}.$$  

(S.14)

Now,

$$\sum_{i=1}^{3} T_i p_i \frac{\partial \hat{z}_i^2}{\partial \hat{p}_k} = p_1 \frac{\partial \hat{z}_1^2}{\partial \hat{p}_k} + (1 + \tau_2)p_2 \frac{\partial \hat{z}_2^2}{\partial \hat{p}_k} + (1 + \tau_3)p_3 \frac{\partial \hat{z}_3^2}{\partial \hat{p}_k}$$

$$= p_1 \frac{\partial \hat{z}_1^2}{\partial \hat{p}_k} + p_2 \frac{\partial \hat{z}_2^2}{\partial \hat{p}_k} + p_3 \frac{\partial \hat{z}_3^2}{\partial \hat{p}_k} + \tau_2 p_2 \frac{\partial \hat{z}_2^2}{\partial \hat{p}_k} + \tau_3 p_3 \frac{\partial \hat{z}_3^2}{\partial \hat{p}_k}.$$  

(S.15)

From the budget (balance-of-trade) identity (in world prices)

$$p_1 \hat{z}_1^2 + p_2 \hat{z}_2^2 + p_3 \hat{z}_3^2 = 0$$

we have

(S.16)

$$\frac{\partial}{\partial \hat{p}_k}(p_1 \hat{z}_1^2 + p_2 \hat{z}_2^2 + p_3 \hat{z}_3^2) = \hat{z}_k^2 + \sum_{i=1}^{3} p_i \frac{\partial \hat{z}_i^2}{\partial \hat{p}_k} = 0.$$

Therefore from (S.15) and (1),

(S.17)

$$\sum_{i=1}^{3} T_i p_i \frac{\partial \hat{z}_i^2}{\partial \hat{p}_k} = -\hat{z}_k^2 + \tau_2 p_2 \frac{\partial \hat{z}_2^2}{\partial \hat{p}_k} + \tau_3 p_3 \frac{\partial \hat{z}_3^2}{\partial \hat{p}_k} \quad (k = 2, 3).$$

Setting (S.14) equal to zero for $j = 2, 3$ we then have from (S.17)

(S.18)

$$\begin{bmatrix} -\hat{z}_2^2 + \tau_2 p_2 \frac{\partial \hat{z}_2^2}{\partial \hat{p}_2} + \tau_3 p_3 \frac{\partial \hat{z}_3^2}{\partial \hat{p}_2} - \hat{z}_3^2 + \tau_2 p_2 \frac{\partial \hat{z}_2^2}{\partial \hat{p}_3} + \tau_3 p_3 \frac{\partial \hat{z}_3^2}{\partial \hat{p}_3} \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{p}_2}{\partial T_2} \\ \frac{\partial \hat{p}_2}{\partial T_3} \end{bmatrix} = (0, 0).$$

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But the matrix \([\partial \bar{p}_j / \partial T_j]\) on the right in (S1.18) is nonsingular provided the 2 \(\times\) 2 submatrix
\[ S = [\bar{S}_{ij}] \quad (i, j = 2, 3) \]
of the trade-Sluttly matrix is nonsingular,\(^1\) as is seen from (S1.8), hence the row vector on the left in (S1.18) must vanish, i.e.,
\[
(S1.19) \quad \begin{bmatrix}
\partial \bar{z}_2^2 / \partial p_2 \\
\partial \bar{z}_3^2 / \partial p_2 \\
\partial \bar{z}_2^3 / \partial p_3 \\
\partial \bar{z}_3^3 / \partial p_3
\end{bmatrix}
\begin{bmatrix}
\tau_2 \\
\tau_3
\end{bmatrix} = \begin{bmatrix}
\bar{z}_2^2 \\
\bar{z}_3^2
\end{bmatrix}, \quad \text{or} \quad \begin{bmatrix}
\partial \bar{z}_2^2 / \partial p_2 \\
\partial \bar{z}_3^2 / \partial p_2 \\
\partial \bar{z}_2^3 / \partial p_3 \\
\partial \bar{z}_3^3 / \partial p_3
\end{bmatrix}
\begin{bmatrix}
\tau_2 \\
\tau_3
\end{bmatrix} = \begin{bmatrix}
\bar{z}_2^2 \\
\bar{z}_3^2
\end{bmatrix}.
\]

