Optimal Allocations with $\alpha$-MaxMin Utilities, Choquet Expected Utilities, and Prospect Theory∗

Patrick Beißner† and Jan Werner‡

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Abstract: The analysis of optimal risk sharing has been thus far largely restricted to non-expected utility models with concave utility functions, where concavity is an expression of ambiguity aversion and/or risk aversion. This paper extends the analysis to $\alpha$-maxmin expected utility, Choquet expected utility, and Cumulative Prospect Theory, which accommodate ambiguity seeking and risk seeking. We introduce a novel methodology of the quasidifferential calculus of Demyanov and Rubinov (1986, 1992) and argue that it is particularly well-suited for the analysis of these three classes of utility functions which are neither concave nor differentiable. We provide characterizations of quasidifferentials of these utility functions, derive first-order conditions for Pareto optimal allocations under uncertainty, and analyze implications of these conditions for risk sharing with and without aggregate risk.

Keywords: ambiguity, risk sharing, non-convex preferences, Pareto optimality, quasidifferential, Clarke subdifferential, $\alpha$-MaxMin expected utility, Choquet expected utility, rank-dependent expected utility, Cumulative Prospect Theory.

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†Research School of Economics, The Australian National University, Canberra, ACT 2600, Australia. Email: patrick.beissner@anu.edu.au
‡Department of Economics, University of Minnesota, Minneapolis, MN 5545, US. Email: jwerner@umn.edu
1 Introduction

Expected utility hypothesis, with risk aversion and common beliefs, leads to clear-cut results on optimal risk sharing with and without aggregate risk. Motivated by the evidence – empirical and experimental – that expected utility fails to properly describe people’s preference in many situations involving risk or uncertainty, the analysis of optimal risk sharing has been extended in the last two decades to non-expected utility models such as the multiple-prior model of Gilboa and Schmeidler (1989), the variational preferences of Maccheroni et al. (2006), the smooth ambiguity model of Klibanoff et al. (2005), the Knightian uncertainty model of Bewley (1986), the Choquet (non-additive) expected utility of Schmeidler (1989), and others.\(^1\) An important assumption in these extensions has been concavity of the utility functions. Concavity implies that preferences exhibit ambiguity aversion and risk aversion.

Ambiguity seeking and risk seeking are two broad behavioral phenomena that have strong empirical support. The most popular models in applied and theoretical research that accommodate ambiguity seeking and mixed attitudes toward ambiguity are the $\alpha$-maxmin expected utility ($\alpha$-MEU), the smooth ambiguity model with non-concave “second-order” utility, the Choquet expected utility (CEU) with non-convex capacity, and the Cumulative Prospect Theory (CPT) of Tversky and Kahneman (1992). The utility functions of these models are non-concave, and - with exception of the smooth model - non-differentiable. This renders the standard methods of differential calculus and convex analysis inapplicable to the analysis of optimal risk sharing.

This paper develops a novel methodology for studying (first-order) optimality conditions for utility functions under uncertainty that are neither concave nor differentiable. The methodology is based on the quasidifferential calculus advanced in the 1980’s by V. Demyanov and A. Rubinov and others (see Demyanov and Rubinov (1986, 1992)). We argue that it is particularly well-suited for $\alpha$-MEU, CEU, and CPT, and superior to the occasionally used Clarke’s (1983) subdifferential.\(^2\) We provide characterizations of quasidifferentials of these three

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\(^1\)We review the existing literature on optimal risk sharing at the end of this section.

\(^2\)See Ghirardato et al. (2004) and Girardato and Siniscalchi (2012).
classes of utility functions, derive first-order conditions for optimal risk sharing, and analyze their implications.

The $\alpha$-MEU is a generalization of the multiple-prior expected utility in form of weighted sum of minimum and maximum of expected utilities over a set of priors. Relative weight between the minimum and the maximum provides parametrization of attitudes toward ambiguity. The maximum term, which stands for ambiguity seeking attitude, leads typically to non-concavity of the resulting utility function. Hypothesizing that subjects follow $\alpha$-MEU, Bossaerts et al. (2010) find evidence of heterogeneous attitudes toward ambiguity in an asset market experiment, see also Ahn et al. (2014). Trautmann and van de Kuilen (2015) survey related evidence in Ellsberg-style experiments. A special case of $\alpha$-MEU is the Hurwicz expected utility of Gul and Pesendorfer (2015) with its strong appeal - applied and theoretical - of flexibility in regard to ambiguity attitudes and their dependence on sources of uncertainty.

The CEU accommodates different ambiguity attitudes by permitting a general non-additive probability measure (or capacity) in the Choquet integral. While a convex capacity reflects ambiguity aversion and a concave reflects ambiguity seeking, a general capacity displays mixed ambiguity attitude. Non-convex capacity leads to non-concave CEU function. An important special case of CEU is the Rank-Dependent Expected Utility (RDEU) of Quiggin (1982) and Yaari (1987) where the capacity is a distortion of a reference probability measure. Convexity of distortion implies convexity of the resulting capacity, and hence ambiguity aversion. Overweighting the worst and the best outcomes, which has been documented in empirical work (see Wakker (2010, Chapter 7) and the references in Gul and Pesendorfer (2015)), makes a non-convex inverse S-shaped distortion function the most important in applications of RDEU.

Lastly, the CPT postulates a utility function that is a sum of two CEU’s of gains and losses. In the most popular formulation with RDEU, the distortion functions are inverse S-shaped. The utility-of-wealth function is convex over losses and concave over gains. Convexity over losses reflects risk-seeking behavior in regard to losses. The resulting utility function of CPT is not concave.

Quasidifferential calculus focuses on directional derivatives, and can be seen as
an extension of sub- and superdifferential calculus of convex analysis (see Rockafellar (1970)) beyond concave and convex functions. It is well known that the directional derivative is a linear function of the directional vector for a (Gateaux) differentiable function. For a concave function, the directional derivative is a sublinear function\(^3\) of the directional vector, while for a convex function, it is superlinear. In quasidifferential calculus, the directional derivative is represented as a sum of sublinear and superlinear functions. There is a pair of convex sets - identified in a non-unique way\(^4\) - such that the sublinear part is the support function (maximum) of one set and the superlinear part is the negative support function (minimum) of the second set. The two sets are called superdifferential and subdifferential because they coincide with those of convex analysis for a concave or a convex function. Examples of quasidifferentiable functions include concave and convex functions, their linear combinations, and maxima and minima of arbitrary collections of differentiable functions.

Important results of quasidifferential calculus are statements of first-order conditions for unconstrained and constrained optimization problems. For example, the necessary first-order condition for unconstrained maximum of a quasidifferentiable function on an open set is that the negative of the subdifferential is a subset of the superdifferential. It is a unified statement of the known first-order conditions for differentiable, concave, and convex functions (see Section 2). A strict form of this condition, which requires that the negative of the subdifferential is a subset of the interior of the superdifferential, is a sufficient condition for local maximum. First-order conditions - necessary, and sufficient - for constrained optimization problems have similar statements featuring Lagrange multipliers.

Quasidifferential calculus is an alternative to the Clarke (1983) method of generalized derivative, or subdifferential.\(^5\) Both methods provide first-order conditions in optimization problems. A drawback of the Clarke’s subdifferential is its

\(^3\)A function is sublinear (superlinear) if it is subadditive (superadditive, respectively) and positively homogeneous.

\(^4\)Sub- and superdifferentials are identified up to an equivalence class of a relation between pairs of convex sets that we introduce in Section 2. All results for quasidifferentiable functions hold independently of the choice of sub- and superdifferentials unless explicitly stated.

\(^5\)We discuss the relationship between the two methods in details in Section 2.1.
lack of additivity. The subdifferential of a sum of two functions need not be equal to the sum of the subdifferentials – it is merely a subset thereof. For example, \( \alpha \)-MEU is a sum of maximum and minimum functions, but there is no known characterization of the Clarke’s subdifferential of it. In contrast, the basic rules of differentiation - in particular, additivity - continue to hold for quasidifferentiation (see Appendix B).

In the first part of the paper, we show that \( \alpha \)-MEU, CEU, and CPT utility, all with arbitrary differentiable utility-of-wealth functions, are quasidifferentiable, and we derive their quasidifferentials. The quasidifferential of an \( \alpha \)-MEU function consists of the superdifferential equal to the minimizing probabilities scaled by marginal utilities and the subdifferential equal to the maximizing probabilities scaled by marginal utilities. The form of the superdifferential is the same as for a multiple-prior expected utility with concave utility of wealth. We show that every CEU with arbitrary capacity is quasidifferentiable, and derive its quasidifferential making use of the representation of the Choquet integral by the Möbius inverse of a capacity. For concave, convex, and neo-additive capacities (see Chateauneuf et al. (2007)), the CEU can be represented as \( \alpha \)-MEU, and the quasidifferential has the simple form of the latter. We derive a novel representation of RDEU with inverse S-shaped distortion function as \( \alpha \)-MEU with different sets of beliefs for minimum and maximum operators. The quasidifferential of such RDEU is similar to the quasidifferential of \( \alpha \)-MEU. Further, we show the CPT utility function with one RDEU for gains and another RDEU for losses - both with inverse S-shaped distortions - is quasidifferentiable.

The second part of the paper is concerned with optimal risk sharing for quasidifferentiable utility functions. An important result is a statement of first-order conditions for an interior Pareto optimal allocation for general quasidifferentiable utility functions. The condition requires that for every profile of vectors in the subdifferentials at an optimal allocation there exists a profile of vectors in the superdifferentials such that the respective sums of sub- and superdifferentials are scale-multiples of the same vector. It is an extension of the standard first-order condition on marginal rates of substitution for differentiable functions and more general conditions for concave or convex functions. Several interesting
implications emerge when our first-order conditions are applied to $\alpha$-MEU. If the utility-of-wealth functions are concave, then every Pareto optimal allocation with $\alpha$-MEU is an optimal allocation for expected utility functions with the same utility-of-wealth functions and heterogeneous beliefs taken from the agents’ sets of priors. For small sets of priors, this is a significant restriction on allocations that can be optimal. Further, we show that there cannot exist a risk-free Pareto optimal allocation in a no-aggregate-risk economy unless the sets of priors have non-empty intersection, that is, unless there is a common prior. Pareto optimal allocations under RDEU with inverse S-shaped distortions have similar properties to those of $\alpha$-MEU. We analyze the first-order conditions for Pareto optimality for CEU and CPT utilities as well.

The paper is organized as follows. Section 2 introduces the quasidifferential calculus and provides a discussion of the relationship with the Clarke’s (1983) theory. In Section 3 we analyze quasidifferentiability of $\alpha$-MEU, CEU, RDEU and CPT. We present the first-order conditions for Pareto optimal allocations in Section 4, and derive some implications for the utility functions from Section 3. Section 5 contains concluding remarks. The appendix consists of three parts: Appendices A and C contain proofs of the results from Sections 3 and 4, while Appendix B provides additional results of quasidifferential calculus.

**Related literature:** Properties of efficient allocations for preferences that exhibit ambiguity aversion have been extensively studied in the literature over the past two decades. Billot et al. (2001) show that if agents with concave multiple-prior expected utilities have at least one prior in common and there is no aggregate risk, then all interior Pareto optimal allocations are risk free. Rigotti et al. (2008) extend that result to other models of convex preferences under ambiguity. Girardato and Siniscalchi (2018) study optimal risk sharing with no aggregate risk assuming supportability of preferred sets at risk-free consumption plans instead of convexity. Even this weaker assumption excludes ambiguity seeking in most models of preferences under ambiguity. General properties of efficient allocations when there is aggregate risk, such as comonotonicity and measurability with respect to aggregate endowment, have been studied in Chateauneuf et al. (2000) and Dana (2004) for ambiguity-averse CEU, and in Strzalecki and Werner (2011)
for general concave utility functions including multiple-prior utilities, variational preferences, and the smooth ambiguity model. Werner (2020) considers participation in risk sharing among agents with multiple-prior expected utilities, and shows that agents with the highest ambiguity (i.e., the largest sets of priors) and low risk aversion are most likely to hold risk-free consumption in any Pareto optimal allocation. de Castro and Chateauneuf (2011) and Strzalecki and Werner (2011) explore efficient risk sharing among ambiguity averse agents when the aggregate risk is unambiguous.

