Notes on Recursive Utility

Consider the setting of consumption in infinite time under uncertainty as in Section 1 (or Chapter 29, LeRoy & Werner, 2nd Ed.)

Let $u_{s_t}$ be the continuation utility at $s_t$. That is, utility function defined on consumption plans for all events that are successors of $s_t$, including $s_t$. Using the notation of Section 2 or Chapter 30, continuation consumption plan from event $s_t$ on is $(c(s_t), c_+(s_t))$.

We write $u_{s_t}(c)$ instead of $u_{s_t}(c(s_t), c_+(s_t))$. We will often suppress events from the notation and simply write $u_t(c)$ for $(F_t$-measurable) date-$t$ continuation utility.

A family of continuation utilities $\{u_t\}_{t=0}^{\infty}$ is Epstein-Zin-Weil recursive if

$$u_t(c) = W(c_t, \mu_t(u_{t+1}(c)))$$

(1)

for every $t$ and every $^1 c$, for some functions $W : \mathcal{R}^2 \rightarrow \mathcal{R}$ and $\mu_t$ mapping $F_{t+1}$-measurable random variables to $F_t$-measurable r.v.’s.

Function $W$ is the aggregator function; $\mu_t$ is the date-$t$ certainty equivalent. Aggregator function assigns utility to current consumption and certainty equivalent of next-period continuation utility. EPZ recursivity (1) re-

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$^1$ This usually involves some specific restrictions on consumption plans.
quires that utility is measured in consumption-equivalent units (so that the certainty equivalent maps uncertain consumption into a certain one).

**Example:** The discounted time-separable expected utility (29.1) gives rise to a family of continuation utilities

\[ U_t(c) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} E_t[v(c_\tau)] \]

Expressed in consumption-equivalent units, the family of utilities is

\[ u_t(c) = v^{-1}\left(\sum_{\tau=t}^{\infty} \delta^{\tau-t} E_t[v(c_\tau)]\right) \]  

Continuation utilities (2) have recursive representation with

\[ W(y, z) = v^{-1}(v(y) + \delta v(z)), \quad \text{and} \quad \mu_t(w) = v^{-1}(E_t[v(w)]) \]

Indeed, for specification (2), relation (1) becomes

\[ u_t(c) = v^{-1}(v(c_t) + \delta E_t[v(u_{t+1}(c))]), \]

which is satisfied by (2).

For power utility \( v(y) = \frac{1}{1-\rho} y^{1-\rho} \) (with relative risk aversion \( \rho \)), the aggregator \( W \) of (3) is \( W(y, z) = (y^{1-\rho} + \delta z^{1-\rho})^{1/(1-\rho)} \) and is the well-known constant elasticity of substitution (CES) function.  

Epstein, Zin and Weil idea is to take different functions \( v \) for \( W \) and for \( \mu \) in (3), so as to separate time preferences from risk preferences. More specific,
EPZ recursive utility obtains by taking

\[ W(y, z) = (y^\alpha + \delta z^\alpha)^{1/\alpha} \]  

(4)

and

\[ \mu_t(w) = [E_t(w^{1-\rho})]^{1/(1-\rho)} \]

(5)

where \( \rho \) and \( \alpha \) are two separate parameters.

Put together, the Epstein-Zin-Weil recursive utility is written as

\[ u_t(c) = \left( c_t^\alpha + \delta [E_t(u_{t+1}(c))^{1-\rho}]^{\alpha/(1-\rho)} \right)^{1/\alpha} \]  

(6)

Notes: (1) Epstein and Zin (1989) prove that recursive utilities (6) are well defined. That is, there exists \( \{u_t\} \) satisfying (6). In Epstein and Zin (1989) the domain of utility functions are intertemporal lotteries. In the setting of dynamic consumption under uncertainty, as in Section 1 (or Chapter 29), the existence is demonstrated in Marinacci and Montrucchio (2010) and Borovicka and Stachurski (2018). Those results impose some restrictions on \( \alpha \) and \( \rho \), on domain of consumption plans, and some limiting condition on utilities \( u_t(c) \) (such as boundedness).