These are the equations of the optimal tariffs. □

An alternative proof proceeds as follows. Formula (S1.12) and the corresponding formula for
\(T_3\), when both set to zero, may be rewritten
\[
(S1.20) \quad \frac{\tau_2 p_2}{\tau_2} \frac{\partial \bar{z}_2^1}{\partial T_2} + \tau_3 p_3 \frac{\partial \bar{z}_3^1}{\partial T_2} = \bar{z}_2^1 \frac{\partial \bar{p}_2}{\partial T_2} + \bar{z}_3^1 \frac{\partial \bar{p}_3}{\partial T_2},
\]
or in matrix notation,
\[
(S1.21) \quad \begin{bmatrix}
\frac{\partial \bar{z}_2^1}{\partial T_2} \\
\frac{\partial \bar{z}_3^1}{\partial T_2} \\
\frac{\partial \bar{z}_2^1}{\partial T_3} \\
\frac{\partial \bar{z}_3^1}{\partial T_3}
\end{bmatrix}
\begin{bmatrix}
\bar{p}_2 \\
\bar{p}_3
\end{bmatrix} = \begin{bmatrix}
\partial \bar{p}_2 / \partial T_2 \\
\partial \bar{p}_2 / \partial T_3 \\
\partial \bar{p}_3 / \partial T_2 \\
\partial \bar{p}_3 / \partial T_3
\end{bmatrix}
\begin{bmatrix}
\tau_2 \\
\tau_3
\end{bmatrix} = \begin{bmatrix}
\bar{z}_2^1 \\
\bar{z}_3^1
\end{bmatrix}.
\]

Now as in (S1.10) we may define, from material balance,
\[
(S1.22) \quad \bar{z}_i^1(T_2, T_3) = -\bar{z}_i^2(T_2, T_3) = -\bar{z}_i^2(\bar{p}_1, \bar{p}_2(T_2, T_3), \bar{p}_3(T_2, T_3), 0, \ell^2)
\]
as in (S1.3), whence by the chain rule
\[
(S1.23) \quad \frac{\partial \bar{z}_i^1}{\partial T_j} = \frac{\partial \bar{z}_i^2}{\partial p_2} \frac{\partial p_2}{\partial T_j} - \frac{\partial \bar{z}_i^2}{\partial p_3} \frac{\partial p_3}{\partial T_j}.
\]

Thus,
\[
(S1.24) \quad \begin{bmatrix}
\frac{\partial \bar{z}_2^1}{\partial T_2} \\
\frac{\partial \bar{z}_3^1}{\partial T_2} \\
\frac{\partial \bar{z}_2^1}{\partial T_3} \\
\frac{\partial \bar{z}_3^1}{\partial T_3}
\end{bmatrix} = - \begin{bmatrix}
\frac{\partial \bar{z}_2^2}{\partial p_2} \frac{\partial p_2}{\partial T_2} + \frac{\partial \bar{z}_3^2}{\partial p_2} \frac{\partial p_3}{\partial T_2} + \frac{\partial \bar{z}_2^3}{\partial p_3} \frac{\partial p_2}{\partial T_2} + \frac{\partial \bar{z}_3^3}{\partial p_3} \frac{\partial p_3}{\partial T_2} \\
\frac{\partial \bar{z}_2^2}{\partial p_2} \frac{\partial p_2}{\partial T_3} + \frac{\partial \bar{z}_3^2}{\partial p_2} \frac{\partial p_3}{\partial T_3} + \frac{\partial \bar{z}_2^3}{\partial p_3} \frac{\partial p_2}{\partial T_3} + \frac{\partial \bar{z}_3^3}{\partial p_3} \frac{\partial p_3}{\partial T_3}
\end{bmatrix}
\]

