Implementing Arrow-Debreu Equilibria by Trading Infinitely-Lived Securities. *

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Abstract

We show that Arrow-Debreu equilibria with countably additive prices in infinite-time economy under uncertainty can be implemented by trading infinitely-lived securities in complete sequential markets under two different portfolio feasibility constraints: wealth constraint, and essentially bounded portfolios. Sequential equilibria with no price bubbles implement Arrow-Debreu equilibria, while those with price bubbles implement Arrow-Debreu equilibria with transfers. Transfers are equal to price bubbles on initial portfolio holdings. Price bubbles arise in sequential equilibrium under the wealth constraint if some securities are in zero supply or negative prices are permitted, but cannot arise with essentially bounded portfolios.

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1. Introduction

Equilibrium models of dynamic competitive economies extending over infinite time play an important role in contemporary economic theory. The basic solution concept for such models is the Arrow-Debreu (or Walrasian) equilibrium. In Arrow-Debreu equilibrium it is assumed that agents simultaneously trade arbitrary consumption plans for the entire infinite and state-contingent future. In applied work, on the other hand, a different market structure and equilibrium concept are used: instead of trading arbitrary consumption plans at a single date, agents trade securities in sequential markets at every date in every event. The importance of Arrow-Debreu equilibrium rests on the possibility of implementing equilibrium allocations by trading suitable securities in sequential markets.

The idea of implementing an Arrow-Debreu equilibrium allocation by trading securities takes its origin in the classical paper by Arrow [3]. Arrow proved that every Arrow-Debreu equilibrium allocation in a two-date economy can be implemented by trading in complete security markets at the first date and spot commodity markets at every date in every event. The implementation is exact—the sets of equilibrium allocations in the two market structures are exactly the same. Arrow's result can be easily extended to a multidate economy with finite time-horizon. Duffie and Huang [7] proved that Arrow-Debreu equilibria can be implemented by trading securities in continuous-time finite-horizon economy.

In this paper we study implementation of Arrow-Debreu equilibrium allocations by sequential trading of infinitely-lived securities in an infinite-time economy. Our results extend those of Kandori [11], from the setting of a representative consumer, and our previous results (Huang and Werner [10]), from the setting of no uncertainty and a single security, to the general setting of multiple consumers, multiple securities, and uncertainty. Wright [20] studied implementation in infinite-time economies with one-period-lived securities.

The crucial aspect of implementation in infinite-time security markets is the choice of feasibility constraints on agents’ portfolio strategies. A feasibility constraint has to be imposed for otherwise agents would be able to borrow in security
markets and roll over the debt without ever repaying it (Ponzi scheme). However, the constraint cannot be too “tight” for it could prevent agents from using portfolio strategies that generate wealth transfers necessary to achieve consumption plans of an Arrow-Debreu equilibrium. Wright [20] employs the wealth constraint which says that a consumer cannot borrow more than the present value of her future endowments. He proved that exact implementation holds with one-period-lived securities—the set of Arrow-Debreu equilibrium allocations and the set of equilibrium allocations in complete sequential markets are the same.

The difficulty in extending implementation results to infinitely-lived securities lies in the possibility of price bubbles in sequential markets. Kocherlakota [12], Magill and Quinzii [15], and Huang and Werner [10] pointed out that the wealth constraint gives rise to sequential equilibria with price bubbles on securities that are in zero supply. We prove that Arrow-Debreu equilibria with countably additive prices can be implemented by trading infinitely-lived securities in complete sequential markets under the wealth constraint with no price bubbles. That is, the set of Arrow-Debreu equilibrium allocations is the same as the set of equilibrium allocations in sequential markets with no price bubbles. Further, we show that sequential equilibria with nonzero price bubbles correspond to Arrow-Debreu equilibria with transfers (and with countably additive prices). Transfers are equal to the value of price bubbles on agents’ initial portfolio holdings.

Our results imply that there are always sequential equilibria with price bubbles under the wealth constraint if some of the securities are in zero supply. This contradict the often asserted claim that price bubbles cannot arise when agents are infinitely lived (see, for example, Blanchard and Fischer [5] who attribute the claim to Tirole [19]). It is also in contrast to recent results of Montruccio and Privileggi [16] who show that price bubbles are a “marginal phenomenon” in representative agent economies. Of course, examples of equilibrium price bubbles have been known before, but our results demonstrate their general existence with zero supply. We also demonstrate that, if one does not exclude negative security prices, then there exist sequential equilibria with price bubbles under the wealth
constraint even if the supply of securities is strictly positive. Santos and Woodford [18] proved that price bubbles cannot exist when all securities are in strictly positive supply and prices are positive.

We consider an alternative portfolio feasibility constraint which requires that the value of borrowing at normalized security prices be bounded from below. We call portfolio strategies satisfying this constraint essentially bounded portfolio strategies. This feasibility constraint has a remarkable property that there cannot be price bubbles in sequential equilibrium regardless of the supply of the securities. We prove that exact implementation of Arrow-Debreu equilibria with countably additive prices (without transfers) holds with essentially bounded portfolio strategies.

It should be emphasized that the concept of Arrow-Debreu equilibrium underlying our implementation results is the one due to Peleg and Yaari [17]. Equilibrium prices assign finite values to consumption plans that are positive and do not exceed the aggregate endowment, but may or may not assign finite values to other consumption plans. The Peleg and Yaari approach should be contrasted with a more standard approach, first proposed by Debreu [6], where the consumption space and the price space are a topological dual pair, and equilibrium prices assign finite values to all consumption plans in the consumption space. Further, equilibrium prices in implementable Arrow-Debreu equilibria must be countably additive. It has been known since Bewley [4] that for some class of economies Arrow-Debreu equilibrium prices may not be countably additive (for an example, see Huang and Werner [10]). Our results indicate that those equilibria cannot be implemented by sequential trading.