(2) Often, discounted expected utility is considered in the scaled form

\[ U_t(c) = (1-\delta) \sum_{\tau=t}^{\infty} \delta^{\tau-t} E_t[v(c_\tau)]. \]

This leads to the aggregator of the form

\[ W(y, z) = v^{-1}((1-\delta)v(y) + \delta v(z)) \]  

and, in particular \( W(y, z) = ((1-\delta)y^\alpha + \delta z^\alpha)^{1/\alpha} \), but otherwise the analysis remains unchanged.
Suppose that there are $J$ infinitely-lived securities with one-period returns at $s_t$ given by $r(s_t)$. Suppressing the notation for events we write $r_t$.

Consider the optimal consumption-portfolio problem of an agent with recursive utility specified by (6). The agent has non-zero endowment only at the initial date $0$. It is more convenient to state the consumption-portfolio problem in terms of consumption $c_t$, wealth $w_t$, and fractions of wealth (net of consumption) $a_t \in \Delta^J$ invested in $J$ securities at date $t$. Short sales are prohibited. Budget constraints are

$$w_{t+1} = (w_t - c_t)r_{t+1}a_t,$$

with $w_0$ given as date-0 endowment plus the value of initial portfolio of securities.

Let $V_0(w_0)$ denote the maximized date-0 utility as a (value) function of initial wealth $w_0$. Similarly, let $V_t(w_t)$ denote the maximized date-$t$ continuation utility as a (value) function of date-$t$ wealth $w_t$. Recursivity of the utility function implies that

$$V_t(w_t) = \max_{c_t,a_t} \left( c_t^\alpha + \delta[E_t(V_{t+1}(w_{t+1}))^{1-\rho}]^{\alpha/(1-\rho)}\right)^{1/\alpha}$$

where $w_{t+1}$ is given by (7).

Because functions $W$ and $\mu_t$ of (6) are homogeneous of degree 1, it follows that value function $V_t$ is linear in wealth, that is, $V_t(w_t) = A_t w_t$ where $A_t > 0$.
is a constant that may depend on date-t. Certainty equivalent of $V_{t+1}$ in (8) can be written as

$$[E_t[(A_{t+1}(w_t - c_t)r_{t+1}a_t)^{1-\rho}]]^{1/(1-\rho)} = (w_t - c_t)[E_t[(A_{t+1}r_{t+1}a_t)^{1-\rho}]]^{1/(1-\rho)}$$

(9)

Let $\{c^*_t, a^*_t\}$ be a solution to the consumption-portfolio problem. Assume that $0 < c^*_t \leq w_t$ and $0 < a^*_t$. It follows from (8) and (9) that $a^*_t$ solves

$$\max_{a_t \in \Delta} E_t[(A_{t+1}r_{t+1}a_t)^{1-\rho}]$$

(10)

The first-order conditions for (10) are

$$E_t[A_{t+1}^{1-\rho}(r^*_{t+1})^{-\rho}(r_{j,t+1} - \bar{r}_{t+1})] = 0,$$

(11)

for every risky security $j$, where $\bar{r}_{t+1}$ denotes one-period risk-free return and $r^*_{t+1} = r_{t+1}a^*_t$ denotes the return on optimal portfolio $a^*_t$.

The optimal consumption $c^*_t$ solves

$$\max_{c_t} [c_t^\alpha + \delta(w_t - c_t)^\alpha(z_t^*)^{\alpha/(1-\rho)}],$$

where $z_t^*$ denotes the maximum value in (10), that is,

$$z_t^* = E_t[(A_{t+1}r^*_{t+1})^{1-\rho}].$$

The first-order condition for $c^*_t$ is

$$(c_t^*)^{\alpha-1} = \delta(w_t - c_t^*)^{\alpha-1}(z_t^*)^{\alpha/(1-\rho)}.$$
or equivalently
\[
(z_t^*)^{\alpha/(1-\rho)} = \frac{1}{\delta} \left( \frac{c_t^*}{w_t - c_t^*} \right)^{\alpha-1}.
\] (12)

It follows from (8) that
\[
(A_tw_t)^\alpha = (c_t^*)^\alpha + \delta(w_t - c_t^*)^{\alpha/(1-\rho)}
\] (13)