\(^1\)This requires only that the trade-indifference surfaces in 3-dimensional space be smooth, i.e., that the direct trade-utility function be twice differentiable.
Now, substituting (S1.24) in (S1.21) and using the material-balance condition \( z_i^1 = -z_i^2 \), we obtain

\[
(S1.25) \quad \begin{bmatrix} \frac{\partial \tilde{p}_2}{\partial T_2} & \frac{\partial \tilde{p}_3}{\partial T_2} \\ \frac{\partial \tilde{p}_2}{\partial T_3} & \frac{\partial \tilde{p}_3}{\partial T_3} \end{bmatrix} \begin{bmatrix} p_2 \frac{\partial z_2^2}{\partial p_2} + p_3 \frac{\partial z_2^3}{\partial p_3} \\ p_2 \frac{\partial z_3^2}{\partial p_2} + p_3 \frac{\partial z_3^3}{\partial p_3} \end{bmatrix} \begin{bmatrix} \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{p}_2}{\partial T_2} & \frac{\partial \tilde{p}_3}{\partial T_2} \\ \frac{\partial \tilde{p}_2}{\partial T_3} & \frac{\partial \tilde{p}_3}{\partial T_3} \end{bmatrix} \begin{bmatrix} z_2^2 \\ z_3^2 \end{bmatrix}.
\]

The matrix that appears on the left of both sides is invertible, since by (S1.8) the trade sub-Slutsky matrix may be assumed to be invertible; hence it may be cancelled from both sides, and we obtain (S1.19) as before. \( \square \)

Formula (S1.19) generalizes Johnson’s (1950) formula. The matrix in that formula is the matrix of own- and cross-elasticities of country 2’s excess demands for goods 2 and 3; in general these elasticities depend on the prices, and the prices in turn depend on the tariffs, via (S1.22). Thus there is a deceptive simplicity in the formula, so it cannot be used to compute the optimal tariffs directly except in the special case in which the elasticities are constant. With known excess-demand functions and nonconstant elasticities, one could presumably devise an iterative procedure using (S1.19) to compute the optimal tariffs.

**Appendix on dynamic stability**

To obtain needed properties of the matrix

\[
(S1.26) \quad A = \begin{bmatrix} \frac{\partial z_1^1}{\partial p_2} + \frac{\partial z_2^2}{\partial p_2} & \frac{\partial z_1^1}{\partial p_3} + \frac{\partial z_2^3}{\partial p_3} \\ \frac{\partial z_3^2}{\partial p_2} + \frac{\partial z_3^3}{\partial p_2} & \frac{\partial z_3^2}{\partial p_3} + \frac{\partial z_3^3}{\partial p_3} \end{bmatrix},
\]

we appeal to the property of dynamic stability. Given any fixed tariff factors \( T_2, T_3 \), and any fixed values of \( \ell^1, \ell^2 \), and \( \tilde{p}_i \), let us denote the world excess demand for commodity \( i \) by

\[
Z_i(p_2, p_3) = z_i^1(\tilde{p}_1, p_2, p_3, T_2, T_3, \ell^1) + z_i^2(\tilde{p}_1, p_2, p_3, \ell^2) \quad (i = 2, 3).
\]

Then we may postulate the dynamic-adjustment system of *tâtonnement* type

\[
(S1.27) \quad \begin{align*}
\dot{p}_2 &= \kappa_2 Z_2(p_2, p_3) \\
\dot{p}_3 &= \kappa_3 Z_3(p_2, p_3)
\end{align*}
\]

where \( \dot{p}_i = dp_i/dt \) (\( t \) is time) and the \( \kappa_i \) denote positive speeds of adjustment. Denoting for brevity \( \bar{p}_i = \tilde{p}_i(T_2, T_3) \) for the given fixed \( T_2, T_3 \), let us take a first-order Taylor approximation of the \( Z_i(\cdot) \) around the equilibrium values \( (\bar{p}_2, \bar{p}_3) \), where \( Z_i(\bar{p}_2, \bar{p}_3) \) denotes \( \partial Z_i/\partial p_j \) evaluated at \( (p_2, p_3) = (\bar{p}_2, \bar{p}_3) \) and \( R_i(\cdot) \) denotes the remainder term:

