
**Time:** $t = 0, 1, \ldots$

**Uncertainty:** Infinite set of states $S$. Information about the state at date $t$ is described by a finite partition $F_t$ of $S$. $F_{t+1}$ is finer than $F_t$ (nondecreasing information). $F_0 = \{S\}$.

This is an infinite event tree.

$\xi_t \in F_t$ denotes an event at date $t$; $\xi_t^- \in F_{t-1}$ is the predecessor of $\xi_t$ at date $t - 1$, that is, $\xi_t \subset \xi_t^-$. 

Securities are traded at each date - **infinitely-lived securities**. Security $j$ pays dividend $x_j(\xi_t)$ at date-t event $\xi_t$ for every $t \geq 1$.

Price of security $j$ at date $t$ in event $\xi_t$ is $p_j(\xi_t)$. A portfolio in event $\xi_t$ is $h(\xi_t)$; $h = \{h_t\}$ is a portfolio strategy.

Dividends are positive, i.e., $x(\xi_t) \geq 0$ for every $\xi_t$. 
Agents.

Consumption plans: \( c(\xi_t) \) in event \( \xi_t \) at date \( t \); \( c_t \) event-contingent consumption plan at date \( t \); \( c = (c_0, c_1, \ldots, ) \).

Agent \( i \)'s utility function \( u^i : C^i \to \mathcal{R} \), where \( C^i \subset \mathcal{R}_{++}^\infty \). \( u^i \) is assumed strictly increasing and continuous in product topology. Examples: discounted time-separable expected utility

\[
    u(c) = \sum_{t=0}^{\infty} \delta^t E[v(c_t)]
\]

for \( v : \mathcal{R}_+ \to \mathcal{R} \) with strictly increasing and continuous \( v \) and \( 0 < \delta < 1 \), recursive utility (in class).

Consumption endowment is \( w^i \). **Initial portfolio** is \( \hat{h}^i_0 \in \mathcal{R}_+^J \).

Aggregate initial portfolio \( \bar{h}_0 = \sum_{i=1}^{I} \hat{h}^i_0 \) is the supply of securities.

**Portfolio Constraints.**

Budget constraints are

\[
    c(\xi_0) + p(\xi_0) h(\xi_0) = w^i(\xi_0) + p(\xi_0) \hat{h}^i_0,
\]

\[
    c(\xi_t) + p(\xi_t) h(\xi_t) = w^i(\xi_t) + [p(\xi_t) + x(\xi_t)] h(\xi_t^-) \quad \forall \xi_t,
\]

for every \( t \geq 1 \).

Additional portfolio constraints must be imposed in order to exclude Ponzi schemes.
• debt constraints

\[ [p(\xi_{t+1}) + x(\xi_{t+1})]h(\xi_t) \geq -D(\xi_{t+1}), \quad \forall \xi_{t+1} \subset \xi_t, \quad (29.5) \]

for every \( \xi_t \). Bounds \( D \) are positive.

• borrowing constraints

\[ p(\xi_t)h(\xi_t) \geq -B(\xi_t), \quad (29.6) \]

for every \( \xi_t \). Bounds \( B \) are positive

• short sales constraints

\[ h_j(\xi_t) \geq -b_j(\xi_t), \quad \forall \ j. \quad (29.6) \]

for every \( \xi_t \), where \( b_j(\xi_t) \) is a positive

There are other possible constraints such as transversality constraint, wealth constraint, solvency constraint, collateral constraint, etc.

We focus on debt constraints (29.5).

Portfolio Choice and First-Order Conditions.

Portfolio choice is

\[ \max_{c,h} u(c) \quad (29.8) \]

subject to (29.2 - 29.3) and (29.5).
First-order conditions for a solution \((c, h)\) such that \(c(\xi_t) > 0, \forall \xi_t\) are

\[
p(\xi_t) = \sum_{\xi_{t+1} \subseteq \xi_t} [p(\xi_{t+1}) + x(\xi_{t+1})] \left[ \frac{\partial \xi_{t+1} u}{\partial \xi_t} + \frac{\mu(\xi_{t+1})}{\partial \xi_t} \right].
\] (29.11)

where \(\mu(\xi_t) \geq 0\) is the Lagrange multiplier associated with the debt constraint.

