Econ 8402: International Trade & Payments
Solutions for Homework Problem Set #3
(Take-Home Final Exam)

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1 Optimal tariff and retaliation

(a) First we derive country 1’s tariff-modified offer function incorporating the tariff factor $T_2$ on its import of commodity 2 in the general case of CES preferences.

The CES trade-demand function for country 1, which is expressed as a function of its domestic prices, is

$$\hat{h}_2^1(p_1^1, p_2^1, D^1; \omega^1) = h_2^1(p_1^1, p_2^1, p_1^1\omega_1^1 + p_2^1\omega_2^1 + D^1) - \omega_2^1 = \frac{p_1^1\omega_1^1 + p_2^1\omega_2^1}{(p_1^1)^{1-\sigma}(p_2^1)^{\sigma} + p_2^1} - \omega_2^1.$$

For any tariff factor $T_2$, country 1’s tariff-modified excess-demand function (which is expressed as a function of world prices) is defined implicitly by

$$\hat{z}_2^1(p_1, p_2, T_2, \omega^1) = \hat{h}_2^1(p_1, T_2 p_2, (T_2 - 1)p_2 \hat{z}_2^1(p_1, p_2, T_2, \omega^1); \omega^1)$$

$$= \frac{p_1^1 \omega_1^1 + T_2 p_2 \omega_2^1 + (T_2 - 1)p_2 \hat{z}_2^1(p_1, p_2, T_2, \omega^1)}{p_1^{1-\sigma}(T_2 p_2)^{\sigma} + T_2 p_2} - \omega_2^1.$$

Collecting terms, this leads to the explicit formula

$$z_2^1 = \hat{z}_2^1(p_1, p_2, T_2, \omega^1) = \frac{p_1 \omega_1^1 - p_1^{1-\sigma}(T_2 p_2)^{\sigma} \omega_2^1}{p_1^{1-\sigma}(T_2 p_2)^{\sigma} + p_2} = \frac{p_1 \omega_1^1 + p_2 \omega_2^1}{p_1^{1-\sigma} T_2^\sigma p_2^\sigma + p_2} - \omega_2^1.$$

Normalizing prices to $(1, r_2)$ where $r_2 = p_2/p_1$, country 1’s tariff-modified inverse excess-demand function $\hat{r}_2(z_2^1; T_2)$ for its import good 2 is now defined implicitly as the solution of the equation

$$\omega_2^1 + z_2^1 = \frac{r_2 \omega_2^1}{T_2^\sigma r_2^\sigma + r_2},$$

and is thus obtainable by extracting the positive root of the equation

$$f(r_2) = (\omega_2^1 + z_2^1)(r_2 + T_2^\sigma r_2^\sigma) - (\omega_1^1 + \omega_2^1 r_2) = 0.$$

Country 1’s tariff-modified offer function is then defined as

$$-z_1^1 = F^1(z_2^1; T_2) = z_2^1 \hat{r}_2(z_2^1; T_2).$$
In the special case \( \sigma = 1 \), (1.2) has the explicit solution

\[
(1.5) \quad \hat{r}_2(z_2^1; T_2) = \frac{\omega_1^1/\omega_2^1}{(1 + T_2)(1 + z_1^2/\omega_2^1)} - 1 = \frac{\omega_1^1}{z_1^2 + \omega_1^1} \cdot \frac{1}{(1 + T_2)(1 + z_1^2/\omega_2^1) - 1} = \frac{\omega_1^1}{1 + (\omega_1^1/\omega_2^1 + 1)T_2}.
\]

Country 1’s tariff-modified offer function is then

\[
(1.6) \quad -z_1^1 = F^1(z_1^1; T_2) = z_1^2 \hat{r}_2(z_2^1; T_2) = \frac{\omega_1^1 z_1^2/\omega_2^1}{(1 + T_2)(1 + z_1^2/\omega_2^1) - 1} = \frac{\omega_1^1}{1 + (\omega_1^1/\omega_2^1 + 1)T_2}.
\]

(b) Country 1’s problem is to maximize its trade-utility function

\[
(1.7) \quad \hat{U}^1(z_1^1, z_2^1) = (\omega_1^1 + z_1^1)(\omega_2^1 + z_2^1)
\]
subject to country 2’s offer (reciprocal demand) function

\[
(1.8) \quad -z_1^1 = F^2(z_1^2) = \frac{\omega_2^2}{2 + \omega_1^2/z_1^2}
\]
together with the material-balance condition \( z_1^1 + z_2^1 = 0 \). Substituting (1.8) into (1.7) and using the material-balance condition, we are to maximize

\[
(1.9) \quad \hat{U}^1(z_1^1) = (\omega_1^1 - z_1^1) \left( \omega_2^1 + \frac{\omega_2^2}{2 + \omega_1^2/z_1^2} \right) = \omega_1^1 \omega_2^1 + \frac{\omega_1^1 \omega_2^2}{2 + \omega_1^2/z_1^2} - \omega_2^1 z_1^2 - \frac{\omega_2^2 z_1^2}{2 + \omega_1^2/z_1^2}.
\]