\[
(S1.28) \quad \begin{align*}
Z_2(p_2, p_3) &= Z_2(\bar{p}_2, \bar{p}_3) + Z_{22}(\bar{p}_2, \bar{p}_3)(p_2 - \bar{p}_2) + Z_{23}(\bar{p}_2, \bar{p}_3)(p_3 - \bar{p}_3) + R_2(\bar{p}_2, \bar{p}_3) \\
Z_3(p_2, p_3) &= Z_3(\bar{p}_2, \bar{p}_3) + Z_{32}(\bar{p}_2, \bar{p}_3)(p_2 - \bar{p}_2) + Z_{33}(\bar{p}_2, \bar{p}_3)(p_3 - \bar{p}_3) + R_3(\bar{p}_2, \bar{p}_3).
\end{align*}
\]

The first terms on the right vanish by definition of equilibrium. Ignoring the remainder terms, defining the deviation of the \( j \)th price from its equilibrium value by

\[
u_j = p_j^2 - \bar{p}_j^2,
\]

and denoting

\[
a_{ij} = Z_{ij}(\bar{p}_2, \bar{p}_3) = \frac{\partial z_i^1}{\partial p_j} + \frac{\partial z_i^2}{\partial p_j},
\]

we obtain

\[
(S1.29) \quad \begin{align*}
\dot{\nu}_2 &= \kappa_2 a_{22}\nu_2 + \kappa_2 a_{23}\nu_3 + \kappa_3 a_{32}\nu_2 + \kappa_3 a_{33}\nu_3 \\
\dot{\nu}_3 &= \kappa_2 a_{23}\nu_2 + \kappa_2 a_{33}\nu_3 + \kappa_3 a_{32}\nu_2 + \kappa_3 a_{33}\nu_3
\end{align*}
\]

\( \square \)
where the derivatives in the third expression are evaluated at the given $T_2, T_3, I^1, I^2,$ and $p_1,$ we have from (S1.27) and (S1.28):

\[
\begin{bmatrix}
\dot{u}_2 \\
\dot{u}_3 \\
\end{bmatrix} = \begin{bmatrix}
\kappa_2 & 0 \\
0 & \kappa_3 \\
\end{bmatrix} \begin{bmatrix}
a_{22} & a_{23} \\
a_{32} & a_{33} \\
\end{bmatrix} \begin{bmatrix}
u_2 \\
u_3 \\
\end{bmatrix}, \quad \text{or} \quad \dot{u} = KAu.
\]

Assume that the eigenvalues $\lambda_2, \lambda_3$ of $KA$ are distinct, and let $v^2, v^3$ denote the corresponding eigenvectors, which are then linearly independent. Then, denoting $V = [v^2, v^3]$ and $\Lambda = \text{diag}\{\lambda_2, \lambda_3\},$ we have

\[
KA = V\Lambda, \quad \text{hence} \quad V^{-1}KA = \Lambda.
\]

Defining $u^* = V^{-1}u,$ it follows that

(S1.30) \[ \dot{u}^* = V^{-1}\dot{u} = V^{-1}KAu = V^{-1}KAVu^* = \Lambda u^*. \]

Dynamic stability requires that $u \rightarrow 0,$ hence $u^* \rightarrow 0,$ as $t \rightarrow \infty.$ From (S1.30) we have

\[
\frac{du_i^*}{dt} = \lambda_i u_i^*, \quad \text{or} \quad d\log u_i^* = \frac{du_i^*}{u_i^*} = \lambda_i dt \quad (i = 2, 3),
\]

which integrates to

\[
u_i^* = \nu_i \exp\{\lambda_i t\} \quad (i = 2, 3).
\]

These converge to zero as $t \rightarrow \infty$ if and only if the real parts of the $\lambda_i$ are negative.