Transversality condition for discounted time-separable expected utility (29.1) is

\[
\lim_{t \to \infty} \sum_{\xi_t \in F_t} \delta^t \pi(\xi_t) v'(c(\xi_t))[(p(\xi_t) + x(\xi_t)) h(\xi_t^-) + D(\xi_t)] = 0.
\] (29.13)

**Equilibrium under Debt Constraints.**

An equilibrium under debt constraints is an allocation \(\{(c^i, h^i)\}\) and a price system \(p\) such that

(i) portfolio strategy \(h^i\) and consumption plan \(c^i\) are a solution to agent \(i\)'s choice problem (29.8)

(ii) markets clear, that is

\[
\sum_i h^i(\xi_t) = \bar{h}_0, \quad \forall \xi_t
\] (29.14)

and

\[
\sum_i c^i(\xi_t) = \bar{w}(\xi_t) + x(\xi_t) \bar{h}_0, \quad \forall \xi_t.
\] (29.15)
14. Arbitrage and Price Bubbles

Let $z(h, p)$ denote the (net) payoff of portfolio strategy $h$ in event $\xi_t$:

$$z(h, p)(\xi_t) = (p(\xi_t) + x(\xi_t))h(\xi_t^-) - p(\xi_t)h(\xi_t).$$

**Arbitrage under debt constraints** is a portfolio strategy $h$ such that

$$z(h, p)(\xi_t) \geq 0, \quad \forall \xi_t \forall t \geq 1, \text{ and } p_0 h_0 \leq 0,$$

with either the payoff or the initial price different from zero, and

$$[p(\xi_{t+1}) + x(\xi_{t+1})]h(\xi_t) \geq 0, \quad \forall \xi_{t+1} \subset \xi_t$$

for every $\xi_t$.

**One-period arbitrage** in event $\xi_t$ is a portfolio $h(\xi_t)$ that has positive one-period payoff $[p(\xi_{t+1}) + x(\xi_{t+1})]h(\xi_t) \geq 0$ for every $\xi_{t+1} \subset \xi_t$ and a negative price $p(\xi_t)h(\xi_t) \leq 0$ at $\xi_t$, with either the gross payoff or the price nonzero.

One period arbitrage is an arbitrage under debt constraints. Ponzi scheme is **not** an arbitrage under debt constraints (unless ...).

**Theorem 30.2.2:** Security prices exclude arbitrage under debt constraints iff they exclude one-period arbitrage in every event.
Event Prices.

Event prices are defined as a sequence \( q \in \mathcal{R}^\infty \) satisfying equations

\[
q(\xi_t) p_j(\xi_t) = \sum_{\xi_{t+1} \subset \xi_t} q(\xi_{t+1})[p_j(\xi_{t+1}) + x_j(\xi_{t+1})]
\]  
(30.1)

for every \( j \) and \( \xi_t \), with \( q(\xi_0) = 1 \).

**Theorem 30.3.1:** Security prices exclude arbitrage under debt constraints iff there exist strictly positive event prices.

Valuation of Dividends and Price Bubbles.

The present value of the dividend stream of security \( j \) in event \( \xi_t \) is

\[
\frac{1}{q(\xi_t)} \sum_{\tau=t+1}^{\infty} \sum_{\xi_{\tau} \subset \xi_t} q(\xi_\tau)x_j(\xi_\tau).
\]  
(30.2)

**Theorem 30.4.1:** Suppose that there exist strictly positive event prices. If security \( j \) is of finite maturity (that is, \( x_{jt} = 0 \) for \( t \geq T \) for some \( T \), and that security is not traded after date \( T \)), then

\[
p_j(\xi_t) = \frac{1}{q(\xi_t)} \sum_{\tau=t+1}^{T} \sum_{\xi_{\tau} \subset \xi_t} q(\xi_\tau)x_j(\xi_\tau).
\]  
(30.5)

**Price bubble** is the difference between the price and the present value of a security. Price bubble at \( \xi_t \) is

\[
\sigma_j(\xi_t) \equiv p_j(\xi_t) - \frac{1}{q(\xi_t)} \sum_{\tau > t} \sum_{\xi_{\tau} \subset \xi_t} q(\xi_\tau)x_j(\xi_\tau)
\]  
(30.6)
The following hold:

\[ 0 \leq \sigma_j(\xi_t) \leq p_j(\xi_t), \quad \forall \xi_t \forall j. \]

If security \( j \) is of finite maturity, then \( \sigma_j(\xi_t) = 0 \) for all \( \xi_t \).