We verify that

\[
(1.10) \quad \frac{d\hat{U}^1}{dz_1^1} = \frac{\omega_1^1 \omega_2^2 \omega_2^1}{2 - \omega_1^2/z_1^2} - \omega_2^2 (\omega_2^1 + 2z_1^1)^2 - \frac{2 \omega_2^2 \omega_1^2 z_1^2}{(\omega_2^1 + 2z_1^1)^2} - 2 \omega_2^2 (z_1^1)^2.
\]

Setting the numerator equal to zero results in the following quadratic equation:

\[
(1.11) \quad [4 \omega_2^1 + 2 \omega_2^2](z_1^2)^2 + [4 \omega_2^2 \omega_1^2 + 2 \omega_2^2 \omega_1^2] z_1^2 + \omega_2^1 (\omega_1^1)^2 - \omega_1^1 \omega_2^2 \omega_1^1 = 0.
\]

For our special case

\[
\begin{bmatrix}
\omega_1^1 & \omega_2^1 \\
\omega_2^1 & \omega_2^2
\end{bmatrix} = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix},
\]
equation (1.11) becomes

\[
8(z_1^2)^2 + 8z_1^2 - 3 = 0,
\]
whose positive root is \( z_1^2 = \sqrt{10}/4 - 1/2 = .290569415044 \). This represents a 58% reduction in country 1’s export of commodity 1 from the free-trade level \( z_1^2 = .5 \). The value of country 2’s offer function at the new equilibrium is

\[
z_2^1 = F^2(z_1^2) = \frac{\omega_2^2}{2 + \omega_1^1/z_1^2} = \frac{2}{2 + 1/z_1^1} = 1 - \frac{2}{\sqrt{10}} = .367544467967
\]
— a 36% decline in country 1’s imports of commodity 2. Country 1’s terms of trade rise from 1 to \( r = p_1/p_2 = 1.26491106406 \), i.e., by 36%.
At the optimal point \((z_1^2, z_2^1) = (\sqrt{10}/4 - 1/2, 1 - 2/\sqrt{10}) = (0.290569415044, 0.367544467968)\), the slope of the tangential trade-indifference curve, which is the domestic price ratio is, from (1.7),

\[
\frac{\partial U}{\partial z_1^2} \frac{\partial^2 z_2^1}{\partial U} = \frac{1 + z_2^1}{2 - z_2^1} = \frac{2 - 2/\sqrt{10}}{5/2 - \sqrt{10}/4} = \frac{\sqrt{10} - 1}{\frac{5}{4}(\sqrt{10} - 1)} = \frac{4}{5} = \frac{p_1}{p_2} = \frac{p_1}{p_2}
\]

(see Figure 1). This, of course, is also the slope of country 2’s offer curve at that point:

\[
\frac{dF^2}{dz_1^2} = \frac{2/(z_1^2)^2}{(2 + 1/z_1^2)^2} \bigg|_{z_1^2 = \sqrt{10}/4 - 1/2} = \frac{2}{2.5} = \frac{4}{5}.
\]

On the other hand, the world price ratio is the slope of the ray through this optimal point, or

\[
\frac{p_1}{p_2} = \frac{\frac{z_2^1}{z_1^2}}{\sqrt{10}/4 - 1/2} = \frac{\sqrt{10} - 2}{\frac{10}{4}(\sqrt{10} - 2)} = \frac{4}{\sqrt{10}}.
\]

Country 1’s optimal tariff factor is then, from (1.14) and (1.12),

\[
T_2 = \frac{p_1}{p_2} = \frac{p_1/p_2}{p_1/p_2} = \frac{4/\sqrt{10}}{4/5} = \frac{5/\sqrt{10}}{2} = 1.58113883008
\]

—a tariff rate of some 58%.

Country 1’s utility at this optimal point is

\[
(2 - z_1^2)(1 + z_2^1) = (1.70943058496)(1.367544467968) = 2.33772233984,
\]

which is some 3.9% higher than its utility of \((2 - .5)(1 + .5) = 2.25\) under free trade. Country 2’s utility at this same point is

\[
(1 + z_1^2)(2 - z_2^1) = (1.290569415044)(1.63245553203) = 2.10679718105,
\]

which is some 6.4% lower than its utility of 2.25 under free trade.