The eigenvalues of $KA$ are the solutions of the characteristic equation

\[
f(\lambda) = |I\lambda - KA| = \begin{vmatrix}
\lambda - \kappa_2 a_{22} & -\kappa_2 a_{23} \\
-\kappa_3 a_{32} & \lambda - \kappa_3 a_{33} \\
\end{vmatrix} = \lambda^2 - \text{tr}(KA)\lambda + \text{det}(KA) = 0,
\]

namely

\[
\lambda = \frac{\text{tr}(KA) \pm \sqrt{[\text{tr}(KA)]^2 - 4\text{det}(KA)}}{2}.
\]

If the discriminant is negative (which can happen only if $\text{det}(KA) > 0$) then $\Re(\lambda) = \text{tr}(KA)/2,$ hence we must have $\text{tr}(KA) < 0.$ If the discriminant is positive then we must have

\[
\text{tr}(KA) + \sqrt{[\text{tr}(KA)]^2 - 4\text{det}(KA)} < 0, \quad \text{i.e.,} \quad \text{tr}(KA) < -\sqrt{[\text{tr}(KA)]^2 - 4\text{det}(KA)},
\]

so that again we have $\text{tr}(KA) < 0.$ Now rewriting the above inequality as

\[
0 < \sqrt{[\text{tr}(KA)]^2 - 4\text{det}(KA)} < -\text{tr}(KA),
\]

or

\[
0 < a < b \quad \text{where} \quad a = \sqrt{[\text{tr}(KA)]^2 - 4\text{det}(KA)} \quad \text{and} \quad b = -\text{tr}(KA),
\]

this implies $a^2 < b^2,$ since $b^2 - a^2 = (b + a)(b - a) > 0.$ Accordingly,

\[
[\text{tr}(KA)]^2 - 4\text{det}(KA) < [\text{tr}(KA)]^2,
\]

i.e., $\text{det}(KA) > 0.$ But $\text{det}(KA) = \text{det}(K)\text{det}(A)$ and $\text{det}(K) > 0,$ hence $\text{det}(A) > 0.$ Now if $\text{tr}(KA) < 0$ for all (positive) speeds of adjustment $\kappa_2, \kappa_3,$ then we must have $a_{22} \leq 0$ and $a_{33} \leq 0.$ These conditions are essentially the “Hicks conditions” for dynamic stability, i.e., that the principal minors of $A$ be alternately nonpositive and nonnegative.\footnote{As stated by Hicks (1939, pp. 315–16, 325), the principal minors should be alternately negative and positive. See also Metzler (1945) and Arrow (1974); Arrow pointed out that the weaker inequalities were all that one could prove for the the minors of the matrix $A.$}
Another sharper stability condition was also obtained by Metzler (1945), namely that if the off-diagonal elements of $A$ are nonnegative, then the system is stable if and only if its principal minors are alternately negative and positive. Such matrices are known as Metzler matrices; stability implies that the diagonal elements of $A$ are nonpositive. Finally, McKenzie (1960, p. 50) showed that if a matrix $B$ has positive diagonal elements and nonpositive off-diagonal elements, then $B^{-1} \text{ has all its elements nonnegative if and only if, for some diagonal matrix } \mathbf{K} \text{ with positive diagonal elements, the matrix } \mathbf{KB} \text{ has a dominant diagonal, i.e., the absolute value of each diagonal element is greater than the sum of the absolute values of the remaining elements in the same column.}$

Putting together these results, we may say that if $A$ is a Metzler matrix, and the system is stable, and if $KA$ has a dominant diagonal for some $K$, then $-A^{-1}$ has positive principal minors and has all its elements nonnegative. We assume these conditions to hold.

From (S1.5) we may conclude that the negative inverse matrix, which is precisely $-A^{-1}$ (see (S1.29)), has positive principal minors and has all its elements nonnegative. Each $\partial \hat{p}_i / \partial T_j \ (i = 2, 3)$ is thus a positive weighted average of the $\partial \hat{z}_i / \partial T_j \ (i = 2, 3)$.