First-order necessary conditions for Pareto optimal allocations without differentiability can be found in Rigotti et al. (2008) for concave utility functions. Those conditions are stated in terms of subjective beliefs but can alternatively be stated in terms of the standard superdifferential of convex analysis, as in Aubin (1998). Ghirardato and Siniscalchi (2018) provide first-order conditions for interior Pareto optimal allocations without concavity using the Clarke’s subdifferential. The difficulty in applying this result to α-MEU or CEU is that there is no known characterization of the Clarke’s subdifferential of these functions. First-order conditions for Pareto optimal allocations with production in terms of the Clarke’s normal cone instead of the subdifferential can be found in Khan and Vohra (1987) and Bonnisseau and Cornet (1988).

There has been some recent interest in general equilibrium theory and welfare theorems in economies with non-convex preferences. Araujo et al. (2018) study the existence of an equilibrium in complete markets under uncertainty when a subset of agents have convex utility functions while the remaining agents have concave utility functions. If the aggregate endowment is sufficiently risky - that is, the aggregate endowment in one state is large enough compared to other states - then there exists an equilibrium. Agents with convex utilities end up with extreme consumption (on the boundary of their consumption sets) in the equilibrium. Araujo et al. (2017) study efficient allocations when there is aggregate risk, continuum of states, and a subset of agents have convex utility functions. They show that (strongly) risk averse agents have comonotone consumption plans in efficient allocations. Richter and Rubinstein (2015) introduce methods of abstract geometric convexity to general equilibrium theory and extend the welfare
theorems by replacing the assumption of convexity in the standard sense by abstract convexity. They provide examples of economies - mostly with indivisible goods - with non-convexities in the standard sense where the extended general equilibrium theory applies.

2 Quasidifferential Calculus

Quasidifferential calculus is an extension of sub- and superdifferential calculus beyond convex and concave functions. It has been developed by V.F. Demyanov and A.M. Rubinov in early 1980’s, see Demyanov and Rubinov (1986, 1992) for self contained expositions. We present the basic concepts and results that will be used later.

Let \( f : X \to \mathbb{R} \) be a real-valued function on an open subset \( X \) of \( \mathbb{R}^S \). Function \( f \) is said to be directionally differentiable at \( x \in X \) in the direction of \( \hat{x} \in \mathbb{R}^S \) if the limit

\[
\lim_{{t \to 0^+}} \frac{f(x + t\hat{x}) - f(x)}{t},
\]

(1)

exists. If the limit exists for every direction \( \hat{x} \in \mathbb{R}^S \), then \( f \) is directionally differentiable at \( x \). If \( f \) is Gateaux differentiable, then directional derivative \( f'(x; \hat{x}) \) is equal to the scalar product \( \nabla f(x) \cdot \hat{x} \), where \( \nabla f(x) \in \mathbb{R}^S \) is the gradient vector.

Function \( f \) is said to be quasidifferentiable at \( x \) if it is directionally differentiable and, furthermore, there exist two compact and convex sets \( A \) and \( B \) in \( \mathbb{R}^S \) such that

\[
f'(x; \hat{x}) = \max_{{z \in A}} \hat{x} z + \min_{{z \in B}} \hat{x} z
\]

(2)

for every \( \hat{x} \in \mathbb{R}^S \). Relation (2) is a representation of the directional derivative by the sum of a sublinear function and a superlinear function. Sets \( A \) and \( B \) in this representation are not unique. For example, the pair \( A - S \) and \( B + S \) satisfies (2) for every convex and compact set \( S \) as well. More generally, any two pairs of convex and compact sets \([A, B]\) and \([A', B']\) give the same representation as long
as\(^6\)

\[ A - B' = A' - B. \]  \(3\)

Equation (3) induces an equivalence relation among pairs of convex and compact sets in \(\mathbb{R}^S\). Equivalence classes of that relation are in one-to-one correspondence to sums of sublinear and superlinear functions on \(\mathbb{R}^S\), see Demyanov and Rubinov (1986). We refer to relation (3) as \textit{DR-equivalence}.\(^7\) Any pair of sets \([A, B]\) from the DR-equivalence class satisfying (2) is denoted by \(\partial f(x)\) for \(A\) and \(\partial f(x)\) for \(B\), and written as

\[ Df(x) = [\partial f(x), \partial f(x)]. \]

Function \(f\) is said to be \textit{subdifferentiable} at \(x\) if it is quasidifferentiable and the superdifferential \(\partial f(x)\) is a singleton set for some DR-equivalent representation of the quasidifferential \(Df(x)\). A subdifferentiable function has sublinear directional derivative. Every convex function is subdifferentiable at every \(x\) with \(\partial f(x)\) being the subdifferential in the sense of convex analysis (and zero superdifferential). Similarly, \(f\) is \textit{superdifferentiable} at \(x\) if it is quasidifferentiable and the subdifferential \(\partial f(x)\) is a singleton set for some representation of \(Df(x)\). The directional derivative of a superdifferentiable function is superlinear. Every concave function is superdifferentiable, with \(\partial f(x)\) being the superdifferential of convex analysis. If the quasidifferential \(Df(x)\) has a representation with singleton sets as sub- and superdifferentials, then \(f\) is Gateaux differentiable at \(x\). Any pair of vectors \((d, d)\) such that \(d + d = \nabla f(x)\) is the quasidifferential of \(f\) at \(x\).

For later use, we demonstrate now that a maximum function over a compact set of parameters is subdifferentiable. Let \(\varphi\) be defined by

\[ \varphi(x) = \max_{y \in Y} f(x, y), \]  \(4\)

where \(f\) is continuous in \((x, y)\) and continuously differentiable in \(x\). The set \(Y \subset \mathbb{R}^n\) is compact. Note that function \(\varphi\) may be neither convex nor concave.

\(^6\)Note that (2) can be written as \(f'(x; \hat{x}) = s_A(\hat{x}) - s_{-B}(\hat{x})\), where \(s_A\) denotes the support function of the set \(A\). Pairs \([A, B]\) and \([A', B']\) satisfy (3) if and only if \(s_{A-B'}(\hat{x}) = s_{A'-B}(\hat{x})\) for every \(\hat{x} \in \mathbb{R}^S\). This can equivalently be written as \(s_A(\hat{x}) - s_{-B}(\hat{x}) = s_A(\hat{x}) - s_{-B}(\hat{x})\) for every \(\hat{x} \in \mathbb{R}^S\). Thus (2) holds for \([A, B]\) if and only if it holds for \([A', B']\).

\(^7\)See Pallaschke and Urbanski (1994) for an extensive discussion of the relation of DR-equivalence and the problem of minimal representation of an equivalence class.
Let $\varphi^*(x)$ denote the set of maximizers in (4) at $x$. It follows from the Danskin’s Envelope Theorem that the directional derivative of $\varphi$ is

$$\varphi'(x, \hat{x}) = \max_{y^* \in \varphi^*(x)} \nabla_x f(x, y^*) \hat{x},$$

for every $\hat{x}$, where $\nabla_x f$ denotes the gradient of $f$ with respect to $x$. Equation (5) implies that $\varphi$ is quasidifferentiable with subdifferential given by

$$\partial \varphi(x) = \text{co} \{ \nabla_x f(x, y^*) : y^* \in \varphi^*(x) \},$$

where $\text{co}$ denotes the convex hull, and zero superdifferential. Therefore $\varphi$ is subdifferentiable at $x$.

Summing up, the class of quasidifferentiable functions includes differentiable, concave, convex functions, and maxima and minima of differentiable functions. Sums, scale multiples, and compositions of quasidifferentiable functions are quasidifferentiable. Further, maxima and minima of finite collections of quasidifferentiable functions are quasidifferentiable as well. Most of the rules of differentiation continue to hold for quasidifferentiation, see Appendix B.

Necessary first-order conditions for solutions to optimization problems can be nicely stated for quasidifferentiable function. For example, a necessary condition for unconstrained maximum $x^*$ of quasidifferentiable function $f$ on $\mathbb{R}^S$ is

$$-\partial f(x^*) \subset \partial f(x^*),$$

which can be equivalently expressed as that for every $\bar{z} \in \partial f(x^*)$ there exists $\bar{z} \in \partial f(x^*)$ such that $\bar{z} + \bar{z} = 0$. The necessary condition for unconstrained minimum is $-\partial f(x^*) \subset \partial f(x^*)$, with interchanged roles of the sub- and superdifferentials. These are unified statements of the standard first-order conditions for differentiable, concave, and convex functions. Strict forms of these conditions - with $\partial f(x^*)$ replaced by its interior for maximum, and $\partial f(x^*)$ replaced by its interior for minimum - are sufficient for local solutions. Note that these first-order conditions do not depend on the choice of DR-equivalent pairs of sets for sub- and superdifferentials.

Necessary first-order conditions for constrained maximization of a quasidifferentiable function can be found in Demyanov and Dixon (1986) for various types
of constraints. To illustrate, we present a first-order condition for maximization of quasidifferentiable utility function $f$ subject to the budget constraint. The budget set is $B(p) = \{x \in \mathbb{R}^S_+ \mid px \leq pe\}$, where $p \in \mathbb{R}^S$ is a vector of prices and $e$ is an endowment. The necessary condition for a strictly positive solution $x^* \in \mathbb{R}^S_+$ is

$$-\partial f(x^*) \subset \partial f(x^*) - \{\lambda p \mid \lambda \geq 0\}. \tag{7}$$

This can be equivalently stated as that for every $\tilde{z} \in \partial f(x^*)$ there exists $\bar{z} \in \partial f(x^*)$ and a multiplier $\lambda^* \geq 0$ such that $\tilde{z} + \bar{z} = \lambda^* p$. Condition (7) with $\partial f(x^*)$ replaced by its interior is sufficient for a local constrained maximum. The relative simplicity of condition (7) stems from the fact that the constraint function $px$ is linear and therefore differentiable.

### 2.1 Clarke Subdifferential and Quasidifferential

The quasidifferential is related to, but different from the Clarke (1983) subdifferential. While quasidifferential calculus is concerned with representation of the standard (Dini) directional derivative (1), the Clarke’s theory introduces extensions of the directional derivative called Clarke lower and upper directional derivatives. The Clarke upper and lower directional derivatives of a Lipschitz continuous function $f$ at $x$ in the direction of $\hat{x}$ are defined, respectively, as

$$f'_+(x; \hat{x}) = \limsup_{y \to x, t \to 0^+} \frac{f(y + t\hat{x}) - f(x)}{t},$$

and

$$f'_-(x; \hat{x}) = \liminf_{y \to x, t \to 0^+} \frac{f(y + t\hat{x}) - f(x)}{t}.$$

The Clarke subdifferential of $f$ at $x$ is

$$\partial_{\text{cl}} f(x) = \text{co}\{ \lim_{k \to \infty} \nabla f(x_k) \text{ for a } \{x_k\} \subset T(f) \text{ s.t. } x_k \to x \}$$

where $T(f) \subset \mathbb{R}^S$ is the set of points of differentiability of $f$.

