Mathematical Appendix III.

Kuhn-Tucker Theorems

I.1 Constrained Maximization: Necessary Conditions.

Function $F : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ is the objective function; functions $g^j : \mathbb{R}^n_+ \rightarrow \mathbb{R}$, for $j = 1, \ldots, k$, are constraint functions. Assume that $F$ and $g^j$ are differentiable, with partial derivatives $\frac{\partial F}{\partial x_i}$ and $\frac{\partial g^j}{\partial x_i}$ denoted by $\partial_i F$ and $\partial_i g^j$, respectively.

The constrained maximization problem (with nonnegativity constraints) is

$$\max_x F(x)$$

subject to

$$g^1(x) \geq 0,$$

$$
\ldots, \n$$

$$g^k(x) \geq 0,$$

$$x_1 \geq 0, \ldots, x_n \geq 0.$$ 

We write the Lagrangian as

$$\mathcal{L}(\lambda^1, \ldots, \lambda^k, x) = F(x) + \sum_{j=1}^{k} \lambda_j g^j(x),$$

where $\lambda_j \geq 0$, for $j = 1, \ldots, k$, are the Lagrange multipliers. We use $\lambda$ to denote the $k$-vector of multipliers.
Kuhn-Tucker conditions for $x^* \geq 0$ and $\lambda^* \geq 0$ are:

for all $i = 1, \ldots, n$ and $j = 1, \ldots, k$,

$$\partial_i F(x^*) + \sum_{j=1}^k \lambda_j^* \partial_i g^j(x^*) \leq 0,$$

and if $x_i^* > 0$, then “$= 0$”, \hspace{1cm} (2a)

$$g^j(x^*) \geq 0,$$

and if $\lambda_j^* > 0$, then “$= 0$”. \hspace{1cm} (2b)

Where do these conditions come from? Think about maximizing Lagrangian $L(\lambda, x)$ with respect to $x$ and minimizing it with respect to $\lambda$, unconstrained, except for $x \geq 0$ and $\lambda \geq 0$. This is the saddle-point. K-T conditions (2) are FOCs for such max-min (or saddle-point) problem.

Theorem (Kuhn-Tucker): If $x^* \geq 0$ is a solution to the constrained maximization problem, and the Constraint Qualification Condition holds, then $x^*$ and some $\lambda^* \geq 0$ satisfy K-T conditions (2).

Constraint Qualification Condition:

(i) Kuhn-Tucker original – don’t touch it.

(ii) $g^j$ concave for all $j$, and Slater’s condition, that is, there is some $x^0 \geq 0$ with $g^j(x^0) > 0$ for all $j$.

(iii) rank condition (see Takayama 1.D.4, or Varian, ch 27),

(iv) $g^j$ linear for all $j$, (Arrow-Hurwicz-Uzawa, see Takayama 1.D.4)
I.2 Sufficiency of Kuhn-Tucker Conditions.

The most standard theorem is:

**Theorem S1:** Suppose that $F$ and $g^1, \ldots, g^k$ are all concave functions. If $x^* \geq 0$ and $\lambda^* \geq 0$ satisfy K-T conditions (2), then $x^*$ is a solution to the constrained maximization problem.

A better theorem is due to Arrow and Enthoven (1961).

**Theorem S2:** Suppose that $F$ and $g^1, \ldots, g^k$ are all quasi-concave functions and some "mild" condition holds. If $x^* \geq 0$ and $\lambda^* \geq 0$ satisfy K-T conditions (2), then $x^*$ is a solution to the constrained maximization problem.

The extra ("mild") condition is not needed if $F$ is concave (and $g^1, \ldots, g^k$ are quasi-concave). See Takayama 1.E for three versions of the condition.

Quasi-concavity (and therefore also concavity) of functions $g^j$ implies that the constraint set, i.e. the set of $x \geq 0$ satisfying $g^1(x) \geq 0, \ldots, g^k(x) \geq 0$, is convex.
I.3 Constrained Minimization

The constrained minimization problem (with nonnegativity constraints) is

\[
\min_x F(x) \quad (3)
\]

subject to \( g^1(x) \leq 0, \ldots, g^k(x) \leq 0, \)
\( x_1 \geq 0, \ldots, x_n \geq 0. \)

The Lagrangian is
\[
\mathcal{L}(\lambda, x) = F(x) + \sum_{j=1}^{k} \lambda_j g^j(x).
\]

**Kuhn-Tucker conditions** for \( x^* \geq 0 \) and \( \lambda^* \geq 0 \) are,

for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, k, \)
\[
\partial_i F(x^*) + \sum_{j=1}^{k} \lambda_j^* \partial_i g^j(x^*) \geq 0, \quad \text{and if } x_i^* > 0, \text{ then } " = 0", \quad (4a)
\]
\[
g^j(x^*) \leq 0, \quad \text{and if } \lambda_j^* > 0, \text{ then } " = 0". \quad (4b)
\]