(c) Country 2’s problem is to maximize its trade-utility function

\[
\tilde{U}(z_1^2, z_2^1) = (\omega_1^2 + z_2^1)(\omega_2^1 + z_2^1)
\]

subject to country 1’s tariff-modified offer function (1.6):

\[
z_1^2 = -z_1^1 = F^1(z_1^1; T_2) = \frac{\omega_1^1}{1 + (\omega_2^1/z_2^1 + 1)T_2}.
\]

Substituting (1.17) into (1.16) and using the material-balance conditions, we are to maximize

\[
\tilde{U}^2(z_1^2) = (\omega_2^1 - z_2^1) \left(\omega_1^1 + \frac{\omega_1^1}{1 + T_2 + T_2\omega_2^1/z_2^1} \right).
\]

Proceeding as before, and setting the derivative of (1.18) equal to zero, we obtain the quadratic equation

\[
a(z_1^2)^2 + bz_2^1 + c = 0,
\]

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where
\[
\begin{align*}
    a &= (1 + T_2)\omega_1^2 + (1 + T_2)^2\omega_2^2 \\
    b &= 2T_2\omega_1^2\omega_2^2 + 2(1 + T_2)T_1\omega_1^2\omega_2^1 \\
    c &= \omega_1^2(\omega_2^2)^2T_2 - \omega_2^2\omega_1^1T_2
\end{align*}
\]
—generalizing the quadratic equation (1.11) for country 2. Inserting into (1.19) the particular values of the \( \omega_i^k \) and country 1’s initial tariff factor \( T_2 = \sqrt{2.5} \), we obtain the solution \( z_2^2 = .223301973654 \), from which we derive \( z_1^2 = .206999497896 \) from (1.17), and thence \( T_1 = 1.3645305422 \) for country 2’s retaliatory optimal tariff factor on good 1.

(d) At the equilibrium values
\[(z_1^2, z_2^2) = (.206999497896, .223301973654)\]
corresponding to the two tariff factors
\[(T_2, T_1) = (1.58113883008, 1.3645305422),\]
the consumption levels are
\[(x_1^1, x_2^1) = (\omega_1^1 - z_1^2, \omega_2^1 + z_2^1) = (1.7930005021, 1.223301973654)\]
in country 1, leading to a utility of 2.19338105297, which is less than the utility level 2.25 corresponding to the consumption levels (1.5, 1.5) achieved under free trade. The corresponding consumption levels in country 2 at the two-step tariff levels are
\[(x_1^2, x_2^2) = (\omega_1^2 + z_1^2, \omega_2^2 - z_2^1) = (1.206999497898, 1.77669802635),\]
leading to a utility level of 2.14447362572, compared to that of 2.25 under free trade. Thus, both countries lose at this stage of the tariff war.

**Alternative derivations for (a)–(c)**

An alternative approach to the above problems is to derive the expression for the world equilibrium \((z_1^2, z_2^2)\) as a function of any pair \((T_2, T_1)\) of tariff factors imposed by countries 1 and 2, and then from this expression find the tariff factor \( T_2 \) for country 1 which maximizes its utility subject to country 2’s tariff factor on good 1 being \( T_1 = 1 \); then, given this tariff factor \( T_2 \), to find country 2’s optimal tariff factor \( T_1 \). This of course defines the sequence of reaction functions which characterize Johnson’s tariff war in the particular case being analyzed.

(a’) The world equilibrium is given by
\[
\begin{pmatrix}
    1/z_1^2 \\
    1/z_2^2
\end{pmatrix} = \begin{pmatrix}
    \omega_1^1 & -\omega_2^1 T_2 \\
    -\omega_1^1 T_1 & \omega_2^2
\end{pmatrix}^{-1} \begin{pmatrix}
    1 + T_2 \\
    1 + T_1
\end{pmatrix}
\]
(1.20)
\[
= \frac{1}{\omega_1^2 \omega_2^2 - \omega_2^1 \omega_1^1 T_2 T_1} \begin{pmatrix}
    \omega_2^2 & \omega_2^1 T_2 \\
    \omega_1^1 T_1 & \omega_1^1
\end{pmatrix} \begin{pmatrix}
    1 + T_2 \\
    1 + T_1
\end{pmatrix}.
\]
The solution may therefore be written as

\[(1.21) \quad z_1^2 = \frac{(\omega_1^3 + \omega_2^3 T_2 T_1)}{\omega_2^2 (1 + T_2) + \omega_2^3 T_2 (1 + T_1)} \quad \text{and} \quad z_2^2 = \frac{(\omega_1^3 + \omega_2^3 T_2 T_1)}{\omega_1^2 T_1 (1 + T_2) + \omega_1^3 (1 + T_1)}.\]