The paper is organized as follows: In section 2 we provide specification of time and uncertainty. In section 3 we introduce the notion of Arrow-Debreu equilibrium and in section 4 we define a sequential equilibrium in security markets. In sections 5 and 6 we state and prove our basic implementation results. In section 7 we present an example that illustrates our results. Section 8 contains more discussion of our results and some remarks on alternative portfolio constraints.
2. Time and uncertainty

Time is discrete with infinite horizon and indexed by \( t = 0, 1, \ldots \). Uncertainty is described by a set \( \mathcal{S} \) of states of the world and an increasing sequence of finite partitions \( \{ \mathcal{F}_t \}_{t=0}^{\infty} \) of \( \mathcal{S} \). A state \( s \in \mathcal{S} \) specifies a complete history of the environment from date 0 to the infinite future. The partition \( \mathcal{F}_t \) specifies sets of states that can be verified by the information available at date \( t \). We take \( \mathcal{F}_0 \) to be the trivial partition. An element \( s^t \in \mathcal{F}_t \) is called a date-\( t \) event.

This description of the uncertain environment can be interpreted as an event tree. Each event \( s^t \) is a node of the event tree. The unique date 0 event \( s^0 \) is the root node. The unique predecessor of \( s^t \) for \( t \geq 1 \) is date-(\( t - 1 \)) event \( s^t_{t-1} \in \mathcal{F}_{t-1} \) such that \( s^t \subset s^t_{t-1} \). Immediate successors of \( s^t \) are all date-(\( t + 1 \)) events \( s^{t+1} \) such that \( s^{t+1} \in \mathcal{F}_{t+1} \) and \( s^{t+1} \subset s^t \). The set and the number of immediate successors of \( s^t \) are denoted by \( \mathcal{F}_{t+1}(s^t) \) and \( \kappa(s^t) \), respectively. We assume that \( \sup_{s^t \in \mathcal{E}} \kappa(s^t) < \infty \), and denote that supremum by \( K \).

The set of all date-\( \tau \) successor events of \( s^t \) for \( \tau > t \), that is all date-\( \tau \) events \( s^\tau \in \mathcal{F}_\tau \) with \( s^\tau \subset s^t \), is denoted by \( \mathcal{F}_\tau(s^t) \). The set of successor events of \( s^t \) at all dates after \( t \) is denoted by \( \mathcal{E}^+(s^t) \). We also write \( \mathcal{E}(s^t) \equiv \{ s^t \} \cup \mathcal{E}^+(s^t) \). The set of all events at all dates is denoted by \( \mathcal{E} \).

3. Arrow-Debreu equilibrium

There is a single consumption good. A consumption plan is a scalar-valued process adapted to \( \{ \mathcal{F}_t \}_{t=0}^{\infty} \). Consumption plans are restricted to lie in a linear space \( \mathcal{C} \) of adapted processes. Our primary choice of the consumption space \( \mathcal{C} \) is the space of all adapted processes (which can be identified with \( \mathcal{R}^{\infty} \)). The cone of nonnegative processes in \( \mathcal{C} \) is denoted by \( \mathcal{C}_+ \); a typical element of \( \mathcal{C} \) is denoted by \( c = \{ c(s^t) \} \) \( s^t \in \mathcal{E} \).

There are \( I \) consumers. Each consumer \( i \) has the consumption set \( \mathcal{C}_+ \), a strictly increasing and complete preference \( \preceq^i \) on \( \mathcal{C}_+ \), and an initial endowment \( \omega^i \in \mathcal{C}_+ \). The aggregate endowment \( \omega \equiv \sum_i \omega^i \) is assumed positive, that is, \( \omega \geq 0 \).

The standard notion of an Arrow-Debreu general equilibrium is extended to our
setting with infinitely many dates as follows: Prices are described by linear functional \( P \) which is positive and well-defined (i.e., finite valued) on each consumer’s initial endowment. We call such functional a *pricing functional*. It follows that a pricing functional is well-defined on the aggregate endowment \( \omega \) and, since it is positive, also on each attainable consumption plan, that is, on each \( c \) satisfying \( 0 \leq c \leq \omega \). It may or may not be well-defined on the entire space \( C \).

The price of one unit consumption in event \( s^t \) under pricing functional \( P \) is

\[
p(s^t) \equiv P(e(s^t)),
\]

where \( e(s^t) \) denotes the consumption plan equal to 1 in event \( s^t \) at date \( t \) and zero in all other events and all other dates. A pricing functional \( P \) is *countably additive* if and only if

\[
P(c) = \sum_{s^t} p(s^t) c(s^t)
\]

for every \( c \) for which \( P(c) \) is well-defined.

An **Arrow-Debreu equilibrium** is a pricing functional \( P \) and a consumption allocation \( \{c^i\}_{i=1}^I \) such that \( c^i \) maximizes consumer \( i \)'s preference \( \preceq^i \) subject to \( P(c) \leq P(\omega^i) \) and \( c \in C_+ \), and markets clear, that is \( \sum_i c^i = \sum_i \omega^i \). An equilibrium pricing functional is normalized so that \( p(s^0) = 1 \).

This concept of Arrow-Debreu equilibrium is due to Peleg and Yaari [17] who also provide sufficient conditions for the existence of an equilibrium with countably additive pricing functional when the consumption space is \( C = \mathbb{R}^\infty \). The conditions are the standard monotonicity and convexity of preferences, as well as continuity of preferences in the product topology. Aliprantis, Brown and Burkinshaw [1, 2] provide an analysis of Peleg-Yaari equilibria in more general consumption spaces.

We will also need the notion of an equilibrium with transfers. For given transfers \( \{e^i\}_{i=1}^I \), where \( e^i \in \mathcal{R} \) and \( \sum_i e^i = 0 \), a pricing functional \( P \) (with \( p(s^0) = 1 \)) and a consumption allocation \( \{c^i\}_{i=1}^I \) are an **Arrow-Debreu equilibrium with transfers** if \( c^i \) maximizes consumer \( i \)'s preference \( \preceq^i \) subject to \( P(c) \leq P(\omega^i) + e^i \) and \( c \in C_+ \), and markets clear. Peleg and Yaari [17] conditions also imply that an Arrow-Debreu equilibrium with transfers exists for small transfers.
4. Sequential equilibrium and price bubbles

We consider $J$ infinitely-lived securities traded at every date. We assume that
the number of securities is greater than or equal to the number of immediate
successors of every event, that is, $J \geq K$. Each security $j$ is specified by a dividend
process $d_j$ which is adapted to $\{F_t\}_{t=0}^\infty$ and nonnegative. The ex-dividend price of
security $j$ in event $s'$ is denoted by $q_j(s')$, and $q_j$ is the price process of security $j$. Portfolio strategy $\theta$ specifies a portfolio of $J$ securities $\theta(s')$ held after trade in
each event $s'$. The payoff of portfolio strategy $\theta$ in event $s'$ for $t \geq 1$ at a price
process $q$ is

$$z(q, \theta)(s') \equiv [q(s') + d(s')]\theta(s') - q(s')\theta(s').$$  \hspace{1cm} (1)