The corresponding saddle-point problem is to minimize Lagrangian \( \mathcal{L}(\lambda, x) \) with respect to \( x \) and maximize it with respect to \( \lambda \) for \( x \geq 0 \) and \( \lambda \geq 0. \)

The Kuhn-Tucker Theorem holds with no change for the constrained minimization problem. However, in constraint qualification conditions concavity of functions \( g^j \), if present, has to be replaced by their convexity. This guarantees convexity of the constraint set described here by inequalities \( g^j(x) \leq 0. \)
Theorems S1 and S2 continue to hold with concavity (quasi-concavity) of functions $F$ and $g^j$ replaced by their convexity (quasi-convexity, respectively).

I.4 Remarks:

- **Applications** of K-T theorems in microeconomics:
  
  (i) Consumer theory: utility maximization subject to budget constraint, and expenditure minimization.
  
  (ii) Welfare economics: Characterization of Pareto optimal allocations as solutions to maximization of a welfare function subject to resource constraints, or maximization of one agent’s utility subject to constraints on other agents’ utilities and resource constraints.
  
  (iii) Producer theory: cost minimization.

- There are versions of K-T theorems for maximization and minimization with mixed constraints, i.e., when some constraints are of the equality form, $g^j(x) = 0$. See Sundaram [2], Section 6.4.

- K-T theorems hold for local maxima (minima) as well.


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I.5 Example: Consider the following constrained maximization problem:

$$\text{maximize } \ln(x_1 + 1) + \ln(x_2 + 1)$$

subject to

$$p_1 x_1 + p_2 x_2 \leq m$$

$$x_1 \geq 0, \quad x_2 \geq 0,$$

where $$p_1 > 0$$, $$p_2 > 0$$ and $$m > 0$$.

In order to derive the solution (as a function of parameters $$p_1, p_2$$ and $$m$$) we write the Kuhn-Tucker first-order conditions (2) as

1. \( \frac{1}{x_1^* + 1} \leq \lambda^* p_1 \leq 0, \) and if \( x_1^* > 0 \), then \( = 0 \).

2. \( \frac{1}{x_2^* + 1} \leq \lambda^* p_2 \leq 0, \) and if \( x_2^* > 0 \), then \( = 0 \).

3. \( p_1 x_1^* + p_2 x_2^* \leq m, \) and if \( \lambda^* > 0 \), then \( = 0 \).

with \( x^* \geq 0 \) and \( \lambda^* \geq 0 \).

Note that (3) holds with equality since it follows from (1) that \( \lambda^* > 0 \).

We solve inequalities (1-3) by considering cases:

Case 1. \( x_1^* > 0, \ x_2^* > 0 \).

Then (1) and (2) hold with equalities. Solving (1), (2) and (3) we find \( x_1^* = \frac{m + p_2 - p_1}{2p_1} \) and \( x_2^* = \frac{m + p_1 - p_2}{2p_2} \) and \( \lambda^* = \frac{2}{m + p_1 + p_2} \). For \( x_1^* \) and \( x_2^* \) to be strictly positive, it has to be that \( m + p_2 > p_1 \) and \( m + p_1 > p_2 \). Thus Case 1 applies with \( x_1^* \) and \( x_2^* \) as listed above if \( m + p_2 > p_1 \) and \( m + p_1 > p_2 \).
**Case 2.** $x_1^* > 0, x_2^* = 0$.

(3) implies that $x_1^* = \frac{m}{p_1}$. Since (1) holds with equality, we solve it for $\lambda^* = \frac{1}{m + p_1}$. Next we need to verify inequality (2). It states

$$1 - \frac{p_2}{m + p_1} \leq 0,$$

and it holds if $p_2 \geq m + p_1$. Thus Case 2 applies (with $x_1^* = \frac{m}{p_1}, x_2^* = 0$) if $p_2 \geq m + p_1$.

**Case 3.** $x_1^* = 0, x_2^* > 0$.

This case is very similar to Case 2. From (3) and (2) we obtain $x_1^* = \frac{m}{p_2}$, $\lambda^* = \frac{1}{m + p_2}$. Verifying inequality (1), we obtain $p_1 \geq m + p_2$. Thus Case 3 applies (with $x_1^* = 0, x_2^* = \frac{m}{p_2}$) if $p_1 \geq m + p_2$.

The case $x_1^* = x_2^* = 0$ cannot hold since it violates equation (3). This concludes our solution to the K-T conditions.

Since utility function is concave and the constraint function is concave (in fact, it is linear) $K - T$ conditions are sufficient (Theorem S1). Hence, the solution to K-T conditions is a constrained maximizer. Further, since the Slater’s condition holds, every constrained maximizer has to satisfy $K - T$ conditions.