For the consumption levels, we have for country 1

\[(1.22) \quad x_1^1 = \omega_1^1 - z_1^2 = \frac{\omega_1^2 (\omega_1^3 + \omega_2^3) T_2 + \omega_2^3 (\omega_1^3 + \omega_1^2) T_1 T_2}{\omega_2^2 + (\omega_2^3 + \omega_2^3) T_2 + \omega_2^3 T_2 T_1},\]

and

\[(1.23) \quad x_2^1 = \omega_1^1 + z_1^2 = \frac{\omega_1^2 (\omega_1^3 + \omega_2^3) + (\omega_2^3 (\omega_1^3 + \omega_1^2) T_1}{\omega_1^2 + (\omega_1^3 + \omega_1^2) T_1 + \omega_1^2 T_1 T_2},\]

and for country 2,

\[(1.24) \quad x_1^2 = \omega_1^2 + z_2^2 = \frac{\omega_2^3 (\omega_1^3 + \omega_1^2) T_1 + (\omega_2^3 (\omega_1^3 + \omega_1^2) T_1 T_2}{\omega_1^2 + (\omega_1^3 + \omega_1^2) T_1 + \omega_1^3 T_1 T_2},\]

and

\[(1.25) \quad x_2^2 = \omega_2^2 - z_2^2 = \frac{\omega_2^3 (\omega_1^3 + \omega_1^2) T_1 + (\omega_2^3 (\omega_1^3 + \omega_1^2) T_1 T_2}{\omega_1^2 + (\omega_1^3 + \omega_1^2) T_1 + \omega_1^2 T_1 T_2}.\]

Now inserting our assumed values \(\omega_1^1 = \omega_2^2 = 2\) and \(\omega_1^2 = \omega_2^1 = 1\), we obtain

\[(1.26) \quad z_1^2 = \frac{4 - T_1 T_2}{2(1 + T_2) + T_2 (1 + T_1)} \quad \text{and} \quad z_2^1 = \frac{4 - T_2 T_1}{2(1 + T_1) + T_1 (1 + T_2)}.\]

The equilibrium consumption levels of the respective countries are

\[(1.27) \quad x_1^1 = \omega_1^1 - z_1^2 = \frac{6 T_2 + 3 T_1 T_1}{2 + 3 T_2 + 2 T_1 T_1}, \quad x_1^1 = \omega_1^1 + z_1^2 = \frac{6 + 3 T_1}{2 + 3 T_1 + T_1 T_2},\]

for country 1, and

\[(1.28) \quad x_2^2 = \omega_2^2 + z_2^2 = \frac{6 + 3 T_2}{2 + 3 T_2 + 2 T_1 T_1}, \quad x_2^2 = \omega_2^2 - z_2^2 = \frac{6 T_1 + 3 T_2 T_1}{2 + 3 T_1 + T_1 T_2},\]

for country 2.

(b') Now if \(T_1 = 1\) initially, country 1’s problem is to maximize \(U(x_1^1, x_1^2) = x_1^1 x_2^1\) with respect to \(T_2\) (subject to \(T_1 = 1\)). From (1.22) and (1.24) with \(T_1 = 1\) the problem is then to maximize

\[(1.29) \quad x_1^1 x_2^1 = \frac{\nu^2 T_2}{\alpha T_2^2 + \beta T_2 + \gamma},\]

with respect to \(T_2\), where

\[(1.30) \quad \nu = \omega_1^2 (\omega_2^3 + \omega_2^3) + \omega_1^3 (\omega_2^3 + \omega_1^2),\]

\(\alpha = \omega_2^2 (2 \omega_1^3 + \omega_2^3),\)

\(\beta = (2 \omega_1^3 + \omega_2^3) (2 \omega_1^3 + \omega_1^2) + \omega_2^3 \omega_1^2,\)

\(\gamma = \omega_2^2 (2 \omega_1^2 + \omega_1^2).

We set

\[
\frac{d}{dT_2} \left( \frac{T_2}{\alpha T_2^2 + \beta T_2 + \gamma} \right) = \frac{\gamma - \alpha T_2^2}{(\alpha T_2^2 + \beta T_2 + \gamma)^2} = 0,
\]

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which has the solution

\[ T_2 = \sqrt{\frac{\gamma}{\alpha}} = \sqrt{\frac{\omega_2^2 (2\omega_1^2 + \omega_2^2)}{\omega_1^2 (2\omega_1^2 + \omega_2^2)}}. \]

Substituting our particular values of the \( \omega_i^k \), we have

\[ x_1^1 x_2^1 \bigg|_{T_1 = 1} = \frac{9T_2}{2 + 4T_2} \cdot \frac{9}{5 + T_2} = \frac{81}{2} \cdot \frac{T_2}{2T_2^2 + 11T_2 + 5} \]

with respect to \( T_2 \), which results in

\[ \frac{d}{dT_2} \left( \frac{T_2}{2T_2^2 + 11T_2 + 5} \right) = \frac{-2T_2^2 + 5}{(2T_2^2 + 11T_2 + 5)^2} = 0, \]

or \( T_2^2 = 5/2 \), hence \( T_2 = \sqrt{2.5} \) as before.