Each consumer $i$ has an initial portfolio $\alpha^i \in \mathcal{R}^J$ at date 0. The dividend
stream $\alpha^i d$ on initial portfolio constitutes one part of consumer $i$’s endowment.
The rest is $y^i \in C$ and becomes available to the consumer at each date in every
event. Thus, it holds

$$\omega^i(s') = y^i(s') + \alpha^i d(s') \quad \forall s'.$$  \hspace{1cm} (2)

The supply of securities is $\alpha = \sum \alpha^i$, and the adjusted aggregate endowment of
goods is $y = \sum y^i$. We assume that $\alpha \geq 0$.

Consumers must face feasibility constraints when choosing their portfolio strategies for otherwise they would use Ponzi schemes. We state a definition of sequential
equilibrium for general sets of feasible portfolio strategies $\Theta^j$ and introduce specific feasibility constraints in sections 5 and 6.

A sequential equilibrium is a price process $q$ and consumption-portfolio allocation $\{\tilde{c}^i, \tilde{\theta}^i\}_{i=1}^J$ such that:

(i) for each $i$, consumption plan $\tilde{c}^i$ and portfolio strategy $\tilde{\theta}^i$ maximize $z^i$ subject
\[ c(s^0) + q(s^0)\theta(s^0) \leq y^i(s^0) + q(s^0)\alpha^i, \]
\[ c(s^t) \leq y^i(s^t) + z(q, \theta)(s^t) \quad \forall s^t \neq s^0, \]  
(3)  
c \in C_+, \; \theta \in \Theta^t;

(ii) markets clear, that is
\[ \sum_i e^i(s^t) = y(s^t) + \alpha d(s^t), \quad \sum_i \theta^i(s^t) = \alpha \quad \forall s^t. \]

Security price process \( q \) is one-period arbitrage free in event \( s^t \) if there does not exist a portfolio \( \theta(s^t) \) such that \( [q(s^t+1) + d(s^t+1)]\theta(s^t) \geq 0 \) for every \( s^t+1 \in \mathcal{F}_{t+1}(s^t) \) and \( q(s^t)\theta(s^t) \leq 0 \), with at least one strict inequality.\(^1\) It is well known that if \( q \) is arbitrage free in every event, then there exist a sequence of strictly positive event prices \( \{\pi_q(s^t)\}_{s^t \in \mathcal{E}} \) with \( \pi_q(s^0) = 1 \) such that
\[ \pi_q(s^t)q_j(s^t) = \sum_{s^t+1 \in \mathcal{F}_{t+1}(s^t)} \pi_q(s^t+1) \left[q_j(s^t+1) + d_j(s^t+1)\right] \quad \forall s^t, j. \]  
(4)  

Security markets are one-period complete in event \( s^t \) at prices \( q \) if the one-period payoff matrix \([q(s^t+1) + d(s^t+1)]_{s^t+1 \in \mathcal{F}_{t+1}(s^t)} \) has rank equal to \( \kappa(s^t) \). Security markets are complete at \( q \) if they are one-period complete at every event.

Suppose that security prices \( q \) are one-period arbitrage free and that markets are complete at \( q \). Then the present value of security \( j \) at \( s^t \) is defined using the unique event prices as
\[ \frac{1}{\pi_q(s^t)} \sum_{s^t \in \mathcal{E}^+(s^t)} \pi_q(s^t) d_j(s^t). \]  
(5)  

If the present value (5) is finite, then the price bubble \( \sigma_{qj}(s^t) \) is defined by
\[ \sigma_{qj}(s^t) \equiv q_j(s^t) - \frac{1}{\pi_q(s^t)} \sum_{s^t \in \mathcal{E}^+(s^t)} \pi_q(s^t) d_j(s^t). \]  
(6)

\(^1\)Note that one-period arbitrage is defined without any reference to the portfolio feasibility constraint. Of the two constraints considered in this paper, the constraint of essentially bounded strategies (Section 6) does not restrict portfolio holdings in any single event while the wealth constraint (Section 5) does. Yet, it remains true that there cannot be a one-period arbitrage in sequential equilibrium under the wealth constraint (see Santos and Woodford [18]).
One can show using (4) that, if the price of security \( j \) is nonnegative in every event, then the present value (5) is finite and does not exceed the price of the security, i.e., \( 0 \leq \sigma_{qj}(s^t) \leq q_j(s^t) \) for every \( s^t \). Also, if the present value of security \( j \) is finite and \( \sigma_{qj}(s^t) \geq 0 \) for every \( s^t \), then \( q_j(s^t) \geq 0 \) for every \( s^t \). We do not exclude the possibility of negative security prices. Absence of one-period arbitrage does not imply that security prices are nonnegative even if, as assumed, dividends are nonnegative.\(^2\) A way to exclude negative security prices is to assume free disposal of securities (see Santos and Woodford [18]).\(^3\)

For use later, we note that (4) and (6) imply that

\[
\sigma_{qj}(s^t) = \frac{1}{\pi_q(s^t)} \sum_{s^{t+1} \in \mathcal{F}_{t+1}(s^t)} \pi_q(s^{t+1}) \sigma_{qj}(s^{t+1}),
\]

(7)
and also that

\[
\sigma_{qj}(s^t) = \lim_{T \to \infty} \frac{1}{\pi_q(s^T)} \sum_{s^{t} \in \mathcal{F}_{T}} \pi_q(s^{T}) q_j(s^{T}).
\]

(8)
for each \( s^t \).

5. **Implementation with the wealth constraint**

A frequently used portfolio feasibility constraint is the wealth constraint. It applies to complete security markets where event prices can be uniquely defined.\(^4\) It prohibits a consumer from borrowing more than the present value of his future endowment. Formally, portfolio strategy \( \theta \) satisfies the \textit{wealth constraint} if

\[
q(s^t)\theta(s^t) \geq -\frac{1}{\pi_q(s^t)} \sum_{s^t \in \mathcal{E}^+} \pi_q(s^t) y^j(s^t) \quad \forall s^t,
\]

(9)
where \( \pi_q \) is the event price system (assumed unique) associated with \( q \). We refer to a sequential equilibrium in which every consumer's set of feasible portfolio strategies is defined by (9) as a sequential equilibrium under the wealth constraint.

