Mathematical Appendix I.

Kuhn-Tucker Theorems

I.1 Constrained Maximization: Necessary Conditions.

Function $F : \mathbb{R}^n_+ \to \mathbb{R}$ is the objective function; functions $g^j : \mathbb{R}^n_+ \to \mathbb{R}$, for $j = 1, \ldots, k$, are constraint functions. Assume that $F$ and $g^j$ are differentiable, with partial derivatives $\frac{\partial F}{\partial x_i}$ and $\frac{\partial g^j}{\partial x_i}$ denoted by $\partial_i F$ and $\partial_i g^j$, respectively.

The constrained maximization problem (with nonnegativity constraints) is

$$\max_x F(x)$$

subject to $g^1(x) \geq 0,$

$\ldots,$

$g^k(x) \geq 0,$

$x_1 \geq 0, \ldots, x_n \geq 0.$

We write the Lagrangian as

$$\mathcal{L}(\lambda^1, \ldots, \lambda^k, x) = F(x) + \sum_{j=1}^{k} \lambda_j g^j(x),$$

where $\lambda_j \geq 0$, for $j = 1, \ldots, k$, are the Lagrange multipliers. We use $\lambda$ to denote the $k$-vector of multipliers.
Kuhn-Tucker conditions for $x^* \geq 0$ and $\lambda^* \geq 0$ are:

for all $i = 1, \ldots, n$ and $j = 1, \ldots, k$,

$$\partial_i F(x^*) + \sum_{j=1}^{k} \lambda^*_j \partial_i g^j(x^*) \leq 0, \quad \text{and if } x^*_i > 0, \text{ then } " = 0", \quad (2a)$$

$$g^j(x^*) \geq 0, \quad \text{and if } \lambda^*_j > 0, \text{ then } " = 0". \quad (2b)$$

Where do these conditions come from? Think about maximizing Lagrangian $L(\lambda, x)$ with respect to $x$ and minimizing it with respect to $\lambda$, unconstrained, except for $x \geq 0$ and $\lambda \geq 0$. This is the saddle-point. K-T conditions (2) are FOCs for such max-min (or saddle-point) problem.

Theorem (Kuhn-Tucker): If $x^* \geq 0$ is a solution to the constrained maximization problem, and the Constraint Qualification Condition holds, then $x^*$ and some $\lambda^* \geq 0$ satisfy K-T conditions (2).

Constraint Qualification Condition:

(i) Kuhn-Tucker original – don’t touch it.

(ii) $g^j$ concave for all $j$, and Slater’s condition, that is, there is some $x^0 \geq 0$ with $g^j(x^0) > 0$ for all $j$.

(iii) rank condition (see Takayama 1.D.4, or Varian, ch 27),

(iv) $g^j$ linear for all $j$, (Arrow-Hurwicz-Uzawa, see Takayama 1.D.4)
I.2 Sufficiency of Kuhn-Tucker Conditions.

The most standard theorem is:

**Theorem S1:** Suppose that $F$ and $g^1, \ldots, g^k$ are all concave functions. If $x^* \geq 0$ and $\lambda^* \geq 0$ satisfy K-T conditions (2), then $x^*$ is a solution to the constrained maximization problem.

A better theorem is due to Arrow and Enthoven (1961).

**Theorem S2:** Suppose that $F$ and $g^1, \ldots, g^k$ are all quasi-concave functions and some “mild” condition holds. If $x^* \geq 0$ and $\lambda^* \geq 0$ satisfy K-T conditions (2), then $x^*$ is a solution to the constrained maximization problem.

The extra (“mild”) condition is not needed if $F$ is concave (and $g^1, \ldots, g^k$ are quasi-concave). See Takayama 1.E for three versions of the condition.

Quasi-concavity (and therefore also concavity) of functions $g^j$ implies that the constraint set, i.e. the set of $x \geq 0$ satisfying $g^1(x) \geq 0, \ldots, g^k(x) \geq 0$, is convex.
I.3 Constrained Minimization

The constrained minimization problem (with nonnegativity constraints) is

$$\min_x F(x)$$

subject to

$$g^1(x) \leq 0, \ldots, g^k(x) \leq 0,$$

$$x_1 \geq 0, \ldots, x_n \geq 0.$$  

The Lagrangian is

$$\mathcal{L}(\lambda, x) = F(x) + \sum_{j=1}^{k} \lambda_j g^j(x).$$

**Kuhn-Tucker conditions** for $$x^* \geq 0$$ and $$\lambda^* \geq 0$$ are,

for all $$i = 1, \ldots, n$$ and $$j = 1, \ldots, k,$$

$$\partial_i F(x^*) + \sum_{j=1}^{k} \lambda^*_j \partial_i g^j(x^*) \geq 0, \quad \text{and if } x^*_i > 0, \text{ then } \ “= \ 0”, \quad (4a)$$

$$g^j(x^*) \leq 0, \quad \text{and if } \lambda^*_j > 0, \text{ then } \ “= \ 0”. \quad (4b)$$