Substituting (12) in (13) we obtain
\[
A_t = \left( \frac{c_t^*}{w_t} \right)^{(\alpha-1)/\alpha}
\] (14)

Writing (14) for \( t + 1 \) instead of \( t \) and making use of budget constraint (7) there results
\[
A_{t+1} = \left( \frac{c_{t+1}^*}{c_t^*} \right)^{(\alpha-1)/\alpha} \left( \frac{c_t^*}{w_t - c_t^*} \right)^{(\alpha-1)/\alpha} \left( r_{t+1}^* \right)^{(1-\alpha)/\alpha}
\] (15)

Substituting (15) in the first-order condition (11), dividing both sides by 
\[
\left( \frac{c_t^*}{w_t - c_t^*} \right)^{(\alpha-1)/\alpha},
\] and making use of \( \frac{(1-\alpha)(1-\rho)}{\alpha} - \rho = \frac{1-\alpha-\rho}{\alpha} \), we obtain
\[
E_t \left[ \left( \frac{c_{t+1}^*}{c_t^*} \right)^{(\alpha-1)(1-\rho)/\alpha} \left( r_{t+1}^* \right)^{(1-\alpha-\rho)/\alpha} \left( r_{j,t+1} - \bar{r}_{t+1} \right) \right] = 0.
\] (16)

Substituting (15) in the first-order condition (12) and simplifying, there results
\[
E_t \left[ \left( \frac{c_{t+1}^*}{c_t^*} \right)^{(\alpha-1)(1-\rho)/\alpha} \left( r_{t+1}^* \right)^{(1-\rho)/\alpha} \right] = 1
\] (17)

Equations (16) and (17) are the Consumption-Based Security Pricing for Epstein-Zin-Weil recursive utility. In a representative-agent economy with
an outstanding portfolio of securities $\hat{h}_0$ and zero consumption endowments at dates $t \geq 1$, the return $r_{t+1}^*$ equals the market return $r_{t+1}^m = \frac{(p_{t+1}^t + x_{t+1}^t)\hat{h}_0}{p_t^t \hat{h}_0}$.

Note that for $\alpha = 1 - \rho$, that is, when the recursive utility reduces to expected (power) utility, we obtain

$$E_t\left[\left(\frac{c_{t+1}^*}{c_t^*}\right)^{-\rho} (r_{j,t+1} - \bar{r}_{t+1})\right] = 0$$

from which the standard CBSP equations (see Chapter 27) can be derived.
I.1 Equilibrium Prices in Security Markets

- Assume throughout this section that utility functions are strictly increasing.

Consumption-Based Security Pricing

This exposition is for time-separable expected utility function

\[ v_0(c_0) + E[v_1(c_1)] \]  

(\ast)

First-order conditions (1.5) for interior optimal consumption \((c_0, c_1)\) are

\[ p_j = \sum_{s=1}^{S} \frac{\pi_s v_1'(c_s) x_{js}}{v_0'(c_0)} \quad \forall \ j \]  

(14.1)

or, using expectation,

\[ p_j = \frac{E[v_1'(c_1)x_j]}{v_0'(c_0)} \quad \forall \ j. \]  

(14.1')

For risk-free security with return \(\bar{r}\), this FOC implies

\[ \bar{r} = \frac{v_0'(c_0)}{E[v_1'(c_1)]}. \]  

(14.3)

For risky security \(j\), we obtain from (14.1') and (14.3)

\[ E(r_j) = \bar{r} - \bar{r} \frac{\text{cov}(v_1'(c_1), r_j)}{v_0'(c_0)}. \]  

(14.6)

(14.6) is the equation of **Consumption-Based Security Pricing** (CSBP).

CBSP holds for any portfolio return \(r\):

\[ E(r) = \bar{r} - \bar{r} \frac{\text{cov}(v_1'(c_1), r)}{v_0'(c_0)}. \]  

(14.7)
Equilibrium Consumption and Expected Returns

Two contingent claims $y$ and $z$ are **co-monotone** if

$$(y_s - y_t)(z_s - z_t) \geq 0, \quad \forall s, t$$

Contingent claims $y$ and $z$ are **strictly co-monotone** if

$$y_s > y_t \text{ iff } z_s > z_t, \quad \forall s, t$$

Strict co-monotonicity implies that $y_s = y_t$ iff $z_s = z_t$. Co-monotonicity does not. $y$ and $z$ are **negatively co-monotone** iff $y$ and $-z$ are co-monotone.