\(^2\)We show later in the paper that negative security prices are possible in sequential equilibrium under the wealth constraint but not with essentially bounded portfolio strategies.

\(^3\)The assumption of free disposal is prohibitively restrictive for many securities. For example, futures markets would not exist if futures contracts could be freely disposed.

\(^4\)Santos and Woodford [18] extend the wealth constraint to incomplete markets.
The main results of this section establish equivalence between countably additive Arrow-Debreu equilibria with transfers and sequential equilibria under the wealth constraint with price bubbles. A special case of this equivalence is when transfers and price bubbles are zero. To ease the exposition, we present this important special case in two separate theorems before stating the general results. All proofs have been relegated to the Appendix.

**Theorem 5.1** Let consumption allocation \( \{c^i\}_{i=1}^I \) and pricing functional \( P \) be an Arrow-Debreu equilibrium. If \( P \) is countably additive, \( P(d_j) < \infty \) for each \( j \), and security markets are complete at prices \( q \) given by

\[
q_j(s^i) = \frac{1}{p(s^i)} \sum_{s^* \in \mathcal{E}^+(s^i)} p(s^*) d_j(s^*) \quad \forall s^i, j,
\]

then there exists a portfolio allocation \( \{\theta^i\}_{i=1}^I \) such that \( q \) and the allocation \( \{c^i, \theta^i\}_{i=1}^I \) are a sequential equilibrium under the wealth constraint.

Theorem 5.1 says that an Arrow-Debreu equilibrium with countably additive pricing can be implemented by sequential trading under the wealth constraint provided that security markets are complete at prices defined by the present value of future dividends (and thus with no price bubbles).

The implementation of countably additive Arrow-Debreu equilibria by sequential trading with no price bubbles is exact.

**Theorem 5.2** Let security prices \( q \) and consumption-portfolio allocation \( \{c^i, \theta^i\}_{i=1}^I \) be a sequential equilibrium under the wealth constraint. If security markets are complete at \( q \) and price bubbles are zero, i.e., \( \sigma_q = 0 \), then consumption allocation \( \{c^i\}_{i=1}^I \) and the pricing functional \( P \) given by

\[
P(c) = \sum_{s^i \in \mathcal{E}} \pi_q(s^i) c(s^i)
\]

are an Arrow-Debreu equilibrium.

Our next result extends Theorem 5.1 to Arrow-Debreu equilibria with transfers that are proportional to initial portfolios.
Theorem 5.3 Let consumption allocation $\{c^i\}_{i=1}^I$ and pricing functional $P$ be an Arrow-Debreu equilibrium with transfers $\{\epsilon^i\}_{i=1}^I$ such that $\epsilon^i = \rho(s^0)\alpha^i$ for some $\rho(s^0) \in \mathcal{R}^J$ with $\rho(s^0)\alpha = 0$. If $P$ is countably additive, $P(d_j) < \infty$ for each $j$, and security markets are complete at prices $q$ given by

$$q_j(s^i) = \frac{1}{p(s^i)} \sum_{s' \in \mathcal{L}^+(s^i)} p(s^i) d_j(s') + \rho_j(s^i) \quad \forall s^i, j,$$

(12)

where $\rho(s^i) \in \mathcal{R}^J$ satisfies $\rho(s^i)\alpha = 0$ for every $s^i$ and

$$\rho_j(s^i) = \frac{1}{p(s^i)} \sum_{s^{i+1} \in \mathcal{L}^{i+1}(s^i)} p(s^{i+1}) \rho_j(s^{i+1}) \quad \forall s^i, j,$$

(13)

then there exists a portfolio allocation $\{\theta^i\}_{i=1}^I$ such that $q$ and allocation $\{c^i, \theta^i\}_{i=1}^I$ are a sequential equilibrium under the wealth constraint.

Security prices in Theorem 5.3 have price bubbles equal to $\rho$. Transfers are equal to price bubbles on initial portfolio holdings. A consumer whose initial portfolio is zero must have zero transfer. If all consumers have zero initial portfolios, then Theorem 5.3 concerns only Arrow-Debreu equilibrium without transfers. However, security prices in implementing sequential equilibria have price bubbles that are constrained only by (13), and may be nonzero (see Magill and Quinzii [15]).

If security prices in sequential markets are required to be positive, then $\rho$ in Theorem 5.3 has to be positive. With $\rho$ positive, condition $\rho(s^0)\alpha = 0$ implies that $\rho_j(s^0) = 0$ for every security $j$ with $\alpha_j > 0$. Then (13) implies that $\rho_j(s^i) = 0$ for such $j$, for every $s^i$. That is, the price bubble on a security with strictly positive supply must be zero. This has been demonstrated by Santos and Woodford [18]. If all securities are in strictly positive supply, then only Arrow-Debreu equilibria without transfers can be implemented in sequential markets with positive prices.

The implementation of countably additive Arrow-Debreu equilibria with transfers is exact.

Theorem 5.4 Let security prices $q$ and consumption-portfolio allocation $\{c^i, \theta^i\}_{i=1}^I$ be a sequential equilibrium under the wealth constraint. If security markets are
complete at \( q \) and \( \sum_{s' \in \mathcal{E}} q(s') d_j(s') < \infty \) for each \( j \), then consumption allocation \( \{c^j\}_{j=1}^J \) and the pricing functional \( P \) given by (11) are an Arrow-Debreu equilibrium with transfers \( \{\sigma_q(s^0)\alpha^j\}_{j=1}^J \). It holds \( \sigma_q(s^0)\alpha = 0 \).

6. Implementation with essentially bounded portfolios

We introduce in this section a portfolio feasibility constraint under which neither price bubbles nor negative security prices can arise in sequential equilibrium, and Arrow-Debreu equilibria without transfers can be implemented in sequential markets. The two important features of this portfolio feasibility constraint are: first, that constant (i.e., with no retrading) portfolio strategies are feasible regardless of security prices, and second, that (under additional conditions on agents’ endowments) the set of budget feasible consumption plans in complete markets at security prices with zero bubbles is the same as under the wealth constraint. The first feature is crucial for eliminating the possibility of price bubbles or negative security prices.