The upper directional derivative is a sublinear function of directional vector $\hat{x}$. It holds

$$f'_+(x; \hat{x}) = \max_{z \in \partial_{\text{cl}} f(x)} \hat{x}z, \tag{8}$$
so that $f'_+(x; \hat{x})$ is the support function of the Clarke subdifferential. The lower directional derivative is superlinear in $\hat{x}$, and it holds

$$f'_-(x; \hat{x}) = \min_{z \in \partial_{\mathbb{H}} f(x)} \hat{x} z. \tag{9}$$

It is well known (see Demyanov and Rubinov (1986, p. 74)) that $f'_-(x; \hat{x}) \leq f'(x; \hat{x}) \leq f'_+(x; \hat{x})$, for every $x$ and $\hat{x}$. It follows, using (8) and (9), that

$$\min_{z \in \partial_{\mathbb{H}} f(x)} \hat{x} z \leq f'(x; \hat{x}) \leq \max_{z \in \partial_{\mathbb{H}} f(x)} \hat{x} z. \tag{10}$$

Thus, Clarke’s subdifferential provides sublinear majorization and superlinear minorization of the directional derivative. The quasidifferential provides an exact representation (2). If function $f$ is convex or concave, then the Clarke’s subdifferential is equal to, respectively, the sub- or the superdifferential of $f$.

As mentioned in the introduction, the Clarke’s subdifferential lacks additivity. Further, the envelope theorem, such as (6), has merely an approximate statement for the subdifferential.

3 \(\alpha\)-MaxMin and Other Utility Functions

We consider a setting with uncertainty described by a finite set of states $S$. Let $\Sigma$ denote the set of all subsets of $S$. There is a single consumption good. State contingent consumption plans (or acts) are vectors $x \in \mathbb{R}_+^S$. The $\alpha$-maxmin expected utility ($\alpha$-MEU) is defined as

$$V(x) = \alpha \min_{P \in \mathcal{P}} E_P[v(x)] + (1 - \alpha) \max_{P \in \mathcal{P}} E_P[v(x)], \tag{11}$$

for $\alpha \in [0, 1]$, utility index $v : \mathbb{R}_+ \to \mathbb{R}$, and a closed and convex set of priors $\mathcal{P} \subseteq \Delta$, where $\Delta$ is the probability simplex on $(S, \Sigma)$.\footnote{Axiomatizations of $\alpha$-MEU have been provided for special sets of priors by Gul and Pesendorfer (2015) and Chateauneuf et al. (2007), and - for general sets - by Frick et al. (2020) in the setting of two preference relations: subjectively rational, and objectively rational. The $\alpha$-MEU of Gul and Pesendorfer (2015) and Chateauneuf et al. (2007) are also CEUs, and will be discussed in Section 3.1.} We assume throughout that $v$ is strictly increasing. The relative weight $\alpha$ in (11) is a parameter of ambiguity.
attitude. For $\alpha = 1$, function $V$ is the ambiguity-averse multiple-prior expected utility of Gilboa and Schmeidler (1989). For $\alpha = 0$, $V$ is the ambiguity-seeking multiple-prior expected utility.

There is an obvious similarity between the form (11) of $\alpha$-MEU and the representation (2) of a directional derivative in quasidifferential calculus. Indeed, $\alpha$-MEU is a sum of superlinear function $\alpha \min_{P \in \mathcal{P}} E_P[\cdot]$ and sublinear function $(1 - \alpha) \max_{P \in \mathcal{P}} E_P[\cdot]$ applied to the utility vector $v(x)$. As for quasidifferential, $\alpha$-MEU is determined up to DR-equivalence relation (3), and hence parameter $\alpha$ and the set of priors $\mathcal{P}$ are typically non-unique.\(^9\) More precisely, two pairs $(\alpha, \mathcal{P})$ and $(\alpha', \mathcal{P'})$ give the same utility function $V$ in (11) if and only if the pair of sets $[\alpha \mathcal{P}, (1 - \alpha) \mathcal{P}]$ is DR-equivalent to $[\alpha' \mathcal{P}', (1 - \alpha') \mathcal{P}']$. Proposition 1 in Frick et al. (2020) provides necessary and sufficient conditions for DR-equivalence of such pairs of sets.

We prove in Lemma C.1 in Appendix C that if the set of priors $\mathcal{P}$ is symmetric around $\pi \in \mathcal{P}$, then $\alpha$-MEU can be represented as either ambiguity-averse or ambiguity-seeking multiple-prior expected utility. A set of priors $\mathcal{P} \subset \Delta$ is symmetric around $\pi \in \mathcal{P}$ if and only if

$$\mathcal{P} - \pi = \pi - \mathcal{P}. \tag{12}$$

Lemma C.1 shows that if (12) holds and $\alpha > \frac{1}{2}$, then the pair $[\alpha \mathcal{P}, (1 - \alpha) \mathcal{P}]$ is DR-equivalent to $[(2\alpha - 1) \mathcal{P} + 2(1 - \alpha) \pi, 0]$. This implies that $\alpha$-MEU with symmetric $\mathcal{P}$ and $\alpha > \frac{1}{2}$ can be written as ambiguity-averse multiple-prior utility with the smaller set of priors $(2\alpha - 1) \mathcal{P} + 2(1 - \alpha) \pi$. Similarly, for $\alpha < \frac{1}{2}$, $\alpha$-MEU with symmetric $\mathcal{P}$ can be written as ambiguity-seeking multiple-prior utility with the smaller set of priors $(1 - 2\alpha) \mathcal{P} + 2\alpha \pi$. Lastly, for $\alpha = \frac{1}{2}$, $\alpha$-MEU with symmetric $\mathcal{P}$ has expected utility representation with $\pi$.\(^{10}\)

Examples of symmetric sets of priors include Euclidean neighborhoods of the form $\mathcal{P} = \{ P \in \Delta : \| P - \pi \| \leq \epsilon \}$, and order intervals such as $\mathcal{P} = \{ P \in \Delta : \pi(A) \leq P(A) \leq (1 + \epsilon) \pi(A), \forall A \subset S \}$, for small $\epsilon$ and $\pi \in \text{int} \, \Delta$. In

\(^{9}\)The fact that $\alpha$-MEU often has non-unique parametric specification $(\alpha, \mathcal{P})$ has been pointed out in Siniscalchi (2006).

\(^{10}\)Rogers and Ryan (2012) showed that symmetric $\mathcal{P}$ and $\alpha = \frac{1}{2}$ are necessary and sufficient for $\alpha$-MEU to have expected utility representation.
particular, every convex set of priors with two states, i.e., $S = 2$, is symmetric.

**Example 1:** Let $S = 2$ and the set of probabilities be $\mathcal{P} = \{(p, 1 - p) : p_l \leq p \leq p_h\}$, where $0 \leq p_l < p_h \leq 1$. This set of probabilities is symmetric around $\pi_m = (p_m, 1 - p_m)$, where $p_m = \frac{1}{2} p_l + \frac{1}{2} p_h$. For $\alpha = \frac{1}{2}$, the set $[\frac{1}{2} \mathcal{P}, \frac{1}{2} \mathcal{P}]$ is DR-equivalent to $[\pi_m, 0]$ and the $\alpha$-MEU is equal to the expected utility with probability measure $\pi_m$. For $\alpha > \frac{1}{2}$, $\alpha$-MEU is equal to the minimum expected utility with the set of priors $\{(p, 1 - p) : \alpha p_l + (1 - \alpha)p_h \leq p \leq \alpha p_h + (1 - \alpha)p_l\}$, while for $\alpha < \frac{1}{2}$ it is equal to the maximum expected utility with the set of priors $\{(p, 1 - p) : \alpha p_h + (1 - \alpha)p_l \leq p \leq \alpha p_l + (1 - \alpha)p_h\}$. \hfill $\square$

Let $\mathcal{P}_{\min}(x) \subset \mathcal{P}$ be the closed and convex subset of priors for which the minimum expected utility is attained in (11). That is,

$$\mathcal{P}_{\min}(x) = \arg \min_{P \in \mathcal{P}} E_P[v(x)]. \quad (13)$$

Similarly, let

$$\mathcal{P}_{\max}(x) = \arg \max_{P \in \mathcal{P}} E_P[v(x)]. \quad (14)$$

The following proposition establishes quasidifferentiability of $\alpha$-MEU and derives its quasidifferential.

**Proposition 1:** The $\alpha$-MEU function $V$ is quasidifferentiable on $\mathbb{R}^S_+$ for every convex and compact $\mathcal{P} \subset \Delta$, every $\alpha \in [0, 1]$, and every continuously differentiable utility index $v$. The sub- and superdifferentials of $V$ at $x \in \mathbb{R}^S_+$ are\textsuperscript{11}

$$\partial V(x) = (1 - \alpha)v'(x)\mathcal{P}_{\max}(x), \quad (15)$$

and

$$\bar{\partial} V(x) = \alpha v'(x)\mathcal{P}_{\min}(x). \quad (16)$$

**Proof:** To demonstrate quasidifferentiability of $\alpha$-MEU function $V$, it suffices to show (by Proposition B.1.1) that the two summands are quasidifferentiable. The second summand, $(1 - \alpha)\max_{P \in \mathcal{P}} E_P[v(x)]$, is the maximum over a compact set of continuously differentiable functions. By the results of Section 2, it is quasidifferentiable. Its quasidifferential is the subdifferential $(1 - \alpha)v'(x)\mathcal{P}_{\max}(x)$, see (6), and zero superdifferential. The first summand, $\alpha \min_{P \in \mathcal{P}} E_P[v(x)]$, is the

\textsuperscript{11}We use the notation $v'(x)\mathcal{P}_{\max}(x)$ for the set $\{z \in \mathbb{R}^S : z_s = v'(x_s)P(s), P \in \mathcal{P}_{\max}(x)\}$. 

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minimum over a compact set of continuously differentiable functions. By the same argument, it is quasidifferentiable with the superdifferential $\alpha v'(x)P_{\min}(x)$ and zero subdifferential. This implies (15).

It follows that the quasidifferential of $\alpha$-MEU function can be written as

$$DV(x) = v'(x)\left[(1 - \alpha)P_{\max}(x), \alpha P_{\min}(x)\right].$$

Clearly, if the sets $P_{\min}(x)$ and $P_{\max}(x)$ are singletons, or DR-equivalent to singletons, then $V$ is differentiable at $x$.

The ambiguity-averse multiple-prior expected utility with $\alpha = 1$ is superdifferentiable with $\partial V(x) = v'(x)P_{\min}(x)$ and $\partial V(x) = \{0\}$. The ambiguity-seeking multiple-prior expected utility with $\alpha = 0$ is subdifferentiable with $\partial V(x) = v'(x)P_{\max}(x)$ and $\partial V(x) = \{0\}$. If the set of priors is symmetric (as in Example 1), then $\alpha$-MEU is superdifferentiable for every $\alpha > \frac{1}{2}$, subdifferentiable for every $\alpha < \frac{1}{2}$, and differentiable for $\alpha = \frac{1}{2}$.

### 3.1 Choquet Expected Utility

Non-additive probabilities provide another way for preferences under uncertainty to accommodate different attitudes toward ambiguity. The mathematical concept to describe non-additive probabilities is a capacity. Capacity is a set function

$$\mu : \Sigma \to [0, 1]$$

such that $\mu(\emptyset) = 0$, $\mu(S) = 1$, and $\mu(A) \leq \mu(B)$ for every $A \subset B$, $A, B \in \Sigma$. The Choquet expected utility (CEU) with utility index $v : \mathbb{R}_+ \to \mathbb{R}$ is defined as the Choquet integral of $v$ under $\mu$, that is,

$$E_\mu[v(x)] = \sum_{k=1}^{S} v(x_{(k)})[\mu(\{s : x_s \geq x_{(k)}\}) - \mu(\{s : x_s \geq x_{(k-1)}\})],$$

where $x_{(k)}$ denotes the $k$-th highest consumption level from among all $x_s$. An axiomatization of CEU has been provided by Schmeidler (1989).