**Proposition, 14.4.1:** If $y$ and $z$ are co-monotone, then $\text{cov}(z, y) \geq 0$. If $y$ and $z$ are strictly co-monotone and nondeterministic, then $\text{cov}(z, y) > 0$.

**Proof:** This follows from

$$\text{cov}(y, z) = \frac{1}{2} \sum_{s=1}^{S} \sum_{t=1}^{S} \pi_s \pi_t (y_s - y_t)(z_s - z_t).$$

Combining Proposition 14.4.1 with CBSP, we obtain

**Theorem, 14.4.2:** If an agent is risk averse, then $E(r) \geq \bar{r}$ for every return $r$ that is co-monotone with optimal consumption. For every return $r$ that is negatively co-monotone with optimal consumption, it holds $E(r) \leq \bar{r}$.

There is strict version of 14.4.2 for strict risk aversion, strictly co-monotone returns, and strict inequalities $E(r) > \bar{r}$ or $E(r) < \bar{r}$.
Volatility of Marginal Rates of Substitution

The first-order condition for expected utility (*) also implies that

$$\sigma \left( \frac{v_1'(c_1)}{v_0'(c_0)} \right) \geq \frac{|E(r_j) - \bar{r}|}{\bar{r}\sigma(r_j)}.$$  \hspace{1cm} (14.15)

where $\sigma(\cdot)$ denotes the standard deviation.

The ratio of risk premium to standard deviation of return is called the **Sharpe ratio**. The marginal rate of substitution between consumption at date 0 and at date 1 in equilibrium is higher than the (absolute value of) the Sharpe ratio of each security divided by the risk-free return.

Inequality (14.15) is known as Hansen-Jagannathan bound.

Further,

$$\sigma \left( \frac{v_1'(c_1)}{v_0'(c_0)} \right) \geq \sup_r \frac{|E(r) - \bar{r}|}{\bar{r}\sigma(r)}.$$  \hspace{1cm} (14.16)

Notes: This part was based on Chapter 14 of LeRoy and Werner (2001).
I.2 Pareto-Optimal Allocations of Risk

Consumption allocation \( \{\tilde{c}^i\} \) Pareto dominates another allocation \( \{c^i\} \) if every agent \( i \) weakly prefers consumption plan \( \tilde{c}^i \) to \( c^i \), that is,

\[
u^i(\tilde{c}^i) \geq u^i(c^i),
\]

and in addition at least one agent \( i \) strictly prefers \( \tilde{c}^i \) to \( c^i \) (so that strict inequality holds for at least one \( i \)).

A feasible consumption allocation \( \{c^i\} \) is \textbf{Pareto optimal} if there does not exist an alternative feasible allocation \( \{\tilde{c}^i\} \) that Pareto dominates \( \{c^i\} \). Feasibility of \( \{c^i\} \) means that

\[
\sum_{i=1}^{I} c^i \leq \bar{w},
\]

where \( \bar{w} = \sum_{i=1}^{I} w^i \) denotes the aggregate endowment.

If \( \{c^i\} \) is interior and utility functions are differentiable, the first-order conditions for Pareto optimality are

\[
\frac{\partial_s u^i(c^i)}{\partial_t u^i(c^i)} = \frac{\partial_s u^k(c^k)}{\partial_t u^k(c^k)} \quad \forall i, k, \forall s, t
\]  

\[(15.6)\]
First Welfare Theorem in Complete Security Markets

**Theorem, 15.3.1:** If security markets are complete and agents’ utility functions are strictly increasing, then every equilibrium consumption allocation is Pareto optimal.

Complete Markets and Options

If there is payoff $z \in \mathcal{M}$ which takes different values in different states, then $S - 1$ options on $z$ complete the markets.

Pareto-Optimal Allocations under Expected Utility

Suppose that agents’ utility functions have expected utility representations with common probabilities.