Further,

\[ q(\xi_t)\sigma_j(\xi_t) = \sum_{\xi_{t+1} \subset \xi_t} q(\xi_{t+1})\sigma_j(\xi_{t+1}) \quad \forall \xi_t \forall j. \quad (30.8) \]

**Price Bubbles in Equilibrium.**

The question is whether price bubbles can be nonzero in equilibrium under debt constraints.

Notation \( c_- \) and \( c_+ \) so that \( c \equiv (c_-(\xi_t), c(\xi_t), c_+(\xi_t)) \).

Agents exhibit *uniform impatience* with respect to \( \hat{w} \) if there exists \( \gamma \) satisfying \( 0 \leq \gamma < 1 \) such that

\[ u^i(c^i_-(\xi_t), c^i(\xi_t) + \hat{w}(\xi_t), \gamma c^i_+(\xi_t)) > u^i(c^i), \quad (30.9) \]

for every \( i \) and every \( \xi_t \) and every \( c^i \) such that \( 0 \leq c^i \leq \hat{w} \).

**Theorem 30.6.1:** Assume that agents’ utility functions exhibit uniform impatience. Suppose that \( (p, \{c^i, h^i\}) \) is an equilibrium in security markets under debt constraints and \( q \) is a sequence of strictly positive event prices associated
with \( p \). If present value of the aggregate endowment is finite,

\[
\sum_{t=0}^{\infty} \sum_{\xi_t \in F_t} q(\xi_t) \bar{w}(\xi_t) < \infty,
\]

(30.10)

then the price bubble is zero for every security that is in strictly positive supply.

**Example 30.6.2**

Time is infinite; so dates are \( t = 0, 1, 2, \ldots \). There is no uncertainty.

There is one security that pays zero dividend at every date.

Two agents \((i = 1, 2)\) with the same utility function

\[
u^i(c) = \sum_{t=0}^{\infty} \beta^t \ln(c_t),
\]

where \( 0 < \beta < 1 \).

Debt bounds are \( D_t = p_t \), so that agents can short sell at most one share of the security.

Suppose that endowments are

\[
w_t^1 = B, \quad w_t^2 = A \quad \text{for } t \text{ even},
\]

\[
w_t^1 = A, \quad w_t^2 = B \quad \text{for } t \text{ odd},
\]

where \( \beta A > B > 0 \). Date-0 endowments will be specified later.

Initial security holdings are \( \hat{h}_0^1 = 1 \) and \( \hat{h}_0^2 = 0 \). The total supply is 1.

Let

\[
\eta = \frac{\beta A - B}{(1 + \beta)}.
\]
There is an equilibrium with strictly positive price

\[ p_t = \frac{1}{3} \eta, \text{ for all } t \geq 0, \quad (\) \]

and

\[ c^1_t = (B + \eta) \text{ for } t \text{ even}, \quad c^1_t = (A - \eta) \text{ for } t \text{ odd} \]
\[ c^2_t = (A - \eta) \text{ for } t \text{ even}, \quad c^2_t = (B + \eta) \text{ for } t \text{ odd}, \]

and

\[ h^1_t = -1 \text{ for } t \text{ even}, \quad h^1_t = 2 \text{ for } t \text{ odd} \]
\[ h^2_t = 2 \text{ for } t \text{ even}, \quad h^2_t = -1 \text{ for } t \text{ odd} \]

This is an equilibrium if \( w^1_0 = B + \frac{1}{3} \eta, \quad w^2_0 = A - \frac{1}{3} \eta. \)