An important feature of CEU is rank-dependence of weights assigned to utilities of consumption in different states (see Wakker (2010, Chapter 10)). Decision weight assigned to $v(x_s)$ in (17) depends on the ranking of $x_s$ among all states. Note that those decision weights add up to one. Different attitudes toward ambiguity can be described in CEU by different properties of the capacity. We will
show later in this section that a convex capacity reflects ambiguity aversion while a concave one reflects ambiguity seeking. Capacities that are neither convex nor concave reflect mixed ambiguity attitudes. For an additive capacity,\footnote{Capacity $\mu$ is additive if $\mu(A \cup B) = \mu(A) + \mu(B)$ for every $A, B \in \Sigma$ such that $A \cap B = \emptyset$, that is, $\mu$ is a probability measure.} CEU is the standard expected utility, and reflects ambiguity neutrality.

A useful concept for establishing quasidifferentiability of CEU is the Möbius inverse of a capacity. The Möbius inverse of capacity $\mu : \Sigma \rightarrow \mathbb{R}$ is a set function $m_\mu : \mathcal{P}(\Sigma) \rightarrow \mathbb{R}$ such that

$$m_\mu(A) = \sum_{\{B \in \Sigma : B \subseteq A\}} (-1)^{|A|-|B|} \mu(B).$$

Equations (18) and (19) define a one-to-one mapping between capacities and set functions satisfying conditions (i) - (iii). A capacity with positive Möbius inverse is called belief function. Belief functions have been extensively studied in Dempster (1967) and in the theory of evidence of Shafer (1976). A capacity is a belief function if and only if it is totally monotone.\footnote{A capacity is totally monotone if $\mu(\bigcup_{i=1}^n A_i) \geq \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} \mu(\bigcap_{i \in I} A_i)$ for every $A_1, \ldots, A_n \in \Sigma$ and every $n$.}

The Choquet integral (17) can be expressed using the Möbius inverse as

$$E_\mu[v(x)] = \sum_{A \in \Sigma} m_\mu(A) \min_{s \in A} v(x_s),$$

see Gilboa and Schmeidler (1994, Section 4). Let $\Delta^A$ denote the set of all probability measures on $(S, \Sigma)$ with the support on $A$. Equation (20) can be re-written as

$$E_\mu[v(x)] = \sum_{A \in \Sigma} m_\mu(A) \min_{P \in \Delta^A} E_P[v(x)].$$
As in Section 3, let \( \Delta_{A_{\text{min}}}(x) \) denote the subset of \( \Delta^A \) for which the minimum of expected utility of \( x \) is attained. That is,

\[
\Delta_{A_{\text{min}}}(x) = \arg \min_{P \in \Delta^A} E_P[v(x)].
\]  

(21)

Further, let \( +_\mu (\Sigma^-) \) denote the subset of the set of events \( \Sigma \) on which the Möbius inverse of \( \mu \) is positive (negative, respectively). We have the following

**Proposition 2:** The CEU function \( E_\mu[v(x)] \) is quasidifferentiable on \( \mathbb{R}^S_{+} \) for every capacity \( \mu \) and every differentiable utility index \( v \). The quasidifferential \( [\partial E_\mu[v(x)], \bar{\partial} E_\mu[v(x)]] \) at \( x \in \mathbb{R}^S_{+} \) is given by

\[
\partial E_\mu[v(x)] = v'(x) \sum_{A \in \Sigma^-_\mu} m_\mu(A) \Delta_{A_{\text{min}}}(x),
\]  

and

\[
\bar{\partial} E_\mu[v(x)] = v'(x) \sum_{A \in \Sigma^+_\mu} m_\mu(A) \Delta_{A_{\text{min}}}(x).
\]  

(22) and (23)

**Proof:** Using (20), \( E_\mu[v(x)] \) can be written as the sum of minimum functions. Each function \( x \mapsto \min_{P \in \Delta^A} E_P[v(x)] \) is quasidifferentiable with the quasidifferential equal to \( v'(x)[0, \Delta_{A_{\text{min}}}(x)] \). Therefore \( E_\mu[v(x)] \) as the finite sum of quasidifferentiable functions is quasidifferentiable, see Proposition B.1.1 in Appendix B. Expressions (22) and (23) result from additivity of quasidifferentials. \( \square \)

Proposition 2 implies that CEU is differentiable at every injective \( x \in \mathbb{R}^S_{+} \), that is, \( x_s \neq x_{s'} \) for every \( s \neq s' \). Indeed, if \( x \) is injective, then \( \Delta_{A_{\text{min}}}(x) \) is a singleton for every \( A \in \Sigma \). Further, it implies that if the Möbius inverse of \( \mu \) is positive, i.e., \( \mu \) is a belief function, then CEU is superdifferentiable. If the Möbius inverse is negative except for singletons (i.e., \( m_\mu(A) \geq 0 \) for every \( A \supset \{s, s'\} \) for some \( s \neq s' \)), then it is subdifferentiable. Indeed, if \( \Sigma^+_\mu \) consists of singletons, then the superdifferential of (23) is a single vector.

There are some well-known capacities for which CEU has a representation as \( \alpha \)-MEU. First, we consider convex and concave capacities. A capacity is convex (or supermodular) if

\[
\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)
\]  

(24)
for every $A, B \in \Sigma$. It is concave (or submodular) if the reverse inequality holds in (24). Every belief function is convex. Every capacity whose Möbius inverse is negative except for singletons is concave.

If capacity $\mu$ is convex, then there exist probability measures on $\Sigma$ that dominate $\mu$. The set of such probability measures is the core of $\mu$ defined by

$$\text{core}(\mu) = \{ P \in \Delta : P(A) \geq \mu(A), \forall A \in \Sigma \}. \quad (25)$$

An example of (non-additive) convex capacity is the null capacity $\mu^N$ defined by $\mu^N(A) = 0$ for every $A \in \Sigma, A \neq S$ and $\mu^N(S) = 1$. The core of the null capacity is the whole probability simplex $\Delta$. The Choquet integral with respect to convex capacity $\mu$ is

$$E_\mu[v(x)] = \min_{P \in \text{core}(\mu)} E_P[v(x)], \quad (26)$$

see Gilboa and Schmeidler (1994). Thus CEU with convex capacity $\mu$ is the ambiguity-averse multiple-prior expected utility with $\text{core}(\mu)$ as the set of priors. By Proposition 1, it is superdifferentiable with the superdifferential equal to $v'(x)\mathcal{P}_{\min}(x)$ for $\mathcal{P} = \text{core}(\mu)$.

If capacity $\mu$ is concave, then the conjugate (or dual) capacity $\bar{\mu}$ defined by $\bar{\mu}(A) = 1 - \mu(A^c)$, where $A^c$ denotes the complement event of $A$, is convex and the core of $\bar{\mu}$ is non-empty. An example of a concave capacity is the unit capacity $\mu^U$ defined by $\mu^U(A) = 1$ for every $A \in \Sigma, A \neq \emptyset$. The conjugate capacity of $\mu^U$ is the null capacity $\mu^N$. CEU with concave capacity is the ambiguity-seeking multiple-prior expected utility with $\text{core}(\bar{\mu})$ as the set of priors, that is

$$E_\mu[v(x)] = \max_{P \in \text{core}(\bar{\mu})} E_P[v(x)]. \quad (27)$$

By Proposition 1, it is subdifferentiable and the subdifferential is equal to $v'(x)\mathcal{P}_{\max}(x)$ for $\mathcal{P} = \text{core}(\bar{\mu})$.

Capacities which are convex combinations of convex capacities and their concave conjugates have been studied in Jaffray and Philippe (1997). For capacity $\mu_\alpha$ defined by

$$\mu_\alpha = \alpha \mu + (1 - \alpha)\bar{\mu} \quad (28)$$

where $\mu$ is a convex capacity and $\alpha \in [0, 1]$, the CEU is

$$E_{\mu_\alpha}[v(x)] = \alpha \min_{P \in \text{core}(\mu)} E_P[v(x)] + (1 - \alpha) \max_{P \in \text{core}(\mu)} E_P[v(x)], \quad (29)$$

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that is, $\alpha$-MEU with the core of $\mu$ as the set of priors. Quasidifferential of CEU (29) can be derived either from Proposition 1 or 2.

A special case of Jaffray and Philippe’s capacity (28) is when the convex capacity $\mu$ is taken as inner capacity $\pi^*$ associated with a probability measure $\pi$ on an algebra $\mathcal{F} \subset \Sigma$ of subsets of states (generated by a partition of $S$). The inner capacity is defined by

$$
\pi^*(A) = \max_{B \subseteq A, B \in \mathcal{F}} \pi(B),
$$

(30)

It follows from Proposition 2.4 in Denneberg (1994) that $\pi^*$ is a convex capacity. The core of $\pi^*$ is the set of all probability measures on $\Sigma$ that coincide with $\pi$ on $\mathcal{F}$. That is\textsuperscript{14}

$$
\text{core}(\pi^*) = \{ P \in \Delta : P(A) = \pi(A), \forall A \in \mathcal{F} \}. 
$$

(31)

A CEU with Jaffrey and Philippe’s capacity obtained from an inner capacity is an $\alpha$-MEU with the set of priors (31). This is the Hurwicz expected utility of Gul and Pesendorfer (2015). The capacity is the Hurwicz capacity.

Another special case of (28) is neo-additive capacity. It obtains when $\mu$ in (28) is taken as $\delta \pi + (1 - \delta)\mu^N$ for an arbitrary probability measure $\pi \in \Delta$ and $\delta \in [0, 1]$, see Chateauneuf et al. (2007) and Eichberger et al. (2012). The core of capacity $\delta \pi + (1 - \delta)\mu^N$ is the set $\delta \pi + (1 - \delta)\Delta$. The CEU (29) for neo-additive capacity $\mu^{\text{neo}}$ can be written as

$$
E_{\mu^{\text{neo}}}[v(x)] = \delta E_\pi[v(x)] + (1 - \delta) \left[ \alpha \min_{P \in \Delta} E_P[v(x)] + (1 - \alpha) \max_{P \in \Delta} E_P[v(x)] \right].
$$

(32)

We conclude this section with an example of CEU with non-convex capacity that illustrates the concepts we introduced, and provides an application of Propositions 1 and 2.

**Example 2:** Consider a capacity $\mu$ on three states given by $\mu(\{s\}) = \eta$, $\mu(\{s, s'\}) = 3\eta$, $\forall s, s', s \neq s'$, for $0 \leq \eta \leq \frac{1}{3}$. One can verify that capacity $\mu$ is convex for every

\textsuperscript{14}The proof of (31) is as follows: Probability measure $P$ is in the core of capacity $\pi^*$ if and only if $P(A) \geq \max\{\pi(B) : B \subseteq A, B \in \mathcal{F}\}$ for every $A \in \Sigma$, see (25). This is equivalent to $P(A) \geq \pi(B)$ for every $A \in \Sigma$ and every $B \subseteq A, B \in \mathcal{F}$. Since algebra $\mathcal{F}$ is generated by a partition of $S$, the latter holds if an only if $P$ is an extension of $\pi$ to $\Sigma$. Thus (31) holds.
The core of $\mu$ is
\[
\text{core}(\mu) = \{\pi \in \Delta : \pi_s \geq \eta, \pi_s + \pi_{s'} \geq 3\eta, \forall s, s', s \neq s'\}.
\]
The core is non-empty for every $\eta \leq \frac{2}{9}$. It is a hexagon for $\eta < \frac{1}{5}$, and a triangle for $\frac{1}{5} \leq \eta \leq \frac{2}{9}$. The Möbius inverse of $\mu$ is
\[
m_\mu(\{s\}) = \eta, \quad m_\mu(\{s, s'\}) = \eta, \quad m_\mu(S) = 1 - 6\eta, \quad \forall s, s', s \neq s'.
\]
It is positive for every $\eta \leq \frac{1}{6}$.

Let us consider risk-free consumption plan $x = (1, 1, 1)$ and the linear utility $v(z) = z$. If $\eta \leq \frac{1}{5}$ so that $\mu$ is convex, then $E_\mu[x]$ is superdifferentiable and the superdifferential at $(1, 1, 1)$ equals the core of $\mu$, by Proposition 1.