(c') Country 2's problem is now to maximize \( U(x_1^2, x_2^2) = x_1^2 x_2^2 \) with respect to \( T_1 \), subject to \( T_2 \) being given, i.e., to maximize

\[ x_1^2 x_2^2 \bigg|_{T_2} = \frac{6 + 3T_2}{2 + 3T_2 + T_2 T_1} \cdot \frac{(6 + 3T_2)T_1}{2 + 3T_1 + T_2 T_1} = \frac{9(T_2 + 2)^2 T_1}{(2 + 3T_2 + T_2 T_1)(2 + 3T_1 + T_2 T_1)} \]

with respect to \( T_1 \). Thus, we are to maximize

\[ \log T_1 - \log(2 + 3T_2 + T_1 T_2) - \log(2 + (3 + T_2)T_1) \]

with respect to \( T_1 \), yielding

\[ \frac{1}{T_1} - \frac{T_2}{2 + 3T_2 + T_2 T_1} - \frac{3 + T_2}{2 + (3 + T_2)T_1} = 0. \]

Multiplying this equation through by the product \( (2 + 3T_2 + T_1 T_2)[2 + (3 + T_2)T_1]T_1 \), we obtain a quadratic equation

\[ (3 + T_2)T_2 T_1^2 - 2(2 + 3T_2) = 0, \]

which of course is a perfect square in \( T_1 \), yielding the solution

\[ T_1 = \sqrt{\frac{2(2 + 3T_2)}{(3 + T_2)T_2}}. \]

This is country 2's reaction function. Substituting the value \( T_2 = \sqrt{2.5} \) in this formula, we obtain for country 1's optimal retaliatory tariff the amount \( T_1 = 1.3645305422 \), in agreement with the above.\(^1\)

\(^1\) Note that if \( T_2 = 1 \), this formula reduces to \( T_1 = \sqrt{10/4} = \sqrt{2.5} \), which was the previous result for country 1. Note also that \( \frac{dT_1}{dT_2} = -\frac{3T_2^2 + 4T_2 + 6}{(T_2^2 + 3T_2)^{3/2}} < 0 \), so that the sequence is oscillatory. At each step after the first, country 1 lowers its tariff rate, and at each step after its first retaliatory tariff, country 2 continues to raise its tariff rate. There is a limiting value of the tariff.
(e)(i) Country 2's tariff-free excess-demand function for it import good (commodity 1) is
\[ z_1^2(p_1, p_2, 0; \omega^2) = \frac{p_1\omega_1^2 + p_2\omega_2^2}{2\omega_1^2} = \frac{\omega_1^2 p_2 - \omega_2^2 p_1}{2p_1}. \]

Thus,
\[ \eta^2 = -\frac{p_1}{z_1^2} \frac{\partial^2 z_1^2}{\partial p_1} = -p_1 \frac{2p_1}{\omega_2^2 p_2 - \omega_1^2 p_1} \cdot \frac{-\omega_2^2 p_2}{2(p_1)^2} = \frac{\omega_2^2 p_2}{\omega_2^2 p_2 - \omega_1^2 p_1} = \frac{1}{1 - \frac{\omega_1^2 p_2}{\omega_2^2 p_2}}. \]

This is certainly not constant—unless, and only if, \( \omega_2^2 = 0. \)

(ii) We are now to show that country 1’s optimal tariff rate \( \hat{\tau}_2 = \hat{T}_2 - 1 \) satisfies Johnson’s formula \( \hat{\tau}_2 = 1/(\eta^2 - 1) \). From (1.33) we have
\[ \eta^2 - 1 = \frac{1}{\frac{\omega_2^2 p_2}{\omega_1^2 p_1}} \text{, or } \frac{1}{\eta^2 - 1} = \frac{\omega_2^2 p_2}{\omega_1^2 p_1} - 1, \]
so this is equivalent to showing that
\[ \hat{T}_2 = \frac{\omega_2^2 p_2}{\omega_1^2 p_1}. \]