We say that portfolio strategy \( \theta \) is bounded from below if

\[
\inf_{s^j, j} \theta_j(s^j) > -\infty. \tag{14}
\]

Portfolio strategy \( \theta \) is essentially bounded from below at \( q \) if there exists a bounded from below portfolio strategy \( b \) such that

\[
q(s^j) \theta(s^j) \geq q(s^j) b(s^j) \quad \forall s^j. \tag{15}
\]

We refer to a (essentially) bounded from below portfolio strategies simply as (essentially) bounded portfolio strategy. Of course, every bounded portfolio strategy is essentially bounded but the converse is not true (unless there is a single security).\(^5\) The set of essentially bounded portfolio strategies is a convex cone.

If security price vector \( q(s^j) \) is positive and nonzero for every event \( s^j \), then portfolio strategy \( \theta \) is essentially bounded if and only if

\[
\inf_{s^j} q(s^j) \theta(s^j) > -\infty, \tag{16}
\]

\(^5\)If there is a single security, (15) and (14) are equivalent, and in that sense the results of this section extend Theorem 9.1 in Huang and Werner [10].
where \( q(s') \equiv q(s') / \sum_j q_j(s') \) is the normalized security price vector.\(^6\)

We refer to sequential equilibrium in which each consumer’s set of feasible portfolios is the set of essentially bounded portfolio strategies (15) as a sequential equilibrium with essentially bounded portfolios.

**Theorem 6.1** If \( q \) is a sequential equilibrium price process with essentially bounded portfolios and if security markets are complete at \( q \), then \( q(s') \geq 0 \) and \( \sigma_q(s') = 0 \) for every \( s' \).

That the price of each security has to be positive follows from the fact that a portfolio strategy of buying the security and holding it forever is bounded and therefore essentially bounded. If the price were negative, then a consumer could buy and hold the security (and do so at an arbitrary scale) and make an arbitrage profit. This is incompatible with an equilibrium. A detailed proof of this and the rest of Theorem 6.1 (as well as all other proofs for this section) can be found in the Appendix.

The following two theorems demonstrate that countably additive Arrow-Debreu equilibria (without transfers) can be implemented by sequential trading with essentially bounded portfolios in exact fashion.

**Theorem 6.2** Let consumption allocation \( \{e^i\}_{i=1}^I \) and pricing functional \( P \) be an Arrow-Debreu equilibrium. If \( P \) is countably additive, \( P(d_j) < \infty \) for each \( j \), security markets are complete at prices \( q \) given by

\[
q_j(s') = \frac{1}{p(s')} \sum_{s' \in \mathcal{E}^+(s')} p(s')d_j(s') \quad \forall s', j, \quad (17)
\]

and there exists an essentially bounded portfolio strategy \( \eta \) such that

\[
\frac{1}{p(s')} \sum_{s' \in \mathcal{E}^+(s')} p(s')y(s') \geq q(s')\eta(s') \quad \forall s', \quad (18)
\]

then there exists a portfolio allocation \( \{\theta^i\}_{i=1}^I \) such that \( q \) and the allocation \( \{e^i, \theta^i\}_{i=1}^I \) are a sequential equilibrium with essentially bounded portfolios.

\(^6\)If \( \theta \) satisfies (15) and \( q \geq 0 \), then \( q(s')|\theta(s')| \geq \sum_j q_j(s')|\theta_j| \) where \( \theta_j = \inf_{s'} \delta_j(s') \). Hence \( q(s')|\theta(s')| \) is bounded below. Conversely, if \( \theta \) satisfies (16), then \( q(s')|\theta(s')| \geq B \) for some \( B \in \mathcal{R} \). With \( b_j(s') = B \) for each \( j \) and \( s' \), \( \theta \) satisfies (15).
Theorem 6.3 Let security prices \( q \) and consumption-portfolio allocation \( \{c^i, \theta^i\}_{i=1}^I \) be a sequential equilibrium with essentially bounded portfolios. If security markets are complete at \( q \) and there exists an essentially bounded portfolio strategy \( \eta \) such that

\[
- \frac{1}{\pi_q(s^i)} \sum_{s^\rho \in \mathcal{E}_{s^i}} \pi_q(s^\rho) y(s^\rho) \geq q(s^i) \eta(s^i) \quad \forall s^i, \tag{19}
\]

then consumption allocation \( \{c^i\}_{i=1}^I \) and pricing functional \( P \) given by

\[
P(c) = \sum_{s^i \in \mathcal{E}} \pi_q(s^i) c(s^i) \tag{20}
\]

are an Arrow-Debreu equilibrium.

Conditions (18) and (19) say that it is feasible to borrow an amount greater than or equal to the present value of aggregate future endowment using an essentially bounded portfolio strategy. Equivalently, it is feasible for each consumer to borrow the present value of his endowment using an essentially bounded portfolio strategy. This implies that the set of all essentially bounded portfolio strategies includes all strategies satisfying the wealth constraint. One can show that for (19) or (18) to hold it is sufficient that there exists a bounded from above and from below portfolio strategy \( b \) such that \( y(s^i) \leq z(q, b)(s^i) \) for all \( s^i \). For this latter condition, it is sufficient that \( y \) is bounded relative to \( d \), that is, that \( y(s^i) \leq \gamma d(s^i) \) for some scale vector \( \gamma \in \mathcal{R}^d \), for all \( s^i \).

7. Example

We consider an infinite-time binomial event-tree. Each event \( s^i \) has two immediate successors \( (s^i, up) \) and \( (s^i, down) \) with conditional probabilities \( \mu(up|s^i) = \mu(down|s^i) = \frac{1}{2} \). There are two consumers with the same utility function

\[
u(c) = \sum_{t=0}^{\infty} \beta^t E[\ln(c_t)], \tag{21}
\]

where expectation is taken with respect to \( \mu \), and \( 0 < \beta < 1 \). Endowments are

\[
\omega^1(s^i, up) = \omega^2(s^i, down) = A, \quad \omega^1(s^i, down) = \omega^2(s^i, up) = B. \tag{22}
\]
for each $s'$, and $\omega^1(s^0) = \omega^2(s^0) = \frac{A + B}{2}$. The aggregate endowment is $A + B$ — risk-free, and constant over time.