Proposition 2 provides a characterization of the quasidifferential of $E_\mu[x]$ for every $\eta \leq \frac{1}{3}$. To see this, we first observe that for $x = (1, 1, 1)$ the set of minimizing probabilities $\Delta^A_{\min}(x)$ of (21) is equal to $\Delta^A$ for every $A \subset S$. If $\eta \leq \frac{1}{6}$, then the set $\Sigma_-$ of events with negative Möbius inverse is empty and the subdifferential in (23) is zero. Let $\Delta_{s,s'}$ be the set of probabilities with support on two states $s$ and $s'$. The superdifferential in (22) equals $\eta(1, 1, 1) + \eta\{\Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}\} + (1 - 6\eta)\Delta$. It can be shown that this set is equal to the core of $\mu$. If $\frac{1}{6} < \eta \leq \frac{1}{5}$, then $\Sigma_-$ consist of the event $S$ and the subdifferential is non-zero and equal to $(1 - 6\eta)\Delta$. The superdifferential is $\eta(1, 1, 1) + \eta\{\Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}\}$. It can be shown that this pair of sets is DR-equivalent to the pair $[0, \text{core}(\mu)]$ arising from Proposition 1.

If $\frac{1}{5} < \eta$, then capacity $\mu$ is not convex. The quasidifferential of $E_\mu[x]$ is, by Proposition 2, a pair of sets $[(1 - 6\eta)\Delta, \eta(1, 1, 1) + \eta\{\Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}\}]$ — a symmetric triangle, and a symmetric hexagon. $\square$

### 3.2 Rank-Dependent Expected Utility

Rank-Dependent Expected Utility (RDEU) is a special case of CEU with the capacity being a distorted probability measure. For a reference (subjective) probability measure $\pi$ on $\Sigma$ and a probability distortion (or weighting) function $w : [0, 1] \to [0, 1]$, assumed strictly increasing and satisfying $w(0) = 0$ and $w(1) = 1$, distorted probability $\pi_w$ is a capacity defined by
\[
\pi_w(A) = w(\pi(A)), \quad \forall A \in \Sigma.
\]
RDEU is the Choquet integral of utility index $v$ with respect to distorted probability $w$, that is,

$$V_{RD}(x) = \sum_{i=1}^{s} v(x(i)) \left[ w(\pi(\{(1), \ldots, (i)\})) - w(\pi(\{(1), \ldots, (i-1)\})) \right],$$

(34)

where $x(k)$ is the $k$-th highest consumption level from among all $x_s$. Proposition 2 implies that RDEU is quasidifferentiable for every differentiable utility index $v$.

The feature of RDEU which distinguishes it from general CEU is probabilistic sophistication, that is, distribution invariance under the reference probability measure $\pi$. Properties of the distortion function in RDEU correspond to certain behavioral phenomena just like properties of the capacity in CEU. For example, convexity of distortion function $w$, which amounts to it being relatively flat for low values of probability and steep for high values, implies underweighting the best outcomes and overweighting the worst outcomes in (34). It reflects pessimism. The resulting capacity $\pi_w$ is convex, and the RDEU function can be expressed as ambiguity-averse multiple-prior utility function (26) with $\mu = \pi_w$. Similarly, concavity of $w$ reflects optimism. The resulting capacity $\pi_w$ is concave, and RDEU is the ambiguity-seeking multiple-prior utility function (27) $\tilde{\mu} = \pi_w$.

Empirical investigations of RDEU point out to inverse S-shaped distortion function, see Wakker (2010, Chapter 7). An inverse S-shaped distortion function $w$ is concave on an interval $[0, B]$ and convex on $[B, 1]$ for some inflection point $B \in [0, 1]$. It reflects overweighting the worst outcomes and overweighting the best outcomes. It plays an important role in the Cumulative Prospect Theory.

Examples of inverse S-shaped distortion functions are normalized power functions and neo-additive functions.

**Example 3:** The normalized-power distortion function introduced by Tversky and Kahneman (1992) is

$$w(p) = \frac{p^r}{(p^r + (1-p)^r)^{1/r}},$$

(35)

with parameter $r \in [0, 1]$, see the left panel in Figure 1.
The neo-additive distortion function is

\[ w_{\text{neo}}(p) = \begin{cases} 
0 & \text{if } p = 0 \\
 ap + b & \text{if } p \in (0, 1) \\
1 & \text{if } p = 1 
\end{cases} \tag{36} \]

with positive parameters \(a, b\) such that \(a + b \leq 1\), see function \(l\) in the left panel in Figure 1. For every neo-additive distortion \(w_{\text{neo}}\) and every \(\pi \in \Delta\), the distorted probability \(\pi_{w_{\text{neo}}}\) is a neo-additive capacity, see (32), with \(a = \delta\) and \(\alpha = (1 - a - b)/(1 - a)\).

Neo-additive distortion with \(a = b = 0\) is denoted by \(w^N\). The distorted probability \(\pi_{w^N}\) is the null-capacity \(\mu^N\) for every \(\pi\). Neo-additive distortion with \(a = 0\) and \(b = 1\) is denoted by \(w^U\), and the distorted probability \(\pi_{w^U}\) is the unit capacity \(\mu^U\) for every \(\pi\).

Figure 1: Decomposition of \(S\)-shaped distortion function \(w\) from (35) with parameter \(r = 0.61\) and inflection point \(B \approx 0.48\). The neo-additive approximation \(l\) is used to construct \(w_0, w_1\) in equation (37).

We derive next a novel representation of RDEU with inverse \(S\)-shaped distortion function. This representation leads to more transparent expressions for the quasidifferential than Proposition 2. Let \(w\) be any inverse \(S\)-shaped distortion function with inflection point \(B\) where \(0 \leq B \leq 1\). We define a neo-additive function \(l = w_{\text{neo}}\) of (36) with parameters \(a = w'(B)\) and \(b = w(B) - Bw'(B)\), see the left panel in Figure 1. Using \(l\), we construct two distortions \(w_0\) and \(w_1\) as

\[ w_0 = 1_{[0,B]} w + 1_{[B,1]} l \quad \text{and} \quad w_1 = 1_{[0,B]} l + 1_{[B,1]} w. \tag{37} \]
see the middle and the right panels in Figure 1. Note that

\[ w = w_0 + w_1 - l. \]  \hspace{1cm} (38)

Distortion function \( w_0 \) is concave on the right-open interval \([0, 1)\). A scale transformation of \( w_0 \) given by

\[ w_0^s(p) = \begin{cases} \frac{w_0(p)}{t_0} & \text{if } p \in [0, 1) \\ 1 & \text{if } p = 1, \end{cases} \]

where \( t_0 = l(1) \), is a concave distortion on \([0, 1)\). Further, \( w_0 = t_0 w_0^s + (1 - t_0) w^N \).

The core of conjugate distortion \( \tilde{w}_0^s(p) = 1 - w_0^s(1 - p) \), denoted by

\[ \mathcal{P}^0 = \text{core}(\pi_{\tilde{w}_0^s}), \]  \hspace{1cm} (39)

is non-empty.

Distortion function \( w_1 \) is convex on the left-open interval \((0, 1]\). A shifted scale transformation of \( w_1 \) given by

\[ w_1^s(p) = \begin{cases} \frac{w_1(p) - t_1}{1 - t_1} & \text{if } p \in (0, 1] \\ 0 & \text{if } p = 0, \end{cases} \]

where \( t_1 = l(0) \), is a convex distortion on \([0, 1)\). It holds \( w_1 = (1 - t_1) w_1^s + t_1 w^U \).

The non-empty core of \( w_1^s \) is denoted by

\[ \mathcal{P}^1 = \text{core}(\pi_{w_1^s}). \]  \hspace{1cm} (40)

The sets \( \mathcal{P}^0 \) and \( \mathcal{P}^1 \) give rise to a representation of RDEU with inverse S-shaped distortion which is closely related to \( \alpha \)-MEU.

**Proposition 3:** For every differentiable and inverse S-shaped distortion \( w \), every probability measure \( \pi \) on \( \Sigma \), and every differentiable utility index \( v \), the rank-dependent expected utility \( V_{RD} \) has a representation

\[ V_{RD}(x) = \eta_0 \max_{P \in \mathcal{P}^0} E_P[v(x)] + \eta_1 \min_{P \in \mathcal{P}^1} E_P[v(x)] - \eta E_E[v(x)], \]  \hspace{1cm} (41)

where \( \mathcal{P}^0 \) and \( \mathcal{P}^1 \) are given by (39) and (40), \( \eta = w'(B), \eta_0 + \eta_1 - \eta = 1 \), and \( \eta_0, \eta_1 \geq 0.15 \) Further, the utility function \( V_{RD} \) in (41) is quasidifferentiable at

\(^{15}\text{Exact expressions for } \eta_0 \text{ and } \eta_1 \text{ can be found in Appendix A.}\)
for every \( x \in \mathbb{R}_+^S \), and the sub- and superdifferentials are given by

\[
\partial V_{RD}(x) = v'(x)\eta_P^0 \mathcal{P}_{\max}(x),
\]

and

\[
\partial V_{RD}(x) = v'(x)[\eta P_{\min}^1(x) - \eta \pi].
\]

**Proof:** See Appendix A.

### 3.3 Cumulative Prospect Theory

The Cumulative Prospect Theory (CPT) of Tversky and Kahneman (1992) is a refinement of RDEU to accommodate reference dependence of preferences. The CPT differentiates between gains and losses, and permits different risk attitudes over gains and losses. The utility index \( v : \mathbb{R} \to \mathbb{R} \) is concave over gains and convex over losses. Further, there are two probability distortion functions - one for gains and one for losses - both inverse S-shaped which reflects overweighting extreme outcomes.

Gains and losses in CPT are defined relative to a *reference point* \( \bar{x} \in \mathbb{R}_+^S \). For an arbitrary consumption plan \( x \in \mathbb{R}_+^S \), gains are \((x - \bar{x})^+ = \max(x - \bar{x}, 0)\), and losses are \((x - \bar{x})^- = \min(x - \bar{x}, 0)\). The CPT utility \( V_{CP} \), with a reference probability measure \( \pi \in \Delta \), two distortion functions \( w^+ \) and \( w^- \) for gains and losses, and a utility index \( v \), is the sum of two rank-dependent expected utilities, that is

\[
V_{CP}(x) = V_{RD}^+(x) + V_{RD}^-(x) = E_{\pi^+}^w[v((x - \bar{x})^+)] + E_{\pi^-}^w[v((x - \bar{x})^-)],
\]

where the Choquet integrals are from (34). Note that the RDEU function over losses in (42) is taken with respect to the conjugate distortion function \( \bar{w}^- \). Thus the decision weight assigned in \( E_{\pi^-}^w[v((x - \bar{x})^-)] \) to a loss-outcome \( x_{(i)} \), with \( x_{(i)} - \bar{x} < 0 \), is \( w^-(\pi\{(i), \ldots, (S)\}) - w^-(\pi\{(i+1), \ldots, (S)\}) \).

Tversky and Kahneman (1992) specification of distortion functions \( w^+ \) and \( w^- \) is in the form of inverse S-shaped normalized-power functions of (35) with parameters \( r^+ = 0.61 \) for \( w^+ \) and \( r^- = 0.69 \) for \( w^- \). Note that the conjugate \( \bar{w}^- \) is inverse S-shaped, as well. The utility index is the power function

\[
v(z) = \begin{cases} 
  z^b & \text{if } z \geq 0 \\
  -\theta(-z)^b & \text{if } z < 0.
\end{cases}
\]

(43)
where $b \in (0,1]$ and $\theta > 1$. It is concave over gains and convex over losses. This induces risk aversion for gains and risk seeking for losses. The parameter $\theta$ reflects loss aversion, see Wakker (2010, Chapter 8). Kahneman and Tversky (1979) found that parameters $b = 0.88$ and $\theta = 2.25$ fit the experimental data best.