From (1.21) and the balanced-trade condition we have, with \( T_1 = 1, \)
\[ \frac{p_2}{p_1} = \frac{z_1^2}{z_2^2} = \frac{\omega_1^2 + 2\omega_1^2}{\omega_2^2 + 2\omega_2^2} \frac{T_2}{(\omega_2^2 + 2\omega_2^2)T_2}, \]
hence from (1.34) and 1.35 we must have
\[ \hat{T}_2 = \frac{\omega_2^2 p_2}{\omega_1^2 p_1} = \frac{\omega_2^2(\omega_1^2 + 2\omega_1^2) + \omega_2^2\omega_1^2 T_2}{\omega_2^2\omega_2^2 + \omega_1^2(\omega_2^2 + 2\omega_2^2)T_2} = \frac{\gamma + \omega_1^2\omega_2^2T_2}{\omega_2^2\omega_2^2 + \omega_1^2(\omega_2^2 + 2\omega_2^2)T_2}, \]
where \( \alpha \) and \( \gamma \) are defined by (1.30). This is equivalent to the quadratic equation \( \alpha T_2^2 = \gamma \), which has the solution (1.31) as before, which for our special values of the \( \omega_1^k \) reduces to \( T_2 = \sqrt{10}/4 \) as before.

factor \( T \) satisfying
\[ T = \sqrt{\frac{6T + 4}{T^2 + 3T^2}}, \]
or
\[ T^4 + 4T^3 - 6T - 4 = 0. \]
This polynomial has the positive root \( T = \sqrt[4]{2} = 1.41421356237 \) (40% tariff rates have in fact been quite typical). The sequence converges to all 11 decimal points in 22 steps.

Formula (1.32) can be generalized by replacing \( \omega_1^k = \omega_2^k = 2 \) by any number \( \omega > 1 \), to obtain
\[ T_1 = \sqrt{\frac{\omega^2 + \omega(\omega + 1)\frac{T_2}{(1 + \omega)(1 + T_2)}}{(1 + \omega)(1 + T_2)}}, \]
(where the superscript to \( \omega \) is now a power), and the limiting solution of the tariff war is the positive root of the polynomial
\[ T^4 + (1 + \omega)T^3 - \omega(1 + \omega)T - \omega^2, \]
which is \( T = \sqrt{\omega} \).
2 Autarky and trade under external economies of scale

(a) Maximizing the utility function subject to the budget constraint \( p_1 x_1 + p_2 y_2 = Y \) (where \( Y \) is national income, equal to \( w l \) where \( w \) is the wage rate), the demand functions are \( x_i = Y / 2p_i \)

Equating demand to supply \((x_i = y_i)\) and total cost to total revenue \((w v_i = p_i y_i)\) we have

\[
v_i = \frac{p_i y_i}{w} = \frac{p_i}{w} x_i = \frac{p_i w l}{2p_i} = \frac{l}{2},
\]

hence from the production functions we obtain the equilibrium solutions

\[
x_1^A = y_1^A = v_1^{\rho_1} = (l / 2)^{\rho_1} \quad (\rho_1 = \frac{1}{2}, \rho_2 = 2)
\]

(where the superscript \( A \) stands for autarky). Setting \( l = 3/4 \) this gives

\[
x_1^A = y_1^A = (3/8)^{1/2} = .375^{1/2} = .612372435696 \quad \text{and} \quad x_2 = y_2 = (3/8)^2 = .375^2 = .140625.
\]

The equilibrium price ratio under autarky is

\[
p^A = \frac{p_1^A}{p_2^A} = \frac{\partial U / \partial x_1}{\partial U / \partial x_2} = \frac{x_2}{x_1} = \left( \frac{l}{2} \right)^{\rho_2 - \rho_1} = \left( \frac{3}{8} \right)^{3/2} = .229639663386,
\]

which falls short of the world price ratio .25. Under the Ohlin definition of comparative advantage, our country has a comparative advantage in commodity 1.

For later reference, utility at this autarky equilibrium is equal to

\[
(2.1) \quad U \left( (3/8)^{1/2}, (3/8)^2 \right) = (3/8)^{5/2} = .0861148737697.
\]

(b.i) At the given world price ratio \( p = 1/4 \), if the country specializes in commodity 1 then the average costs must satisfy

\[
c_1 = w y_1 = p_1 \quad \text{and} \quad c_2 = w y_2^{-1/2} > p_2
\]

hence

\[
\frac{c_1}{c_2} = y_1 y_2^{1/2} < \frac{p_1}{p_2} = \frac{1}{4}.
\]

This is clearly satisfied if \( y_2 = 0 \). Thus \( v_1 = l \) and \( y_1 = \sqrt{l} = \sqrt{3/2} = 0.866025403784 \).