References


2 Effect of technological change on countries’ absolute and relative standards of living

(a) The condition for world equilibrium may be written

(S2.1) \[ \hat{h}_2(p_1, p_2(A^1), 0; A^1 l_1^1, A^1 l_2^1) + \hat{h}_2(p_1, p_2(A^1), 0; l_1^2, l_2^2) = 0, \]

where $\hat{h}_j^k(\cdot)$ is country $k$’s trade-demand function for commodity $j$. This equation implicitly defines the function $\hat{p}_2(A^1)$, the remaining variables $p_1$ and $l_k^i \ (i, k = 1, 2)$ being fixed by hypothesis. Differentiating (S2.1) with respect to $A^1$ and assuming $A^1$ to have the value 1 in the initial equilibrium, we obtain

(S2.2) \[
\frac{\partial \hat{p}_2}{\partial A^1} = \left( \frac{\partial \hat{h}_2^1}{\partial l_1^1} + \frac{\partial \hat{h}_2^2}{\partial l_2^2} \right) - \left( \frac{\partial \hat{h}_2^1}{\partial p_2} + \frac{\partial \hat{h}_2^2}{\partial p_2} \right).
\]

From the definition of country 1’s trade-demand function,

\[
\hat{h}_j^1(p_1, p_2, 0; l_1^1, l_2^2) = h_j^1(p_1, p_2, \Pi^1(p_1, p_2, l_1^1, l_2^2)) - y_j^1(p_1, p_2, l_1^1, l_2^2),
\]

8
(where $\Pi^1(\cdot)$ is country 1’s domestic-product function and $\hat{y}_j^1 = \partial \Pi^1 / \partial p_j$ is its Rybczynski function for commodity $j$), we have

$$(S2.3) \quad \frac{\partial \hat{h}_j^1}{\partial l_i^1} = \frac{\partial h_j^1}{\partial Y^1} w_i^1 - \frac{\partial \hat{y}_j^1}{\partial l_i^1},$$

where $Y^k = \Pi^k$ is country $k$’s national income (equal to its domestic product, trade being assumed to be balanced) and $w_i^k$ is the wage rate of the $i$th factor in country $k$. Therefore we have for the numerator of $(S2.2)$, using the homogeneity of degree 1 of the Rybczynski function $\hat{y}_j^1(p_1, p_2, l_1^1, l_2^1)$ in the factor endowments,

$$(S2.4) \quad \sum_{i=1}^2 \frac{\partial \hat{h}_j^1}{\partial l_i^1} l_i^1 = \frac{\partial h_j^1}{\partial Y^1} [w_1^1 l_1^1 + w_2^1 l_2^1] - \left[ \frac{\partial \hat{y}_j^1}{\partial l_1^1} l_1^1 + \frac{\partial \hat{y}_j^1}{\partial l_2^1} l_2^1 \right]
= \frac{\partial h_j^1}{\partial Y^1} Y^1 - \hat{y}_j^1
= x_j^1 - \hat{y}_j^1 = z_j^1,$$

using the homogeneity of the demand function in income on the last line. Thus, $(S2.2)$ becomes

$$(S2.5) \quad \frac{d\bar{p}_2}{dA^1} = -\frac{z_2^1}{\frac{\partial \hat{h}_2^1}{\partial p_2} + \frac{\partial \hat{h}_2^1}{\partial p_2}} = \frac{p_2}{\eta^1 + \eta^2 - 1},$$

using the fact that $(p_2 / \hat{h}_2^1) \partial h_2^1 / \partial p_2 = \eta^2 - 1$ (cf. “Notes on the Theory of Tariffs”, formula (3.10)). Since stability implies that $\eta^1 + \eta^2 - 1 > 0$, $(S2.5)$ implies that the technical improvement in country 1 worsens country 1’s terms of trade (and thus improves country 2’s terms of trade). □

(b) It remains to assess the effect of this price change on the living standards of the two countries. Let country $k$’s welfare or standard-of-living function be defined by

$$(S2.6) \quad W^k(A^1) = \hat{V}^k(p_1, \bar{p}_2(A^1), 0; A^{k1}_1, A^{k2}_2)$$

where $A^2 = 1$ and $\hat{V}^k$ is country $k$’s indirect trade-utility function

$$(S2.7) \quad \hat{V}^k(p_1, p_2, 0, A^{k1}_1, A^{k2}_2) = V^k(p_1, p_2, \Pi^k(p_1, p_2, A^{k1}_1, A^{k2}_2))$$

and $V^k(\cdot)$ is the ordinary indirect utility function. Differentiating $(S2.6)$ with respect to $A^1$ we obtain

$$(S2.8) \quad \frac{dW^1}{dA^1} = \frac{\partial \hat{V}^k}{\partial p_2} \frac{d\bar{p}_2}{dA^1} + \frac{\partial \hat{V}^k}{\partial l_1^1} l_1^1 + \frac{\partial \hat{V}^k}{\partial l_2^1} l_2^1, \quad \frac{dW^2}{dA^1} = \frac{\partial \hat{V}^k}{\partial p_2} \frac{d\bar{p}_2}{dA^1}.$$