The unique Arrow-Debreu equilibrium consists of allocation $\{e^1, e^2\}$ given by

$$c^1(s') = c^2(s') = \frac{A + B}{2} \quad \forall s',$$

supported by a countably additive pricing functional $P$ given by

$$p(s') = \beta^t \mu(s').$$

There are two infinitely-lived securities. Security 1 has risk-free dividends

$$d_1(s') = 1 \quad \forall s', \quad t \geq 1.$$  

Security 2 has state-dependent dividends

$$d_2(s', up) = u, \quad d_2(s', down) = d,$$

for each $s'$. We assume that $u > d$. Initial portfolios are $\alpha^1 = (1, 1)$ and $\alpha^2 = (-1, 0)$. Adjusted goods endowments are $y^1 = \omega^1 - d_1 - d_2$ and $y^2 = \omega^2 + d_1$.

Theorem 5.1 says that allocation (23) can be implemented in a sequential equilibrium under the wealth constraint with security prices (10). They are

$$q_1(s') = \frac{\beta}{1 - \beta}, \quad q_2(s') = \frac{u + d}{2} \frac{\beta}{1 - \beta},$$

for every $s'$. Security markets are complete at $q$, and there are no price bubbles.

For small enough transfers $e^1, e^2$ (with $e^1 + e^2 = 0$), there is an Arrow-Debreu equilibrium with transfers. It consists of consumption allocation

$$\tilde{c}^1(s') = \frac{A + B}{2} + e^1(1 - \beta) \quad \tilde{c}^2(s') = \frac{A + B}{2} + e^2(1 - \beta),$$

for every $s'$, supported by the pricing functional (24).

If transfers are proportional to initial portfolios, that is, if $e^1 = \rho(s^0) \alpha^1$ and $e^2 = \rho(s^0) \alpha^2$ for some $\rho(s^0) \in \mathbb{R}^2$ such that $\rho(s^0) \alpha = 0$, then Theorem 5.3 implies that allocation (28) can be implemented as a sequential equilibrium under the wealth
constraint. Clearly, arbitrary transfers are proportional to initial portfolios (via 
\( \rho(s^0) = (\epsilon^1, 0) \)) and hence every allocation (28) can be implemented. Equilibrium 
security prices can be obtained from (12). They are 
\[ \tilde{q}_1(s') = \frac{\beta}{1 - \beta} + \rho_1(s'), \quad \tilde{q}_2(s') = \frac{u + d}{2} \frac{\beta}{1 - \beta} + \rho_2(s'), \] 
(29)
for some \( \rho \) such that \( \rho(s^0) = (\epsilon^1, 0) \),
\[ \rho(s') = \beta \left( \frac{1}{\tilde{q}} \rho(s', up) + \frac{1}{2} \rho(s', down) \right), \] 
(30)
and the security markets are complete at \( \tilde{q} \). Market completeness holds if the 2-
by-2 payoff matrix \( [\tilde{q}(s^{t+1}) + d(s^{t+1})]_{s^{t+1} \in \mathcal{F}_{t+1}(s')} \) has full rank for every \( s' \). If \( \rho \) is
restricted to be positive, then necessarily \( \rho_2(s') = 0 \) for every \( s' \) so that there can
be positive price bubbles only on zero-supply security 1.

It follows from Section 6 that only the Arrow-Debreu equilibrium allocation
(23) can be implemented in sequential equilibrium with essentially bounded port-
folios. Arrow-Debreu equilibrium allocations with transfers (28) cannot be imple-
mented. In particular, security prices (29) are not equilibrium prices with essen-
tially bounded portfolios since they permit arbitrage (see Theorem 6.1).

8. Concluding remarks

Since Arrow-Debreu equilibria with or without transfers are Pareto optimal,
the results of Sections 5 and 6 imply that sequential equilibria in complete security
markets under the two portfolio constraints are Pareto optimal. Also, existence
results of Arrow-Debreu equilibria with countably additive pricing – see Section
3 – can be used to derive existence of sequential equilibria in complete markets.
Of course, the caveat here is that market completeness depends on endogenous
equilibrium prices. Our assumed condition that there are more securities than im-
mediate successors of every event \( (J \geq K) \) is necessary for market completeness.
It is also sufficient for a generic set of (appropriately parametrized) economies, see
Magill and Quinzii [15]. It is worth pointing out that, in the context of Theorem
5.3, the freedom of choosing price bubbles in implementation of Arrow-Debreu
equilibria with transfers can be used to assure that security markets are complete. This is so because price bubbles $\rho$ can be chosen arbitrarily subject to the “martingale property” (13) and the condition of zero bubble on the supply of securities (see the example of Section 7).

An often used feasibility constraint on portfolio strategies is the *transversality condition*. In our setting the transversality condition is written as

$$\lim \inf_{T \to \infty} \sum_{s^T \in \mathcal{F}_T(s')} \pi_q(s^T) q(s^T) \theta(s^T) \geq 0 \quad \forall s'.$$

(31)

Hernandez and Santos [9] proved that consumers' budget sets in sequential markets are the same under the wealth constraint and the transversality condition, as long as $\sum_{s' \in \mathcal{E}} \pi_q(s') y^i(s') < \infty$. Therefore, all implementation results of Section 5 remain valid when the wealth constraint is replaced by the transversality condition (and under an additional assumption that $\sum_{s' \in \mathcal{E}} \pi_q(s') y(s') < \infty$).

Hernandez and Santos [9] and Magill and Quinzii [15] provide other specifications of portfolio constraints that lead to the same budget sets as the wealth constraint. For further discussion of equivalent portfolio constraints in the setting with one-period-lived securities, see Florenzano and Gourdel [8], Magill and Quinzii [14], and Levine and Zame [13].
Appendix

We first establish the equivalence between budget sets in complete security markets under the wealth constraint and in Arrow-Debreu markets. Let $B_w(q; y^i, \alpha^i)$ denote the set of budget feasible consumption plans in sequential markets at prices $q$ under the wealth constraint. We have $c \in B_w(q; y^i, \alpha^i)$, if $c \in C_+$ and there exists a portfolio strategy $\theta$ such that (3) holds, with $\Theta^i$ defined by the wealth constraint (9). Let $B_{AD}(P; \omega^i, \epsilon^i)$ denote the set of budget feasible consumption plans in Arrow-Debreu markets at pricing functional $P$ with transfer $\epsilon^i$. We have $c \in B_{AD}(P; \omega^i, \epsilon^i)$, if $c \in C_+$ and $P(c) \leq P(\omega^i) + \epsilon^i$. We write $B_{AD}(P; \omega^i)$ for $B_{AD}(P; \omega^i, 0)$. Throughout the Appendix, $\omega^i$, $y^i$ and $\alpha^i$ are related by (2).