The corresponding saddle-point problem is to *minimize* Lagrangian $$\mathcal{L}(\lambda, x)$$ with respect to $$x$$ and *maximize* it with respect to $$\lambda$$ for $$x \geq 0$$ and $$\lambda \geq 0.$$ The Kuhn-Tucker Theorem holds with no change for the constrained minimization problem. However, in constraint qualification conditions concavity of functions $$g^j$$, if present, has to be replaced by their convexity. This guarantees convexity of the constraint set described here by inequalities $$g^j(x) \leq 0.$$
Theorems S1 and S2 continue to hold with concavity (quasi-concavity) of functions $F$ and $g^j$ replaced by their convexity (quasi-convexity, respectively).

I.4 Remarks:

- **Applications** of K-T theorems in microeconomics:
  
  (i) Consumer theory: utility maximization subject to budget constraint, and expenditure minimization.

  (ii) Welfare economics: Characterization of Pareto optimal allocations as solutions to maximization of a welfare function subject to resource constraints, or maximization of one agent’s utility subject to constraints on other agents’ utilities and resource constraints.

  (iii) Producer theory: cost minimization.

- There are versions of K-T theorems for maximization and minimization with mixed constraints, i.e., when some constraints are of the equality form, $g^j(x) = 0$. See Sundaram [2], Section 6.4.

- K-T theorems hold for *local* maxima (minima) as well.


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Consider the following constrained maximization problem:

\[
\text{maximize} \quad \ln(x_1 + 1) + \ln(x_2 + 1)
\]

subject to \( p_1 x_1 + p_2 x_2 \leq m \)

\[x_1 \geq 0, \quad x_2 \geq 0,\]

where \( p_1 > 0, p_2 > 0 \) and \( m > 0 \).

In order to derive the solution (as a function of parameters \( p_1, p_2 \) and \( m \)) we write the Kuhn-Tucker first-order conditions (2) as

\[
\begin{align*}
(1) \quad & \frac{1}{x_1^* + 1} - \lambda^* p_1 \leq 0, \quad \text{and if} \quad x_1^* > 0, \text{then} \quad "= 0". \\
(2) \quad & \frac{1}{x_2^* + 1} - \lambda^* p_2 \leq 0, \quad \text{and if} \quad x_2^* > 0, \text{then} \quad "= 0". \\
(3) \quad & p_1 x_1^* + p_2 x_2^* \leq m, \quad \text{and if} \lambda^* > 0, \text{then} \quad "= 0".
\end{align*}
\]

with \( x^* \geq 0 \) and \( \lambda^* \geq 0 \).

Note that (3) holds with equality since it follows from (1) that \( \lambda^* > 0 \).

We solve inequalities (1-3) by considering cases:

**Case 1.** \( x_1^* > 0, x_2^* > 0 \).

Then (1) and (2) hold with equalities. Solving (1), (2) and (3) we find \( x_1^* = \frac{m + p_2 - p_1}{2p_1} \) and \( x_2^* = \frac{m + p_1 - p_2}{2p_2} \) and \( \lambda^* = \frac{2}{m + p_1 + p_2} \). For \( x_1^* \) and \( x_2^* \) to be strictly positive, it has to be that \( m + p_2 > p_1 \) and \( m + p_1 > p_2 \). Thus Case 1 applies with \( x_1^* \) and \( x_2^* \) as listed above if \( m + p_2 > p_1 \) and \( m + p_1 > p_2 \).
Case 2. $x_1^* > 0, x_2^* = 0$.

(3) implies that $x_1^* = \frac{m}{p_1}$. Since (1) holds with equality, we solve it for
\[ \lambda^* = \frac{1}{m + p_1}. \]
Next we need to verify inequality (2). It states
\[ 1 - \frac{p_2}{m + p_1} \leq 0, \]
and it holds if $p_2 \geq m + p_1$. Thus Case 2 applies (with $x_1^* = \frac{m}{p_1}, x_2^* = 0$) if $p_2 \geq m + p_1$.

Case 3. $x_1^* = 0, x_2^* > 0$.

This case is very similar to Case 2. From (3) and (2) we obtain $x_1^* = \frac{m}{p_2}, \lambda^* = \frac{1}{m + p_2}$. Verifying inequality (1), we obtain $p_1 \geq m + p_2$. Thus Case 3 applies (with $x_1^* = 0, x_2^* = \frac{m}{p_2}$) if $p_1 \geq m + p_2$.

The case $x_1^* = x_2^* = 0$ cannot hold since it violates equation (3). This concludes our solution to the K-T conditions.

Since utility function is concave and the constraint function is concave (in fact, it is linear) K – T conditions are sufficient (Theorem S1). Hence, the solution to K-T conditions is a constrained maximizer. Further, since the Slater’s condition holds, every constrained maximizer has to satisfy K – T conditions.