Now from $(S2.7)$ we see that

$$(S2.9) \quad \frac{\partial \hat{V}^k}{\partial l_i^k} = \frac{\partial V^k}{\partial Y_i^k} w_i^k$$
(since $A^1 = 1$ in the initial equilibrium), hence the last two terms in the first equation of (S2.8) become

\[(S2.10) \quad \frac{\partial \hat{\nu}^1}{\partial \hat{\nu}^1 l_1} + \frac{\partial \hat{\nu}^1}{\partial \hat{\nu}^1 l_2} = \frac{\partial V^1}{\partial Y^1} (w_1^1 l_1 + w_2^1 l_2) = \frac{\partial V^1}{\partial Y^1} Y^1.\]

Moreover, from (S2.7) it is clear (since there are no nontradables in this model) that

\[(S2.11) \quad \frac{\partial \hat{\nu}^k}{\partial D^k} = \frac{\partial V^k}{\partial Y^k},\]

and furthermore, from the assumed homogeneity of the utility function,

\[(S2.12) \quad V^k(p_1, p_2, Y^k) = Y^k V^k(p_1, p_2, 1) = Y^k / C(p_1, p_2),\]

where $C(p_1, p_2) = 1 / V^k(p_1, p_2, 1)$ (the same for each country $k$) may be interpreted as a cost-of-living function. Finally, from Antonelli’s partial differential equation, we have, using (S2.11) and (S2.12),

\[(S2.13) \quad \frac{\partial \hat{\nu}^k}{\partial p_j} = - \frac{\partial \hat{\nu}^k}{\partial D^k} \hat{h}_j^k = \frac{\partial V^k}{\partial Y^k} \hat{z}_j^k = - \frac{\hat{z}_j^k}{C(p_1, p_2)}.\]

Putting together (S2.9), (S2.10), (S2.11), (S2.12), (S2.13), and (S2.5), the equations (S2.8) become

\[(S2.14) \quad \frac{dW^1}{dA^1} = \frac{1}{C} \left[ Y^1 - \frac{p_2 \hat{h}_2^1}{\eta^1 + \eta^2 - 1} \right],\]

\[(S2.15) \quad \frac{dW^2}{dA^1} = \frac{1}{C} \left[ -p_2 \hat{h}_2^2 / \eta^1 + \eta^2 - 1 \right].\]

(i) From the first of these equations we see that country 1 gains, i.e., $dW^1 / dA^1 > 0$, if and only if

\[(S2.16) \quad \frac{p_2 \hat{h}_2^1}{Y^1} < \frac{\eta^1 + \eta^2 - 1}{2},\]

i.e., if and only if the share of its imports (which, because of balanced trade, is also the share of its exports) in its national income falls short of the “Marshall-Lerner” expression.

(ii) From the second equation of (S2.14) we have, unambiguously, $dW^2 / dA^1 > 0$, i.e., country 2 necessarily gains from the technical improvement in country 1.

(iii) Finally, subtracting the second equation of (S2.14) from the first, we obtain

\[(S2.17) \quad \frac{p_2 \hat{h}_2^1}{Y^1} < \frac{\eta^1 + \eta^2 - 1}{2},\]

i.e., the share of country 1’s imports in its national income (which, by balanced trade, is equal to the share of its exports in its national income) falls short of one-half the “Marshall-Lerner” expression. □
3 Foreign aid as a substitute for trade restrictions