Lemma A.1 Let $P$ and $q$ be such that

$$\pi_q(s^i) = p(s^i) \quad \forall s^i. \quad (32)$$

If $P$ is countably additive, $P(y^i) < \infty$ and $P(d_j) < \infty$ for each $i$ and $j$, and security markets are complete at $q$, then

$$B_w(q; y^i, \alpha^i) = B_{AD}(P; \omega^i, \alpha^i \sigma_q(s^0)). \quad (33)$$

Proof: Suppose that $c \in B_w(q; y^i, \alpha^i)$. Multiplying both sides of the budget constraint (3) at $s^i$ by $\pi_q(s^i)$ and summing over all $s^i$ for $t$ ranging from 0 to arbitrary $r$, and using (4), we obtain

$$\sum_{t=0}^{r} \sum_{s^t \in F_t} \pi_q(s^t)c(s^t) + \sum_{s^r \in F_r} \pi_q(s^r)q(s^r)\theta(s^r) \leq \sum_{t=0}^{r} \sum_{s^t \in F_t} \pi_q(s^t)y^i(s^t) + q(s^0)\alpha^i. \quad (34)$$

Adding $\sum_{t=r+1}^{\infty} \sum_{s^t \in F_t} \pi_q(s^t)y^i(s^t)$ to both sides of (34), there results

$$\sum_{t=0}^{r} \sum_{s^t \in F_t} \pi_q(s^t)c(s^t) + \sum_{s^r \in F_r} \pi_q(s^r)q(s^r)\theta(s^r) + \sum_{s^t \in F^+(s^r)} \pi_q(s^t)y^i(s^t) \leq \sum_{s^t \in F} \pi_q(s^t)y^i(s^t) + q(s^0)\alpha^i. \quad (35)$$
If the use is made of the wealth constraint, (35) implies that

$$\sum_{t=0}^{\tau} \sum_{s^t \in \mathcal{F}_t} \pi_q(s^t)c(s^t) \leq \sum_{s^t \in \mathcal{E}} \pi_q(s^t)y^i(s^t) + q(s^0)\alpha^i. \quad (36)$$

Taking limits in (36) as $\tau$ goes to infinity yields

$$\sum_{s^t \in \mathcal{E}} \pi_q(s^t)c(s^t) \leq \sum_{s^t \in \mathcal{E}} \pi_q(s^t)y^i(s^t) + q(s^0)\alpha^i. \quad (37)$$

Since $\sum_{s^t \in \mathcal{E}} \pi_q(s^t)d_j(s^t) < \infty$ for every $j$, the price bubble $\sigma_q(s^0)$ is well-defined. If the use is made of (6) and (2), inequality (37) can be written as

$$\sum_{s^t \in \mathcal{E}} \pi_q(s^t)c(s^t) \leq \sum_{s^t \in \mathcal{E}} \pi_q(s^t)\omega^i(s^t) + \alpha^i\sigma_q(s^0), \quad (38)$$

or simply as $P(c) \leq P(\omega^i) + \alpha^i\sigma_q(s^0)$. Thus $c \in B_{AD}(P; \omega^i, \alpha^i\sigma_q(s^0))$.

Suppose now that $c \in B_{AD}(P; \omega^i, \alpha^i\sigma_q(s^0))$. Since security markets are complete at $q$, for each $s^t$ there exists portfolio $\theta(s^t)$ such that

$$[q(s^{t+1}) + d(s^{t+1})] \theta(s^t) = \sum_{s^\tau \in \mathcal{E}(s^{t+1})} \frac{\pi_q(s^\tau)}{\pi_q(s^{t+1})}[c(s^\tau) - y^i(s^\tau)] \quad \forall s^{t+1} \in \mathcal{F}_{t+1}(s^t). \quad (39)$$

The sum on the right-hand side of (39) is finite since $\sum_{s^\tau \in \mathcal{E}(s^{t+1})} \pi_q(s^\tau)c(s^\tau) \leq P(c)$. Multiplying both sides of (39) by $\pi_q(s^{t+1})$, summing over all $s^{t+1} \in \mathcal{F}_{t+1}(s^t)$, and using (4), we obtain

$$q(s^t)\theta(s^t) = \sum_{s^\tau \in \mathcal{E}(s^t)} \frac{\pi_q(s^\tau)}{\pi_q(s^t)}[c(s^\tau) - y^i(s^\tau)] \quad \forall s^t. \quad (40)$$

It follows from (39) and (40) that

$$c(s^t) + q(s^t)\theta(s^t) = y^i(s^t) + [q(s^t) + d(s^t)]\theta(s^t) \quad \forall s^t \neq s^0. \quad (41)$$

Thus $c$ and $\theta$ satisfy the sequential budget constraint (3) at each $s^t \neq s^0$. Equation (40) for $s^0$ and (37) imply the date-0 budget constraint. Since $c \geq 0$, equation (40) implies that $\theta$ satisfies the wealth constraint. Thus $c \in B_w(q; y^i, \alpha^i)$.
Proof of Theorem 5.1: For $q$ defined by (10), event prices $\pi_q$ satisfy (32) and $\sigma_q = 0$. Since $P(\omega^i) < \infty$ and $P(d_j) < \infty$, it follows that $P(y^i) < \infty$. Lemma A.1 implies that $B_w(q; y^i, \alpha^i) = B_{AD}(P; \omega^i, \alpha^i)$, and so consumption plan $c^i$ is optimal in sequential markets. It remains to be shown that portfolio strategies that generate $\{c^i\}$ clear security markets.