The power function is “problematic” (see Wakker (2010, p. 267)) because of infinite derivative at zero. This leads to problems in studying loss aversion, but also makes it not suitable for the study of optimal choices. We shall assume that the utility index $v$ is differentiable for every $z \neq 0$ and has well-defined right- and left-hand derivatives at zero. An example of such function occasionally used in CPT is the shifted power function

$$v(z) = \begin{cases} 
\frac{(1+z)^b}{b} - \frac{1}{b} & \text{if } z \geq 0 \\
-\frac{(1-z)^b}{b} + \frac{\theta}{b} & \text{if } z < 0.
\end{cases}$$

with $b \in (0,1]$ and $\theta > 1$, see Wakker (2010, p. 271). It is concave on gains, convex on losses, and superdifferentiable at zero.

In the analysis of quasidifferentiability of the CPT utility function (42) we shall restrict attention to inverse S-shaped distortion functions, and rely on the results of Section 3.2. The difficulty in establishing quasidifferentiability of the CPT function and deriving its quasidifferential lies in non-differentiability of the gain and the loss functions at the reference point. Let us consider first the RDEU function $E_{w^+}[v((x-x)^+)]$ for gains. If $w^+$ is inverse S-shaped, then the gain-RDEU function has the max-plus-min representation (41) of Proposition 3 with sets $P_0$ and $P_1$ of equations (39) and (40) for distortion $w^+$. If function $v$ is strictly increasing, differentiable for $z \neq 0$, and has well-defined right- and left-hand derivatives at zero, then the gain function $v((z-x)^+)$ for $z \in \mathbb{R}$ is quasidifferentiable because it is the maximum of two quasidifferentiable functions $v(z-x)$ and 0. It is in fact subdifferentiable, see Section 2.\footnote{The subdifferential $\partial v((z-x)^+)$ of the gain function is $v'(z-x)$ for $z > x$, 0 for $z < x$, and the interval $[0, v'_+(0)]$ for $z = x$, where $v'_+(0)$ is the right-hand derivative at 0.}

The function $E_P[v((x-x)^+)]$ is the sum of quasidifferentiable functions, hence it is quasidifferentiable for every $P \in \Delta$. Further, the minimum function $\min_{P \in P_1^1} E_P[v((x-x)^+)]$
and the maximum function \( \max_{P \in P_+} E_P[v((x - \bar{x})^+)] \) are quasidifferentiable as well. This follows from Proposition B.3 in Appendix B. Note that Proposition B.3 can be applied because the sets \( P_0^+ \) and \( P_1^+ \) are cores of convex distortions, and hence convex polytopes. The same arguments apply to the loss-RDEU function \( E_{P-}[v((x - \bar{x})^-)] \) with the only difference that the loss \( v((z - \bar{z})^-) \) is superdifferentiable as the minimum two quasidifferentiable functions.\(^17\)

Summing up, we have

**Proposition 4:** For every differentiable and inverse S-shaped distortions \( w^+ \) and \( w^- \), every probability measure \( \pi \) on \( \Sigma \), and every utility index \( v \) that is differentiable on \( \mathbb{R} \setminus \{0\} \) and has well-defined right- and left-hand derivatives at 0, the CPT utility function (42) is quasidifferentiable.\(^18\)

The quasidifferential of the CPT utility function satisfying the assumptions of Proposition 4 can be derived applying the rules of quasidifferentiation - in particular Proposition B.3 in Appendix B - to representations (41) of the RDEU functions \( V_{RD_+}^+(x) \) and \( V_{RD_-}^-(x) \) in (42). Because of non-differentiability of the CPT gain and loss functions, expressions for quasidifferentials of \( V_{RD_+}^+(x) \) and \( V_{RD_-}^-(x) \) are more complex than those in Proposition 3 as can be seen from Proposition B.3. We omit exact derivations. For consumption plans that involve strictly positive gains in every state or strictly positive losses in every state, the quasidifferential of a CPT function is the quasidifferential of the respective RDEU summand as derived in Section 3.2.

### 4 Pareto Optimal Allocations

Suppose that there are \( I \) agents with preferences over state-contingent consumption plans in \( \mathbb{R}_+^S \) described by strictly increasing utility functions \( V_i \). Let \( e \in \mathbb{R}_+^S \)

\(^17\)The superdifferential \( \partial v((z - \bar{z})^-) \) of the loss function is \( v'(z - \bar{z}) \) for \( z < \bar{z} \), 0 for \( z > \bar{z} \), and the interval \([0, v'_-(0)]\) for \( z = \bar{z} \), where \( v'_-(0) \) is the left-hand derivative at 0.

\(^18\)Proposition 4 can be extended to CPT utility functions with arbitrary distortions. The argument relies on the representation (20) of the Choquet integrals in the definition of RDEU functions \( V_{RD_+}^+(x) \) and \( V_{RD_-}^-(x) \). These functions can be represented as weighted sums of minimum functions, and therefore are quasidifferentiable.
denote the aggregate endowment of the economy. Recall that an allocation \( \{x_i\} \), where \( x_i \in \mathbb{R}_+^S \) for every \( i \), is feasible if \( \sum_{i=1}^I x_i \leq e \). A feasible allocation is Pareto optimal if there is no other feasible allocation \( \{\tilde{x}_i\} \) such that \( V_i(\tilde{x}_i) \geq V_i(x_i) \) with at least one strict inequality. Since the utility functions need not be concave, Pareto optimal allocations cannot be characterized as solutions to the problem of maximizing weighted sum of individual utilities subject to the feasibility constraint. Instead, we consider the problem of maximizing one agent’s utility subject to constraints on other agents’ utilities and feasibility. Choosing agent 1 without loss of generality, we have

\[
\max_{\{x_i\} \in \mathbb{R}_+^SI} V_1(x_1) \tag{45}
\]

subject to

\[
V_i(x_i) \geq \bar{v}_i, \ i = 2, \ldots, I, \\
\sum_{i=1}^I x_i \leq e,
\]

for some bounds \( \bar{v}_i \in \mathbb{R} \). Every allocation solving (45) is Pareto optimal. Conversely, every Pareto optimal allocation is a solution to (45) for some bounds \( \bar{v}_i \).

The following necessary first-order conditions for a Pareto optimal allocation are derived from (45).

**Proposition 5:** Suppose that utility functions \( V_i \) are quasidifferentiable. If \( \{x_i\} \) is an interior Pareto optimal allocation, then for every profile \( \tilde{z}_i \in \partial V_i(x_i) \) there exist a corresponding profile \( \bar{z}_i \in \partial V_i(x_i) \), positive multipliers \( \lambda_i \in \mathbb{R}_+ \), and a positive vector \( q \in \mathbb{R}_+^S \), not all zero, such that

\[
\lambda_i[\tilde{z}_i + \bar{z}_i] = q, \tag{46}
\]

for every \( i \). Further, the complementary slackness conditions hold.\(^{19}\)

Proposition 5 follows from Proposition 1.1 in Gao (2000a) which we reproduce and discuss in Appendix A. A strict form of condition (46) which is sufficient for local Pareto optimality can be found in Appendix A as well. Proposition 5 holds independently of the DR-equivalent representation of the quasidifferentials.

\(^{19}\) Those are \( \lambda_i(V_i(x_i) - \bar{v}_i) = 0 \) for \( i = 2, \ldots, I \) and \( q_s(\sum_{i=1}^I x_{i,s} - e_s) = 0 \) for \( s = 1, \ldots, S \). We omit the slackness conditions from all subsequent refinements of Proposition 5.
If every utility function $V_i$ is differentiable at $x_i$, then the first-order condition (46) states that $\lambda_i \nabla V_i(x_i) = q$ for the gradient vector $\nabla V_i(x_i)$, for every $i$, which is the standard condition of common marginal rates of substitution. If every function $V_i$ is concave, so that the quasidifferential has the representation $[0, \partial V_i(x)]$ with zero subdifferential, then condition (46) states that there exist a profile $\tilde{z}_i \in \partial V_i(x)$ such that $\lambda_i \tilde{z}_i = q$. This is the standard necessary and sufficient condition for Pareto optimality of an interior allocation for concave utility functions, see Aubin (1998).

In the reminder of this section we present statements of Proposition 5 specialized to all agents with $\alpha$-MEU or all with RDEU, and discussions of applications to CEU and CPT. Settings with mixed utility functions can be easily analyzed using those results.

### 4.1 $\alpha$-MaxMin Expected Utilities

Suppose that agents have $\alpha$-MEU functions with agent-specific weights $\alpha_i \in [0, 1]$ and sets of priors $\mathcal{P}_i \subset \Delta$, assumed closed and convex. That is,

$$V_i(x) = \alpha_i \min_{P \in \mathcal{P}_i} E_P[v_i(x)] + (1 - \alpha_i) \max_{P \in \mathcal{P}_i} E_P[v_i(x)],$$

where functions $v_i : \mathbb{R}^+ \rightarrow \mathbb{R}$ are strictly increasing and continuously differentiable. Using Propositions 1 and 5, we obtain

**Proposition 6:** If $\{x_i\}$ is an interior Pareto optimal allocation for $\alpha$-MEU, then for every profile of beliefs $P_i \in \mathcal{P}_i^{\max}(x_i)$ there exist a corresponding profile of beliefs $\tilde{P}_i \in \mathcal{P}_i^{\max}(x_i)$, strictly positive multipliers $\lambda_i \in \mathbb{R}^+$, and a positive vector $q \in \mathbb{R}^S_+$, $q \neq 0$, such that for every $i$

$$\lambda_i v_i'(x_i)[\alpha_i \tilde{P}_i + (1 - \alpha_i)P_i] = q. \quad (47)$$

If all agents have ambiguity-averse $\alpha$-MEU with $\alpha_i = 1$, then condition (47) says that there exists a profile of beliefs $\tilde{P}_i \in \mathcal{P}_i^{\min}(x_i)$ such that

$$\lambda_i v_i'(x_i)\tilde{P}_i = q,$$

for every $i$. This is the first-order condition of Rigotti et al. (2008) for Pareto optimality of an interior allocation for ambiguity-averse multiple-prior expected utilities. Proposition 6 extends the result to non-concave utility indices $v_i$.  

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Proposition 6 implies that every interior Pareto optimal allocation for $\alpha$-MEU functions with concave utility indexes is Pareto optimal for expected utility functions with heterogeneous beliefs taken from the agents’ sets of priors.

**Corollary 1:** Suppose that $v_i$ is concave for every $i$. If $\{x_i\}$ is an interior Pareto optimal allocation for $\alpha$-MEU, then there exists a profile of beliefs $P_i \in \mathcal{P}_i$ such that the allocation $\{x_i\}$ is Pareto optimal for expected utility functions $E_{P_i}[v_i(x)]$.

**Proof:** For arbitrary beliefs $\bar{P}_i$ and $\bar{P}_i$ satisfying (47), define probability measures $P_i = \alpha_i \bar{P}_i + (1 - \alpha_i) \bar{Q}_i$. Note that $P_i \in \mathcal{P}_i$. The allocation $\{x_i\}$ satisfies the first-order conditions for Pareto optimality for expected utilities with beliefs $P_i$. Because of concavity of $v_i$, those conditions are sufficient and hence the allocation is Pareto optimal.

For small sets of priors with non-empty intersection, the set of Pareto optimal allocations with heterogeneous beliefs taken from those sets is a limited set of allocations. Another corollary to Proposition 6 establishes necessity of a common prior for the existence of a risk-free Pareto optimal allocation. Of course, there can be a risk-free allocation only if the aggregate endowment is risk free, that is, there is no aggregate risk.