The budget equation for consumers is then

\[
.25 x_1 + x_2 = .25 y_1 = \sqrt{3}/8 = .216506350946.
\]

Utility maximization subject to this budget constraint implies \( x_2 = .25 x_1 \), hence

\[
x_1^S = \sqrt{3}/4 = .433012701892 \quad \text{and} \quad x_2^S = \sqrt{3}/16 = .108253175473
\]

where the superscript \( S \) stands for specialization. The country’s welfare is then

\[
U \left( \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{16} \right) = \frac{3}{64} = .046875.
\]
This is clearly lower than the utility level of \(0.0861148737698\) under autarky. The country’s net imports are given by
\[
(\bar{z}_1^S, \bar{z}_2^S) = (x_1, x_2) - (y_1, y_2) = (\sqrt{3}/4, \sqrt{3}/16) - (\sqrt{3}/2, 0),
\]
which evaluate to \((\bar{z}_1^S, \bar{z}_2^S) = (-0.433012701892, 0.108253175473)\) (see Figure 2).

(b.ii) For an equilibrium under diversification to exist, the two equations
\[
y_1^2 + y_2^{1/2} = l \quad \text{and} \quad y_1 y_2^{1/2} = p
\]
must be satisfied. The first is the equation of the country’s production-possibility frontier, obtained by substituting the production functions into the resource-allocation constraint. The second is the equality of price ratios to the ratios of average costs, which holds when both commodities are produced. Combining these two equations we obtain the cubic equation
\[
y_1^3 - l y_1 + p = 0.
\]
Substituting in our numerical values \(l = 3/4\) and \(p = 1/4\), this becomes
\[
y_1^3 - 0.75y_1 + 0.25 = (y_1 + 1)(y_1^2 - y_1 + 0.25) = (y_1 + 1)(y_1 - 0.5)^2 = 0.
\]
The first factorization is obtained by noting that since \(l + p = 1\), \(y_1 = -1\) is a root of the cubic polynomial, and the quadratic is easily seen to have two repeated real roots.

Thus the equilibrium outputs under diversification are
\[
y_1^D = 0.5 \quad \text{and} \quad y_2^D = (0.75 - y_1^2)^2 = 0.25
\]
(where the superscript D stands for diversification). The value of national income is then \(Y = 0.25(0.5) + 0.25 = 0.375\). The budget equation for consumers is
\[
0.25x_1 + x_2 = 0.375.
\]
Utility maximization subject to this constraint leads to
\[
x_2 = 0.25x_1.
\]
Combining these two equations we obtain
\[
x_1^D = 3/4 = 0.75 \quad \text{and} \quad x_2^D = 3/16 = 0.1875
\]
whence utility is \(x_1x_2 = 9/64 = 0.140625\), which exceeds the utility under autarky. We verify that
\[
(x_1^D, x_2^D) = (0.75, 0.1875) \quad \text{and} \quad (y_1^D, y_2^D) = (0.5, 0.25)
\]
imply
\[
(\bar{z}_1^D, \bar{z}_2^D) = (x_1^D, x_2^D) - (y_1^D, y_2^D) = (0.25, -0.0625).
\]
Thus the country exports commodity 2 and imports commodity 1.
3 Effect of a capital inflow on a small open economy

In the model with uniform currencies and free trade, the condition for equality of demand and supply of nontradables is

\[ h_3(p_1, p_2, p_3, \Pi(p_1, p_2, p_3, l) + D) - \hat{y}_3(p_1, p_2, p_3, l) = 0. \]

For exogenous \( p_1, p_2, D, l \) this equation defines implicitly the function

\[ p_3 = \hat{p}_3(p_1, p_2, D, l). \]

Substituting (3.3) in (3.2) and differentiating (3.2) through with respect to \( D \), we get

\[ \frac{\partial \hat{p}_3}{\partial D} = \frac{c_3}{s_{33} - t_{33}}, \]

where \( c_3 = \partial h_3 / \partial Y \) and \( s_{33} \) and \( t_{33} \) are the Slutsky and transformation terms for the compensated demand for and the supply of the nontradable with respect to its price.

Defining the exchange rate, \( \chi \), as the price of the open economy’s currency in terms of the foreign currency, we have \( p_j^* = \chi p_j \) \((j = 1, 2)\), where \( p_j \) is the price of commodity \( j \) in the domestic currency and \( p_j^* \) its price expressed in foreign currency.

Expressing our country’s condition for equality of demand and supply of the nontradable in terms of the exogenous variables \( p_1^*, p_2^*, D^*, \) and \( l \), we have from (3.2)

\[ h_3(p_1^*/\chi, p_2^*/\chi, p_3, \Pi(p_1^*/\chi, p_2^*/\chi, p_3, l) + D^*/\chi) - \hat{y}_3(p_1^*/\chi, p_2^*/\chi, p_3, l) = 0. \]

Since \( \Pi \) is homogeneous of degree 1 in \( p_1, p_2, \) and \( p_3 \); \( h_3 \) is homogeneous of degree 0 in \( p_1, p_2, p_3, \) and \( D; \) and \( \hat{y}_3 \) is homogeneous of degree 0 in \( p_1, p_2, \) and \( p_3 \); multiplying these four variables by \( \chi \) we obtain

\[ F_1(\chi, p_3; D^*) \equiv h_3(p_1^*, p_2^*, \chi p_3, \Pi(p_1^*, p_2^*, \chi p_3, l) + D^*) - \hat{y}_3(p_1^*, p_2^*, \chi p_3, l) = 0, \]

the exogenous variables \( p_1^*, p_2^*, l \) being suppressed since they are assumed constant throughout.