Verifying the equilibrium: (i) markets clear at every date, (ii) budget and short-sales constraints are all satisfied. Further, (iii) first-order condition for agent who is unconstrained is

\[ \frac{\beta^t}{c^1_t} p_t = \frac{\beta^{t+1}}{c^1_{t+1}} p_{t+1}, \quad (30.23) \]

It holds. First-order condition for the constrained agent is

\[ \frac{\beta^t}{c^1_t} p_t \geq \frac{\beta^{t+1}}{c^1_{t+1}} p_{t+1}, \quad (\) \]

It is satisfied, too. Transversality condition (29.13) holds.
Event prices are \( q_t = 1 \) for every \( t \). Present value of the aggregate endowment is infinite.

**Proposition 30.6.3:** If \( \inf_{t \geq 0} \inf_{\xi_t \in F_t} \hat{w}(\xi_t) > 0 \) and \( \sup_{t \geq 0} \sup_{\xi_t \in F_t} \hat{w}(\xi_t) < \infty \), then the discounted time-separable expected utility with continuous and strictly increasing von Neumann-Morgenstern utility function exhibits uniform impatience with respect to \( \hat{w} \).

Consider the following **natural debt bounds** given by

\[ N^i(\xi_t) = -\frac{1}{q(\xi_t)} \sum_{\tau=t}^{\infty} \sum_{\xi_\tau \subset \xi_t} q(\xi_\tau) w^i(\xi_\tau) \quad \forall \xi_t, \quad (31.7) \]

where \( q \) is a sequence of event prices.

Debt constraints with natural bounds \( N^i(\xi_t) \) prevent agents from holding debt in excess of present value of their future endowments.

**Theorem 31.4.2:** Let \( p \) and \( \{c^i, h^i\} \) be a security market equilibrium under natural debt constraints. If security markets are complete at \( p \) and price bubbles are zero, then \( \{c^i\} \) and \( Q \) given by

\[ Q(c) \equiv \sum_{t=0}^{\infty} \sum_{\xi_t \in F_t} q(\xi_t)c(\xi_t) \quad (31.1) \]

are an Arrow-Debreu equilibrium. Further, consumption allocation \( \{c^i\} \) is Pareto optimal.
Appendix I. Recursive Utility

Consider the setting of consumption in infinite time under uncertainty as in Section 13 (or Chapter 29, LeRoy & Werner, 2nd Ed.) Let $u$ be a utility function.

We say that $u$ induces **continuation utility** at $\xi_t$ if there exists a utility function $u_{\xi_t}$ on consumption plans for event $\xi_t$ and all successor events of $\xi_t$ such that

$$u_{\xi_t}(c_+(\xi_t)) \leq u_{\xi_t}(c'_+(\xi_t)) \iff u(\bar{c}_-(\xi_t), c_+(\xi_t)) \leq u(\bar{c}_-(\xi_t), c'_+(\xi_t))$$

(1)

for every $c, c', \bar{c}$. Condition (1) requires that preferences be independent of unrealized events and past consumption.

**Notation:** $c_+(\xi_t)$ denotes the consumption plan for all events that are successors of $\xi_t$ including event-$\xi_t$; $c_-(\xi_t)$ denotes the consumption plan for all nodes not in the subtree of $\xi_t$; so that $c = (c_-(\xi_t), c_+(\xi_t))$.

We write $u_{\xi_t}(c)$ instead of $u_{\xi_t}(c_+(\xi_t))$. If the notation $\xi_t$ for events is suppressed, we write $u_t(c)$ for ($F_t$-measurable) date-t continuation utility.

Suppose that utility function $u$ induces continuation utility $u_t$ for every $t$. If

$$u_t(c) = W(c_t, \mu_t(u_{t+1}(c)))$$

(2)

for every $c$ and $t$, for some functions $W : \mathcal{R}^2 \to \mathcal{R}$ and $\mu_t$ mapping $F_{t+1}$-measurable random variables to $F_t$-measurable r.v.’s, then $u$ is said to be re-
cursive. Function $W$ is the **aggregator** function; $\mu_t$ is the **date-$t$ certainty equivalent**. Aggregator function assigns utility to current consumption and certainty equivalent of next-period continuation utility.