**Corollary 2:** If there exists an interior risk-free Pareto optimal allocation for $\alpha$-MEU, then

$$\bigcap_{i=1}^{l} \mathcal{P}_i \neq \emptyset. \tag{48}$$

**Proof:** Let $\{x_i\}$ be an interior risk-free Pareto optimal allocation and let $P_i \in \mathcal{P}_i$ be as defined in the proof of Corollary 1. Since $x_i$ is state-independent, the first-order condition

$$\lambda_i v'_i(x_i) P_i = q$$

implies that $P_i = P$ and $P \in \mathcal{P}_i$ for every $i$, hence (48) holds.

The contraposition to Corollary 2 is that if there is no common prior, then there cannot be a risk-free Pareto optimal allocation even if there is no aggregate risk. Billot et al. (2000) show that the common prior condition (48) is necessary and sufficient for all Pareto optimal allocations to be risk free, if there is no aggregate risk, for ambiguity-averse MEU with $\alpha = 0$. Extensions of that result...
to general convex preferences can be found in Rigotti et al. (2008), see also Ghirardato and Siniscalchi (2018).

The next corollary shows limitations to the possibility of subdifferentiable \( \alpha \)-MEU at a Pareto optimal allocation. Recall from Section 2 that a function is subdifferentiable at \( x \) if its quasidifferential has a representation with zero superdifferential. An \( \alpha \)-MEU with \( \alpha = 0 \) is subdifferentiable everywhere and has set-valued subdifferential (equal to the set of priors) at any risk-free consumption plan.

**Corollary 3:** Let \( \{x_i\} \) be an interior Pareto optimal allocation for \( \alpha \)-MEU. If there are two or more agents whose utility functions are subdifferentiable at their respective consumption plans \( x_i \), then their utility functions are differentiable at \( x_i \).

**Proof:** See Appendix A.

Corollary 3 implies that, for any interior Pareto optimal allocation \( \{x_i\} \), there can be at most one agent \( i \) whose utility function is subdifferentiable but not differentiable at \( x_i \). Further, if there are at least two agents with ambiguity-seeking MEU with \( \alpha = 0 \) and there is no aggregate risk, then no Pareto optimal allocation can be risk free.

### 4.2 Rank-Dependent Expected Utilities and Other Utility Functions

Proposition 5 applies to CEU considered in Section 3.2. Equations (22) and (23) of Proposition 2 provide representations of sub- and superdifferentials of CEU that can be used in conjunction with Proposition 5. If agents’ capacities are Jaffray and Philippe’s capacities of (28), then the CEUs are \( \alpha \)-MEUs, and the results of Section 4.1 apply. We recall that concave, convex, neo-additive, and Hurwicz capacities are special cases of Jaffray and Philippe’s capacities.

As argued in Sections 3.2, RDEU functions with inverse S-shaped distortion functions are an important subclass of CEUs because of their empirical significance and their role in the Cumulative Prospect Theory. Since RDEU functions with inverse S-shaped distortions have max-plus-min representation (see Propo-
sition 3), some properties of Pareto optimal allocations established in Section 4.1 for α-MEU extend to RDEU. To simplify notation, we assume that agents have common reference probability measure \( \pi \). Utility indexes \( v_i \) are assumed differentiable. Using Propositions 3 and 5, we obtain

**Proposition 7:** If \( \{x_i\} \) is an interior Pareto optimal allocation for RDEU with differentiable and inverse S-shaped distortion functions \( w_i \), then for every profile of beliefs \( P_i \in \mathcal{P}_{\max}^0(x_i) \) there exist a corresponding profile of beliefs \( P_i \in \mathcal{P}_{\min}^1(x_i) \), strictly positive multipliers \( \lambda_i \in \mathbb{R}_+ \), and a positive vector \( q \in \mathbb{R}_+^S \), \( q \neq 0 \), such that for every \( i \)

\[
\lambda_i v'_i(x_i)[\eta_{0,i} P_i + \eta_{1,i} P_i - \eta_i \pi] = q,
\]

where \( \eta_{0,i}, \eta_{1,i}, \) and \( \eta_i \) are as in Proposition 3.

As in Section 4.1, Proposition 7 implies that every interior Pareto optimal allocation for RDEU with inverse S-shaped distortions is Pareto optimal for expected utility functions with heterogeneous beliefs.

**Corollary 4:** Suppose that \( v_i \) is concave for every \( i \). If \( \{x_i\} \) is an interior Pareto optimal allocation for RDEU with differentiable and inverse S-shaped distortion functions, then there exists a profile of beliefs \( P_i \in \eta_{0,i} \mathcal{P}_{\max}^0 + \eta_{1,i} \mathcal{P}_{\min}^1 - \eta_i \pi \) such that the allocation \( \{x_i\} \) is Pareto optimal for expected utility functions \( E_{P_i}[v_i(x)] \).

Proposition 5 applies to CPT utility functions, too. CPT utility functions are sums of rank-dependent expected utilities of gains and losses, albeit with non-differentiabilities at the reference point, see Section 3.3. For Pareto optimal allocations that involve strictly positive gains in every state for every agent, the first-order conditions are those of Proposition 7 for gain-RDEU. The same holds for optimal allocations with losses in every state, if such allocations exist. We illustrate the application of Proposition 5 to CPT utilities with an example of an Edgeworth box. The example showcases the roles of the reference point, risk-seeking for losses, and loss aversion for optimal allocations.

**Example 4:** There are two agents and two states. The aggregate endowment is \( e = (6, 6) \). Agents have the same CPT utility functions with reference belief \( \pi = (\frac{1}{2}, \frac{1}{2}) \), state-dependent reference point for gains and losses \( \bar{x} = (2, 1) \), and the utility index \( v \) of the form (44) with parameters \( b = \frac{1}{2} \) and \( \theta = 2 \), where the latter
reflects loss aversion. To simplify, we abstract from distortion of probabilities, that is, we take \( w^+ = w^- = id \). With no distortion, the CPT utility function is the expected utility, and the distinction between gains and losses is unnecessary. The agents’ utility functions are therefore

\[
V^i_{CP}(x) = E_{\pi}[v(x - \bar{x})], \quad i = 1, 2.
\]

Figure 2 shows the Edgeworth box under consideration. Indifference curves are plotted for five different utility levels for each agent.\(^{20}\)

\(^{20}\)For nonlinear distortion functions, there would be additional kinks on the 45°-degree lines for each agent.
experience gains in both states. There are Pareto optimal allocations in that region and they look like typical optimal allocations for concave utility functions. One such allocation is the equal-sharing allocation of \((3, 3)\) for each agent which is the center of the box. Utility functions are differentiable at this point and the first-order condition is the standard equality of marginal rates of substitution. Further, there are Pareto optimal allocations such as \((1.5, 1), (4.5, 5)\) near the bottom-left corner and the corresponding one near the top-right corner where one of the agents experiences losses. Since consumption of agent 1 in state 2 equals her reference level, her utility function is not differentiable and the indifference curve has a kink. Her utility function is superdifferentiable because of loss aversion. The first-order condition of Proposition 5 is that the vector of marginal utilities of agent 2 lies in the superdifferential of the utility of agent 1. Allocations where one agent’s consumption is equal to the reference point in both states are Pareto optimal as well. The utility function is superdifferentiable at the reference point.

Note that even though utility functions are non-differentiable on a small set (with measure zero) of points, those points are of critical importance for optimal allocations.

\[ \square \]

5 Concluding Remarks

We introduced the methodology of quasidifferential calculus to the analysis of optimality conditions for non-differentiable and non-concave utility functions arising in contemporary decision theory. Quasidifferential calculus offers transparent statements of the first-order conditions in a way that unifies and extends the well-known conditions for differentiable, concave, and convex functions. We argued that it is better suited than the more standard method of the Clarke subdifferential, in particular, for such utility functions as \(\alpha\)-MEU, CEU, RDEU, and CPT.

We presented first-order conditions for Pareto optimal allocations under uncertainty for these utility functions. The result leads to interesting implications in regard to optimal risk sharing with quasidifferentiable utilities. For example, a necessary condition for existence of a risk-free Pareto optimal allocation in an
economy with no aggregate risk and arbitrary $\alpha$-MEU functions is that the sets of priors have non-empty intersection.

$\alpha$-MEU and CEU models are often considered in settings of infinitely many states. Since quasidifferential calculus has been developed in general Banach spaces of functions (see Palaschke and Rolewicz (1997)), the results of this paper can be extended to infinite state spaces. We leave technical details of such extensions for future research.

A Proofs

Proof of Proposition 3: Using (38), we obtain that

$$V_{RD}(x) = E_{\pi_{w0}}[v(x)] + E_{\pi_{w1}}[v(x)] - E_{\pi_{l}}[v(x)]$$

(49)

for any probability measure $\pi \in \Delta$.

We shall derive representations of the three summands in decomposition (49). First, we recall that distortion $w_0$ can be written as

$$w_0 = t_0 w_0^s + (1 - t_0) w^N,$$

where $t_0 = l(1)$. Since the scaled distortion $w_0^s$ is concave and the null distortion $w^N$ is convex, we have

$$E_{\pi_{w0}}[v(x)] = t_0 \max_{p \in P^0} E_p[v(x)] + (1 - t_0) \min_{p \in \Delta} E_p[v(x)],$$

(50)

where $P^0$ is from (39). Similarly, distortion $w_1$ can be written as

$$w_1 = (1 - t_1) w_1^s + t_1 w^U,$$

where $t_1 = l(0)$. The scaled distortion $w_1^s$ is convex and the unit distortion $w^U$ is concave. we have

$$E_{\pi_{w1}}[v(x)] = (1 - t_1) \min_{p \in P^1} E_p[v(x)] + t_1 \max_{p \in \Delta} E_p[v(x)],$$

(51)

where $P^1$ is from (40).
Recall that neo-additive distortion $l$ is given by (36) with parameters $a = w'(B)$ and $b = w(B) - Bw'(B)$. It holds $t_0 = a + b$, and $t_1 = b$. The RDEU under $l$ is (see (32))

$$E[x_l(v(x))] = aE[x_l(v(x))] + (1 - a - b) \min_{P \in \Delta} E_P[v(x)] + b \max_{P \in \Delta} E_P[v(x)]$$  \hspace{1cm} (52)

Combining (49) - (52) yields

$$V_{RD}(x) = (a + b) \max_{P \in \mathcal{P}_0} E_P[v(x)] + (1 - b) \min_{P \in \mathcal{P}_1} E_P[v(x)] - aE[x_l(v(x))].$$  \hspace{1cm} (53)

Equation (53) is the representation (41) with $\eta = w'(B), \eta_0 = w'(B) + w(B) - Bw'(B)$, and $\eta_1 = 1 - w(B) + Bw'(B)$.

Note that each summand in (41) is quasidifferentiable. The quasidifferentiability and the form of the quasidifferential of $V_{RD}$ follow from Corollary B.1 and the summation rule in Proposition B.1.1.

**PROOF OF PROPOSITION 5:** Consider the following constrained maximization problem:

$$\max_x f_0(x)$$  \hspace{1cm} (54)

subject to

$$f_i(x) \geq 0, \ i = 1, \ldots, m$$  \hspace{1cm} (55)

where $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 0, \ldots, m$ are quasidifferentiable.

**Proposition A.1:** (Gao (2000a)): If $x^*$ is a solution to (54), then for every profile $z_i \in \partial f_i(x^*)$, there exist corresponding profile $\bar{z}_i \in \partial f_i(x^*)$ and positive multipliers $\lambda_i \in \mathbb{R}_+, \ i = 0, \ldots, n$, not all zero, such that

$$\sum_{i=0}^m \lambda_i [\bar{z}_i + z_i] = 0,$$  \hspace{1cm} (56)

and $\lambda_i f_i(x^*) = 0$ for every $i \geq 1$.