Note that (3.5) defines implicitly only the product variable \( \chi p_3 \) (the price of our country’s nontradable expressed in units of the foreign currency), as a function of the exogenous variable \( D^* \) (as well, of course, of \( p_1^*, p_2^*, \) and \( l \)). A second equation is needed to separate out the effects on \( \chi \) and \( p_3 \). This is the equation

\[ \bar{c}_1 p_1^*/\chi + \bar{c}_2 p_2^*/\chi + \bar{c}_3 p_3 - \bar{p} = 0 \]

expressing the open economy’s monetary authority’s policy of stabilization of the general price level. Multiplying this equation through by \( \chi \) we obtain

\[ F_2(\chi, p_3; D^*) \equiv \bar{c}_1 p_1^* + \bar{c}_2 p_2^* + \bar{c}_3 \chi p_3 - \chi \bar{p} = 0. \]

Equations (3.5) and (3.6) together define implicitly the two functions

\[ \chi = \hat{\chi}(D^*) \quad \text{and} \quad p_3 = \hat{p}_3(D^*). \]
Taking differentials of the pair of equations (3.5) and (3.6) we obtain

\[
\begin{pmatrix}
\frac{\partial F_1}{\partial \chi} & \frac{\partial F_1}{\partial p_3} \\
\frac{\partial F_2}{\partial \chi} & \frac{\partial F_2}{\partial p_3}
\end{pmatrix}
\begin{pmatrix}
\frac{d\hat{\chi}}{dD^*} \\
\frac{d\hat{p}_3}{dD^*}
\end{pmatrix}
+ \begin{pmatrix}
\frac{\partial F_1}{\partial D^*} \\
\frac{\partial F_2}{\partial D^*}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

Computing the partial derivatives of the \(F_i\)s in the matrices of (3.7), we obtain

\[
\begin{pmatrix}
(s_{33} - t_{33})p_3 & (s_{33} - t_{33})\chi \\
\bar{c}_3p_3 - \bar{p} & \bar{c}_3\chi
\end{pmatrix}
\begin{pmatrix}
\frac{d\hat{\chi}}{dD^*} \\
\frac{d\hat{p}_3}{dD^*}
\end{pmatrix}
= \begin{pmatrix}
-c_3 \\
0
\end{pmatrix},
\]

where \(c_3 = \partial h_3/\partial Y\). The Jacobian determinant in (3.8) is simply \((s_{33} - t_{33})\chi\bar{p} < 0\). Consequently, the solution of (3.8) is

\[
\begin{pmatrix}
\frac{d\hat{\chi}}{dD^*} \\
\frac{d\hat{p}_3}{dD^*}
\end{pmatrix}
= \frac{1}{(s_{33} - t_{33})\chi\bar{p}} \begin{pmatrix}
\bar{c}_3\chi & -(s_{33} - t_{33})\chi \\
\bar{p} - \bar{c}_3p_3 & (s_{33} - t_{33})p_3
\end{pmatrix}
\begin{pmatrix}
-c_3 \\
0
\end{pmatrix}
\]

\[
= \frac{c_3}{-(s_{33} - t_{33})} \begin{pmatrix}
\bar{c}_3 & 0 \\
\bar{p} - \bar{c}_3p_3 & \chi\bar{p}
\end{pmatrix}
\]

As long as the nontradable is a superior good (i.e., \(c_3 > 0\)), the capital inflow results in both an appreciation of the country’s currency and a rise in the domestic price of its nontradable. The intuition behind this is that the rise in disposable income resulting from the capital inflow will increase the demand for the nontradable, which will require resources to move out of the tradables sector into the nontradables sector. The currency appreciation automatically lowers the domestic prices of the tradables, and the price-level-stabilization policy compensates with a rise in the domestic price of the tradable, encouraging the resource reallocation. Note that if the country’s preferences are homothetic, then we have \(\partial h_3/\partial Y = h_3/Y\), hence at the equilibrium point, \(c_3 = \bar{c}_3\).
Figure 1

Elasticity of country 2’s offer function
Figure 2
Autarky and trade equilibria under external economies and diseconomies of scale