**Example:** Continuation utility for the discounted time-separable expected utility (29.1) is

$$u_t(c) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} E_t[v(c_\tau)]$$

It is actually more convenient to consider a different ordinally-equivalent continuation utility

$$u_t(c) = v^{-1} \left( \sum_{\tau=t}^{\infty} \delta^{\tau-t} E_t[v(c_\tau)] \right)$$  (3)

Expected utility (29.1) with continuation utilities (3) has recursive representation with

$$W(y, z) = v^{-1}(v(y) + \delta v(z)), \quad \text{and} \quad \mu_t(w) = v^{-1}(E_t[v(w)])$$  (4)

For power utility $v(y) = \frac{1}{1-\rho} y^{1-\rho}$ (with relative risk aversion $\rho$), the aggregator $W$ of (4) is $W(y, z) = (y^{1-\rho} + \delta z^{1-\rho})^{1/(1-\rho)}$ and is the well-known constant elasticity of substitution (CES) function. ◊

**Epstein-Zin-Weil recursive utility** obtains by taking

$$W(y, z) = (y^\alpha + \delta z^\alpha)^{1/\alpha}$$  (5)

and

$$\mu_t(w) = [E_t(w^{1-\rho})]^{1/(1-\rho)}$$  (6)
where $\rho$ and $\alpha$ are two separate parameters.

Notes: Epstein and Zin (1989) prove that such recursive utility function is well defined. More precisely, there exists $u$ with recursive representation given by (5-6). In Epstein and Zin (1989) the domain of utility functions are intertemporal lotteries. See Marinacci and Montrucchio (2010) for setting as in Section 13 or Chapter 29.
Consumption-Based Security Pricing for Recursive Utility

Suppose that there are $J$ infinitely-lived securities with one-period return on security $j$ at $\xi_t$ denoted by $r_j(\xi_t)$. Suppressing the notation for events we write $r_t$ for the vector of $J$ one-period returns.

Consider the optimal consumption-portfolio problem of an agent with recursive utility specified by (5-6). The agent has non-zero endowment only at the initial date 0. It is more convenient to state the consumption-portfolio problem in terms of consumption $c_t$, wealth $w_t$, and fractions of wealth (net of consumption) $a_t \in \Delta^J$ invested in $J$ securities at date $t$. Short sales are prohibited. Budget constraints are

$$ w_{t+1} = (w_t - c_t)r_{t+1}a_t, $$

with $w_0$ given as date-0 endowment plus the value of initial portfolio of securities.

Let $V_0(w_0)$ denote the maximized date-0 utility as a (value) function of initial wealth $w_0$. Similarly, let $V_t(w_t)$ denote the maximized date-$t$ continuation utility as a (value) function of date-$t$ wealth $w_t$. Recursivity of $u$ implies that

$$ V_t(w_t) = \max_{c_t, a_t} \left( c_t^\alpha + \delta [E_t(V_{t+1}(w_{t+1}))^{1-\rho}]^{\alpha/(1-\rho)} \right)^{1/\alpha} $$

(8)

where $w_{t+1}$ is given by (7) and depends on $a_t$ and $c_t$.

Because functions $W$ and $\mu_t$ of (5-6) are homogeneous of degree 1, it follows that value function $V_t$ is linear in wealth, that is, $V_t(w_t) = A_t w_t$ where $A_t > 0$. 