We note that the multiplier $\lambda_i$ may depend on the selected profile $\bar{z}_i \in \partial f_i(x)$. Proposition A.1 is an extension of the Fritz John’s first-order conditions for differentiable functions in non-linear programming, see Takayama (1985). Neither Proposition A.1 nor the John’s result require a constrained qualification condition known from the Kuhn-Tucker theorems, but they feature a multiplier on the
objective function that could be zero. We apply Proposition A.1 to the Pareto problem (45). The function \( e_s - \sum_{i=1}^{I} x_{i,s} \) of the feasibility constraint in state \( s \) is differentiable. Using \( q \in \mathbb{R}^S_+ \) for the vector of multipliers of feasibility constraints, we obtain the first-order conditions (46).

A strict form of the first-order condition (46) is sufficient for local Pareto optimality. An allocation is locally Pareto optimal if it cannot be improved upon by a feasible allocation that lies in a small neighborhood of that allocation for every agent. The strict form of (46) requires that for every profile \( z_i \in \partial V_i(x_i) \) there exist \( \tilde{z}_i \in \text{int}\partial V_i(x_i) \) for every \( i \), and positive multipliers \( \lambda_i \) and a positive vector \( q \in \mathbb{R}^S_+ \), not all zero, such that (46) holds. The result follows from Proposition 3.1 in Gao (2000b).

**Proof of Corollary 3:** If \( \alpha \)-MEU function \( V_i \) is subdifferentiable, then the pair of sets \( [(1 - \alpha)P_{\max}(x_i), \alpha P_{\min}(x_i)] \) is DR-equivalent to \( [A_i, 0] \) for some compact and convex set \( A_i \). It can be easily seen that \( A_i \subseteq \mathcal{P} \).

In order to prove the first part, let \( i \) and \( j \) be the two agents whose utility functions are subdifferentiable with respective subdifferentials \( v'_i(x_i)A_i \) and \( v'_j(x_j)A_j \), and zero superdifferentials. To simplify the exposition, we disregard agents other than \( i \) and \( j \) in our arguments. In particular, a Pareto optimal allocation for \( I \) agents is Pareto optimal for any pair of agents.

Proposition 6 says that for any selection of \( a_i \in A_i \) and \( a_j \in A_j \), there exist multipliers \( \lambda_i \) and \( \lambda_j \), and vector \( q \) such that

\[
v'_i(x_i)a_i = \lambda_i q \quad \text{and} \quad v'_j(x_j)a_j = \lambda_j q \quad (57)
\]

Let us consider arbitrary \( a'_i \in A_i \). We shall prove that \( a'_i = a_i \). Applying Proposition 6 to the pair \( a'_i \in A_i \) and \( a_j \in A_j \), there exist \( \lambda'_i, \lambda'_j, \) and \( q' \) such that

\[
v'_i(x_i)a'_i = \lambda'_i q' \quad \text{and} \quad v'_j(x_j)a_j = \lambda'_j q' \quad (58)
\]

Using the equations for agent \( j \) in (57) and (58), it follows that vectors \( p \) and \( p' \) are scale-multiples of each other, that is \( p' = (\lambda'_j/\lambda_j) p \). This implies that \( a_i \) and \( a'_i \) are scale-multiples of each other. Since they both lie in the probability simplex \( \Delta \), they must be equal. Therefore the subdifferential \( A_i \) is a singleton and \( V_i \) is
differentiable at $x_i$. The same argument with reversed roles for $i$ and $j$ shows that the set $A_j$ must be singleton. This concludes the proof of the first part. The second part follows easily from the first since $\alpha$-MEU with $\alpha = 0$ is subdifferentiable everywhere and non-differentiable at every risk-free consumption plan. □

B Rules of Quasidifferential Calculus

The quasidifferential of a function is a pair of compact and convex sets. We define first some algebraic operations on pairs of sets. Let $A, B, C, D$ be convex and compact sets in $\mathbb{R}^S$. The operations of addition and multiplication by a scalar are defined as follows:

$$[A, B] + [C, D] = [A + C, B + D]$$

and

$$c[A, B] = \begin{cases} [cA, cB] & \text{if } c \geq 0 \\ [cB, cA] & \text{if } c < 0. \end{cases}$$

The rules of quasidifferentiation are extensions of the well-known rules of the classical differential calculus. A more detailed and systematic account can be found in Demyanov and Rubinov (1992), Chapters 10-12.

**Proposition B.1:** Suppose that functions $f_k : \mathbb{R}_+^S \to \mathbb{R}$ are quasidifferentiable at $x \in \mathbb{R}_+^S$ for every $k = 1, \ldots, m$. Let $Df_k(x) = [\partial f_k(x), \bar{\partial} f_k(x)]$ be the quasidifferential of $f_k$ and $a_k \in \mathbb{R}$ for $k = 1, \ldots, m$. The following rules hold:

1. (Sum) Let $f = \sum_{k=1}^m a_k f_k$. Then $f$ is quasidifferentiable at $x$, and

$$Df(x) = \sum_{k=1}^m a_k Df_k(x).$$

2. (Product) Let $f = f_1 \cdot f_2$. Then $f$ is quasidifferentiable at $x$ and

$$Df(x) = f_1(x)Df_2(x) + f_2(x)Df_1(x).$$

**Proof of Proposition B.1:** Part (1) follows from Theorem 10.2 (i) and (ii) in Demyanov and Rubinov (1986). For part (2), see Theorem 10.2 (iii). □
Theorem 12.2 of Demyanov and Rubinov (1986) provides an exact formula for quasidifferential of a composition of two quasidifferentiable functions. We reproduce it in Proposition B.2. Note that the chain rule for the Clarke subdifferential calculus yields only upper bounds on the Clarke subdifferential of the composition, see Section 2.1. in Clarke (1983).

**Proposition B.2:** (Composition) Suppose that functions \(f_k : \mathbb{R}^S_+ \to \mathbb{R}\) are quasidifferentiable at \(x \in \mathbb{R}^S_+\) for every \(k = 1, \ldots, m\). If function \(g : \mathbb{R}^m \to \mathbb{R}\) is uniformly quasidifferentiable\(^{21}\) at \(y = (f_1(x), \ldots, f_m(x))\), then the composition \(V : \mathbb{R}^S \to \mathbb{R}\) defined by

\[
V(x) = g(f_1(x), \ldots, f_m(x))
\]

is quasidifferentiable at \(x\), and

\[
\partial V(x) = \left\{ \sum_{k=1}^m (z_k + \bar{z}_k)w_k - z_k \gamma_k - \bar{z}_k \bar{\gamma}_k : w \in \partial g(y), z_k \in \partial f_k(x), \bar{z}_k \in \bar{\partial} f_k(x) \right\}
\]

\[
\bar{\partial} V(x) = \left\{ \sum_{k=1}^m (z_k + \bar{z}_k)\bar{w}_k + z_k \gamma_k + \bar{z}_k \bar{\gamma}_k : \bar{w} \in \bar{\partial} g(y), z_k \in \partial f_k(x), \bar{z}_k \in \bar{\partial} f_k(x) \right\}
\]

where \(\gamma, \bar{\gamma} \in \mathbb{R}^m\) are arbitrary vectors such that \(\gamma \leq \partial g(y) \cup (-\bar{\partial} g(y)) \leq \bar{\gamma}\).

The next result is taken from Demyanov and Rubinov (1992), Theorem 2.2.

**Proposition B.3:** Suppose that functions \(f_k : \mathbb{R}^S_+ \to \mathbb{R}\) are quasidifferentiable at \(x \in \mathbb{R}^S_+\) for every \(k = 1, \ldots, m\). Let

\[
\varphi(x) = \max_{k=1,\ldots,m} f_k(x), \quad \text{and} \quad \psi(x) = \min_{k=1,\ldots,m} f_k(x)
\]

and \(\varphi^*(x) = \arg \max_k f_k(x), \quad \text{and} \quad \psi^*(x) = \arg \min_k f_k(x)\).

Then

\[
(i) \, D \varphi(x) = \left[ \text{co} \left\{ \bigcup_{k \in \varphi^*} \left( \partial f_k(x) - \sum_{i \in \varphi^*} \bar{\partial} f_i(x) \right) \right\}, \sum_{k \in \varphi^*(x)} \bar{\partial} f_k(x) \right] \quad (59)
\]

\(^{21}\)I.e., \(g\) is (a.) uniformly directionally differentiable at \(y\) and (b.) quasidifferentiable at \(y\). (a.) holds for instance if \(g\) is Lipschitz continuous around \(y\), see Proposition 3.4, p.29, in Demyanov and Rubinov (1986).
\[(ii) \quad D\psi(x) = \left[ \sum_{k \in \psi^*(x)} \partial f_k(x), \ co \left\{ \bigcup_{k \in \psi^*(x)} \left( \partial f_k(x) - \sum_{i \in \psi^*(x) \setminus k} \partial f_i(x) \right) \right\} \right]. \tag{60} \]

**Corollary B.1:** If every function \( f_k \) is differentiable, then

\[
D\varphi(x) = [co \{ \nabla f_k(x) : k \in \varphi^*(x) \}, 0] \tag{61}
\]

and

\[
D\psi(x) = [0, co \{ \nabla f_k(x) : k \in \psi^*(x) \}]. \tag{62}
\]

**Proof:** If \( f_k \) is differentiable for every \( k \), then we can set \( \partial f_k(x) = \nabla f_k(x) \) and \( \partial f_k(x) = 0 \) in equation (59) of Proposition B.3, and this results in (61). If we set \( \partial f_k(x) = \nabla f_k(x) \) and \( \partial f_k(x) = 0 \) in equation (60), we obtain (62).

We proved in Section 2 that results similar to (61) and (62) hold for an arbitrary family of continuously differentiable functions, see (6).

### C Symmetric Sets of Priors

Recall that a set of priors \( \mathcal{P} \subset \Delta \) is symmetric around \( \pi \in \mathcal{P} \) if

\[
\mathcal{P} - \pi = \pi - \mathcal{P}. \tag{63}
\]

We have

**Lemma C.1:** If \( \mathcal{P} \subset \Delta \) is symmetric around \( \pi \in \mathcal{P} \), then pair of sets \( [\alpha \mathcal{P}, (1 - \alpha)\mathcal{P}] \) is DR-equivalent to

\[
\begin{align*}
(i) & \quad [0, (1 - 2\alpha)\mathcal{P} + 2\alpha\pi], \text{ if } \alpha < \frac{1}{2}, \\
(ii) & \quad [\pi, 0], \text{ if } \alpha = \frac{1}{2}, \\
(iii) & \quad [(2\alpha - 1)\mathcal{P} + 2(1 - \alpha)\pi, 0], \text{ if } \alpha > \frac{1}{2}.
\end{align*}
\]
We prove part (i). Using definition (3) of DR-equivalence, we need to show that
\[ \alpha \mathcal{P} - (1 - 2\alpha)\mathcal{P} - 2\alpha \pi = -(1 - \alpha)\mathcal{P}. \] (64)

The left-hand side of equation (64) can be rearranged as
\[ \alpha \mathcal{P} - (1 - 2\alpha)[\mathcal{P} - \pi] - \pi = \alpha \mathcal{P} - (1 - 2\alpha)[\mathcal{P} + \pi] - \pi = (1 - \alpha)\mathcal{P} - 2(1 - \alpha)\pi. \] (65)

where we used (63).

The right-hand side of equation (64) can be rearranged as
\[ (1 - \alpha)[-\mathcal{P} + \pi] - (1 - \alpha)\pi = (1 - \alpha)\mathcal{P} - 2(1 - \alpha)\pi. \] (66)

where, again, we used (63). The right-most expressions in (65) and (66) are equal. This proves that (64) holds true. Proofs of (ii) and (iii) are similar. □

Note that the sets \((1 - 2\alpha)\mathcal{P} + 2\alpha \pi\), for \(\alpha < \frac{1}{2}\), and \((2\alpha - 1)\mathcal{P} + 2(1 - \alpha)\pi\), for \(\alpha > \frac{1}{2}\), are strict subsets of \(\mathcal{P}\).

References