(a) Country 1 receives either foreign aid in the amount $A$, or tariff revenues from its duty on imports of commodity 2 from country 2 (or both). We may define its excess demand for commodity 2 as the solution of the functional equation

\[ z^1_2(p_1, p_2, A, T_2, l^1) = \hat{h}^1_2(p_1, T_2 p_2, A + (T_2 - 1)p_2 z^1_2(p_1, p_2, A, T_2, l^1); l^1) \]

where $p_j$ is the world price of commodity $j$ ($p^1_j$ being the price of commodity $j$ on country 1’s domestic markets) and $T_2 - 1$ is the ad valorem tariff rate expressed as a proportion of the world price of commodity 2. $l^1 = (l^1_1, l^1_2)$ denotes the vector of country 1’s factor endowments. Country 2’s excess demand for commodity 2 is

\[ z^2_2(p_1, p_2, A, T_2, l^2) = \hat{h}^2_2(p_1, p_2, -A; l^2). \]

World equilibrium is defined by

\[ \hat{z}^1_2(p_1, p_2, A, T_2, l^1) + \hat{z}^2_2(p_1, p_2, -A, T_2, l^2) = 0. \]

This equation implicitly defines the functions

\[ p_2 = \tilde{p}_2(A, T_2), \quad p^1_2 = T_2 \tilde{p}_2(A, T_2), \quad \text{and} \]

\[ \tilde{z}^1_2(A, T_2) = \hat{z}^1_2(p_1, \tilde{p}_2(A, T_2), A, T_2, l^1) \]

where we omit the variables $p_1, l^1, l^2$ which are assumed constant.

Initially, country 1 imposes the tariff and receives no foreign aid. Its tariff revenues, equal to its trade deficit, are equal to

\[ R^1(T_2) \equiv (T_2 - 1)\tilde{p}_2(0, T_2)\hat{z}^1_2(0, T_2). \]

Country 2 now offers foreign aid to country 1 in the amount of $A = R^1(T_2)$ provided country 1 rescinds its tariff. Country 1’s potential welfare in the tariff equilibrium is

\[ W^1(0, T_2) = \tilde{V}^1(p_1, \tilde{p}^1_2(0, T_2), R^1(T_2); l^1), \]

where $\tilde{V}^1(p^1_1, p^1_2, D^1; l^1)$ is country 1’s indirect trade-utility function (here $p^1_1 = p_1$). Its potential welfare with the foreign aid but without the tariff is

\[ W^1(A, 1) = \tilde{V}^1(p_1, \tilde{p}^2_2(A, 1), A; l^1). \]

By assumption, the third argument of $\tilde{V}^1$ is the same in both cases, i.e., $R^1(T_2) = A$. The only arguments that can differ, then, are the second: $p^1_2$. The question then reduces to how country 1’s indirect trade utility varies as a function of the domestic price of its import good.

From Antonelli’s partial differential equation

\[ \frac{\partial \tilde{V}^1}{\partial p^1_j} = - \frac{\partial \tilde{V}^1}{\partial D^1} \hat{h}^1_j, \]

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since \( \partial V^1 / \partial D^1 > 0 \) by local nonsatiation, and \( h^1 > 0 \), it follows that for given \( D^1 \) and \( p_1 \), country 1’s potential welfare is a decreasing function of the price \( p^1_2 \) of its import good on the home market. Thus, country 1 is “better off” with the foreign aid if and only if \( \bar{p}^1_2(A, 1) < \bar{p}^1_2(0, T_2) \), where \( A = R^1(T_2) \). □

(b) (i) If a transfer from country 2 to country 1 has the “orthodox” effect of improving country 1’s terms of trade, then

(S3.9) \[ \bar{p}_2(A, 1) < \bar{p}_2(0, 1). \] □

(ii) If a tariff imposed by country 1 does not lead to the “Metzler paradox”, then

(S3.10) \[ \bar{p}^1_2(0, T_2) \geq \bar{p}^1_2(0, 1) \] for \( T_2 > 1 \).

From (S3.9), (S3.10), and (S3.4), we have

(S3.11) \[ \bar{p}^1_2(A, 1) = \bar{p}_2(A, 1) < \bar{p}_2(0, 1) = \bar{p}^1_2(0, 1) \leq \bar{p}^1_2(0, T_2). \] □