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is a constant that may depend on date-$t$. Certainty equivalent of $V_{t+1}$ in (8) can be written as

$$\left[E_t[(A_{t+1}(w_t - c_t)r_{t+1}a_t)^{1-\rho}]\right]^{1/(1-\rho)} = (w_t - c_t)\left[E_t[(A_{t+1}r_{t+1}a_t)^{1-\rho}]\right]^{1/(1-\rho)}$$

(9)

Let $\{c_t^*, a_t^*\}$ be a solution to the consumption-portfolio problem. Assume that $0 < c_t^* \leq w_t$ and $0 < a_t^*$. It follows from (8) and (9) that $a_t^*$ solves

$$\max_{a_t \in \Delta^j} E_t[(A_{t+1}r_{t+1}a_t)^{1-\rho}]$$

(10)

The first-order conditions for (10) are

$$E_t[A_{t+1}^{1-\rho}(r_{t+1}^*)^{-\rho}(r_{j,t+1} - \bar{r}_{t+1})] = 0,$$

(11)

for every risky security $j$, where $\bar{r}_{t+1}$ denotes one-period risk-free return and $r_{t+1}^* = r_{t+1}a_t^*$ denotes the return on optimal portfolio $a_t^*$.

The optimal consumption $c_t^*$ solves

$$\max\{c_t^{\alpha} + \delta(w_t - c_t)^{\alpha}(z_t^*)^{\alpha/(1-\rho)}\},$$

where $z_t^*$ denotes the maximum value in (10), that is,

$$z_t^* = E_t[(A_{t+1}r_{t+1}^*)^{1-\rho}].$$

The first-order condition for $c_t^*$ is

$$(c_t^*)^{\alpha-1} = \delta(w_t - c_t^*)^{\alpha-1}(z_t^*)^{\alpha/(1-\rho)}$$

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or equivalently
\[(z_t^*)^{\alpha/(1-\rho)} = \frac{1}{\delta} \left( \frac{c_t^*}{w_t - c_t^*} \right)^{\alpha-1}. \tag{12}\]

It follows from (8) that
\[(A_tw_t)^\alpha = (c^*_t)^\alpha + \delta(w_t - c_t^*)^\alpha (z_t^*)^{\alpha/(1-\rho)} \tag{13}\]

Substituting (12) in (13) we obtain
\[A_t = \left( \frac{c_t^*}{w_t} \right)^{(\alpha-1)/\alpha} \tag{14}\]

Writing (14) for \(t+1\) instead of \(t\) and making use of budget constraint (7) there results
\[A_{t+1} = \left( \frac{c_{t+1}^*}{c_t^*} \right)^{(\alpha-1)/\alpha} \left( \frac{c_t^*}{w_t - c_t^*} \right)^{(\alpha-1)/\alpha} (r_{t+1}^*)^((1-\alpha)/\alpha) \tag{15}\]

Substituting (15) in the first-order condition (11), dividing both sides by \((\frac{c_t^*}{w_t - c_t^*})^{(\alpha-1)/\alpha}\), and making use of \(\frac{(1-\alpha)(1-\rho)}{\alpha} - \rho = \frac{1-\alpha-\rho}{\alpha}\), we obtain
\[E_t \left[ \left( \frac{c_{t+1}^*}{c_t^*} \right)^{(1-\rho)/(1-\alpha)} (r_{t+1}^*)^((1-\alpha-\rho)/\alpha) (r_{j,t+1} - \tilde{r}_{t+1}) \right] = 0. \tag{16}\]

Substituting (15) in the first-order condition (12) and simplifying, there results
\[E_t \left[ \left( \frac{c_{t+1}^*}{c_t^*} \right)^{(1-\rho)/(1-\alpha)} (r_{t+1}^*)^((1-\rho)/\alpha) \right] = 1 \tag{17}\]

Equations (16) and (17) are the Consumption-Based Security Pricing for Epstein-Zin-Weil recursive utility. In representative-agent economy with
an outstanding portfolio of securities \( \hat{h}_0 \) and no consumption endowments at any date \( t \geq 1 \), the return \( r_{t+1}^* \) equals the market return \( r_{t+1}^m = \frac{(p_{t+1} + x_{t+1})\hat{h}_0}{p_t\hat{h}_0} \).

Note that for \( \alpha = 1 - \rho \), that is, when the recursive utility reduces to expected (power) utility, equation (16) becomes

\[
E_t \left[ \left( \frac{c_{t+1}^*}{c_t^*} \right)^{-\rho} (r_{j,t+1} - \bar{r}_{t+1}) \right] = 0.
\]

This is the standard CBSP equation of Chapter 27 for power utility.