**Theorem, 15.5.1:** If agents are strictly risk averse, then at every Pareto-optimal allocation their date-1 consumption plans are co-monotone with each other and with the aggregate endowment.
The proof of Theorem 15.5.1 draws on the concept of greater risk (Ch. 10). An easier argument is available when agents’ utility functions are differentiable. It applies to interior allocations and shows that each agent’s date-1 consumption plan is **strictly co-monotone** with the aggregate endowment. The argument is as follows:

Pareto optimal allocation \( \{c^i\} \) must be a solution to the maximization of weighted sum of utilities \( \sum_{i=1}^{I} \mu^i u^i(\tilde{c}^i) \) subject to feasibility \( \sum_{i=1}^{I} \tilde{c}^i \leq \bar{w} \), for some weights \( \mu^i > 0 \). If the allocation is interior and \( u^i \) has expected utility form (*) with the same probabilities, then first-order conditions for this constrained maximization imply that

\[
\mu^i v^i_1(c^i_s) = \mu^k v^k_1(c^k_s), \quad \forall i, k, \forall s.
\]

Since date-1 marginal utilities \( v^i_1 \) are strictly decreasing, it follows that if

\[
c^i_s \geq c^i_t
\]

for some \( i \), for states \( s \) and \( t \), then

\[
c^k_s \geq c^k_t
\]

for every \( k \). Hence \( c^1_1 \) and \( c^k_1 \) are strictly co-monotone for every \( i \) and \( k \).
If consumption plans \( \{c^i\} \) are co-monotone with each other and satisfy \( \sum_{i=1}^{I} c^i \equiv \bar{w} \), then \( \bar{w}_s = \bar{w}_t \) implies that \( c^i_s = c^i_t \) for every \( i \). It follows now from Theorem 15.5.1 that

**Corollary, 15.5.3:** If agents are strictly risk averse and the aggregate date-1 endowment is state independent for a subset of states, then each agent’s date-1 consumption at every Pareto-optimal allocation is state independent for that subset of states.

Co-monotonicity of consumption plans implies that the variance of aggregate consumption (which equals the aggregate endowment) is greater than the sum of variances of individual consumption plans.

\[
\text{var}\left(\sum_{i=1}^{I} c^i\right) \geq \sum_{i=1}^{I} \text{var}(c^i).
\]
Equilibrium Expected Returns in Complete Markets

If security markets are complete and agents are strictly risk averse, then equilibrium date-1 consumption plans are co-monotone with the aggregate date-1 endowment \( \bar{\omega}_1 \). Then any return that is co-monotone with \( \bar{\omega}_1 \) is also co-monotone with every agent’s date-1 consumption plan in equilibrium. Using Theorem 14.4.2 from Section 7, we obtain

**Theorem, 15.6.1:** If security markets are complete, all agents are strictly risk averse, and have strictly increasing utility functions, then \( E(r) \geq \bar{r} \) for every return \( r \) that is co-monotone with the aggregate date-1 endowment. For every \( r \) that is negatively co-monotone with the aggregate endowment, it holds \( E(r) \leq \bar{r} \).

Of course, \( \bar{\omega}_1 \) is co-monotone with itself. The return on \( \bar{\omega}_1 \) is \( r_m = \bar{w}_1/q(\bar{w}_1) \) – the market return. Thus

\[
E(r_m) \geq \bar{r}
\]

[This holds in equilibrium with incomplete markets, too.]
Pareto-Optimal Allocations under Linear Risk Tolerance

**Theorem, 15.7.1:** If every agent’s risk tolerance is linear with common slope $\gamma$, i.e.,

$$T^i(y) = \alpha^i + \gamma y,$$

then date-1 consumption plans at any Pareto-optimal allocation lie in the span of the risk-free payoff and the aggregate endowment.

The consumption set of agent $i$ is $\{c \in R^S : T^i(c_s) > 0, \text{ for every } s\}$.

Pareto-Optimal Allocations under Multiple-Prior Expected Utility

**Theorem, 15.8.1:** If there is no aggregate risk, agents have strictly concave utility functions and at least one common probability belief, i.e.,

$$\bigcap_{i=1}^{I} P_i \neq \emptyset.$$

then each agent’s date-1 consumption at every Pareto-optimal allocation is risk free.
**Proof of Theorem 15.3.1:** Let \( p \) and \( \{c^i\} \) be an equilibrium in complete security markets. Consumption plan \( c^i \) maximizes \( u^i(c_0, c_1) \) subject to

\[
c_0 \leq w_0^i - qz
\]

\[
c_1 \leq w_1^i + z, \quad z \in \mathcal{M} = \mathcal{R}^S,
\]

where \( q \) is the (unique) vector of state prices.

The above budget constraints are equivalent to a single budget constraint

\[
c_0 + qc_1 \leq w_0^i + qw_1^i.
\]

Suppose that allocation \( \{c^i\} \) is not Pareto optimal, and let \( \{\tilde{c}^i\} \) be a feasible Pareto dominating allocation. Then (since \( u^i \) is strictly increasing)

\[
\tilde{c}_0^i + q\tilde{c}_1^i \geq w_0^i + qw_1^i
\]

for every \( i \), with strict inequality for agents who are strictly better-off. Summing over all agents, we obtain

\[
\sum_{i=1}^I \tilde{c}_0^i + \sum_{i=1}^I q\tilde{c}_1^i > \tilde{w}_0 + q\tilde{w}_1
\]

which contradicts the assumption that allocation \( \{\tilde{c}^i\} \) is feasible.
I.3 Effectively Complete Security Markets

A consumption allocation \( \{c^i\} \) is **attainable through security markets** if the net trade \( c^i_1 - w^i_1 \) lies in the asset span \( M \) for every agent \( i \).

Security markets are **effectively complete** if every Pareto-optimal allocation is attainable through security markets.

**Theorem, 16.4.1:** *If consumption sets are \( \mathcal{R}_+^{S+1} \), then every equilibrium consumption allocation in effectively complete security markets is Pareto optimal.*

**Examples of Effectively Complete Markets**

All with (1) strictly risk averse expected utilities with common probabilities, (2) endowments in the asset span, i.e., security markets economy, (3) the risk-free payoff in the asset span, and (4) with consumption restricted to being positive (except for LRT).

I. **There is no aggregate risk.**

II. **All options on aggregate endowment** are in the asset span.

III. **Agents’ utility functions have Linear Risk Tolerance with common slope** (and are time separable), that is, \( T^i(y) = \alpha^i + \gamma y \) for all \( i \).
In III consumption sets are \( \{ c \in \mathcal{R}^S : T^i(c_s) > 0, \forall s \} \). Theorem 16.4.1 applies despite these sets not being \( \mathcal{R}^{S+1}_+ \), see Proposition 16.7.1. Case III holds if all agents have quadratic utility functions.

**Representative Agent under LRT**

Consider effectively complete markets with LRT utilities (III). Equilibrium consumption allocation is Pareto optimal. Let state prices \( q \) be equal to (common) marginal rates of substitution. It holds

\[
\left( \frac{q_s}{q_t} \right)^{-\gamma} = \left( \frac{\pi_s}{\pi_t} \right)^{-\gamma} \frac{\sum_i \alpha^i + \gamma \bar{w}_s}{\sum_i \alpha^i + \gamma \bar{w}_t}
\]

(16.20)

for \( \gamma \neq 0 \), and

\[
\ln \left( \frac{q_s}{q_t} \right) = \ln \left( \frac{\pi_s}{\pi_t} \right) + \frac{1}{\sum_i \alpha^i} \left( \bar{w}_t - \bar{w}_s \right)
\]

(16.22)

for \( \gamma = 0 \) (negative exponential utility).

Eq. (16.20) and (16.22) imply two things:

- **Theorem, 16.7.1:** Equilibrium security prices in effectively complete markets with LRT utilities do not depend on the distribution of agents’ endowments.
The same state prices, and hence security prices, would obtain if there were a single agent with LRT utility function with risk tolerance

$$T(y) = \sum_{i=1}^{I} \alpha^i + \gamma y$$

(16.23)

and endowment equal to the aggregate endowment $\bar{w}$.

We refer to the single agent of (16.23) as the representative agent of the security markets economy with LRT utilities.

**Conclusion:** Equilibrium security prices in a heterogeneous-agent economy are the same as in the representative-agent economy for every allocation of endowments in the heterogeneous-agent economy.