Let $\theta^i$ be portfolio strategy defined by (39) with $c^i$, that is, satisfying

$$[q(s^{i+1}) + d(s^{i+1})] \theta^i(s^i) = \sum_{s^r \in \mathcal{E}(s^{i+1})} \frac{p(s^r)}{p(s^{i+1})} c^i(s^r) - y^i(s^r) \quad \forall s^{i+1} \in \mathcal{F}_{i+1}(s^i),$$

(42)

for each $s^i$ and each $i$. Such $\theta^i$ generates $c^i$ and satisfies $i$'s wealth constraint. Summing (42) over all $i$ and using $\sum_i c^i = \sum_i \omega^i$ and (2), we obtain

$$[q(s^i) + d(s^i)] \sum_i \theta^i(s^i) = \sum_{s^r \in \mathcal{E}(s^i)} \frac{p(s^r)}{p(s^i)} d(s^r) \alpha \quad \forall s^i \neq s^0.$$ 

(43)

It follows from (10) and (43) that

$$[q(s^i) + d(s^i)] \left[ \sum_i \theta^i(s^i) - \alpha \right] = 0 \quad \forall s^i \neq s^0. \quad (44)$$

If there are no securities with redundant one-period payoffs, then (44) implies that $\sum_i \theta^i(s^i) = \alpha$ for all $s^i$. Otherwise, if there are redundant securities, then portfolio strategies $\{\theta^i\}$ can be modified without changing their payoffs so that the market clearing holds. $\Box$

Proof of Theorem 5.2: In a sequential equilibrium under the wealth constraint, the present value $\frac{1}{\pi_q(s^i)} \sum_{s^r \in \mathcal{E}(s^i)} \pi_q(s^r) y^i(s^r)$ must be finite, for otherwise there would not exist an optimal portfolio strategy for consumer $i$. Therefore $P(y^i) < \infty$. Further, since price bubbles are zero, it follows that $P(d_j) < \infty$ (see Section 4). Lemma A.1 can be applied and it implies the conclusion. $\Box$

Proof of Theorem 5.3: For security prices $q$ defined by (12) event prices $\pi_q$ satisfy (32), and $\sigma_q = \rho$. As in the proof of Theorem 5.1, one can show that $P(y^i) < \infty$. It follows from Lemma A.1 that each consumption plan $c^i$ is optimal in sequential markets.
The proof that portfolio strategies that generate \( \{e^i\} \) also clear security markets is the same as in Theorem 5.1 with one minor modification. With \( q \) defined by (12), the right-hand side of (44) is \(-\rho(s^i)\alpha\), which equals zero by assumption. □

**Proof of Theorem 5.4:** The same argument as in the proof of Theorem 5.2 implies that \( P(y^i) < \infty \). It follows from Lemma A.1 that \( P \) and \( \{e^i\}_{i=1}^l \) are an Arrow-Debreu equilibrium with transfers \( \{\sigma_q(s^0)e^i\}_{i=1}^l \). Walras’ Law \( \sum_i P(e^i) = \sum_i P(\omega^i) + \sigma_q(s^0)\alpha \) and market-clearing \( \sum_i e^i = \sum_i \omega^i \) imply that \( \sigma_q(s^0)\alpha = 0 \). □

**Proof of Theorem 6.1:** We first show that \( q \geq 0 \). Suppose, by contradiction, that \( q_j(s^t) < 0 \) for some \( j \) and \( s^t \). Let \( c^i \) be equilibrium consumption plan and \( \theta^i \) equilibrium portfolio strategy of consumer \( i \). Consider a portfolio strategy \( \hat{\theta}^i \) that results from holding \( \theta^i \) and purchasing one share of security \( j \) in event \( s^t \) and holding it forever. Since \( \theta^i \) is essentially bounded, so is \( \hat{\theta}^i \). Further, since \( q_j(s^t) < 0 \) and \( d_j \geq 0 \), \( \hat{\theta}^i \) generates consumption plan that is greater than or equal to \( c^i \) in every event and strictly greater in event \( s^t \). This contradicts optimality of \( c^i \).

Since \( q \geq 0 \), it follows from the discussion in Section 4 that the present value (5) of each security is finite and the price bubble is well-defined and nonnegative in every event. To prove that \( \sigma_q(s^t) = 0 \) for every \( t \) it suffices to show that \( \sigma_q(s^0) = 0 \). The rest follows from (7).

Since security markets are complete at \( q \), for each \( s^t \) and each \( j \) there exists a portfolio \( \xi_j(s^t) \) such that

\[
[q(s^{t+1}) + d(s^{t+1})] \xi_j(s^t) = \sum_{s^\tau \in \mathcal{E}_{t+1}} \frac{\pi_q(s^\tau)}{\pi_q(s^{t+1})} d_j(s^\tau), \quad \forall s^{t+1} \in \mathcal{F}_{t+1}(s^t). \tag{45}
\]

Multiplying both sides of (45) by \( \pi_q(s^{t+1}) \), summing over all \( s^{t+1} \in \mathcal{F}_{t+1}(s^t) \) and using (4), we obtain

\[
q(s^t) \xi_j(s^t) = \sum_{s^\tau \in \mathcal{E}_t(s^t)} \frac{\pi_q(s^\tau)}{\pi_q(s^t)} d_j(s^\tau). \tag{46}
\]

Since \( d_j \geq 0 \), it follows that \( q(s^t) \xi_j(s^t) \geq 0 \). Therefore \( \xi_j \) is essentially bounded. Using (45) and (46), we obtain

\[
[q(s^t) + d(s^t)] \xi_j(s^t) - q(s^t) \xi_j(s^t) = d_j(s^t), \quad \forall s^t \neq s^0, \tag{47}
\]

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Thus, the payoff of \( \xi^j \) equals dividend \( d_j \), that is, \( z(q, \xi^j) = d_j \). Date-0 price of \( \xi^j \) equals the present value of \( d_j \) (see (46)).

Let \( \eta^j \) denote a portfolio strategy of selling one share of security \( j \) at date 0 and never buying it back. For \( \xi^j + \eta^j \), we have \( z(q, \xi^j + \eta^j)(s^t) = 0 \) for every \( s^t \neq s^0 \), and \( q(s^0)[\xi^j(s^0) + \eta^j(s^0)] = -\sigma_{qj}(s^0) \). Consider \( \hat{\theta}^i = \theta^i + \xi^j + \eta^j \). Since \( \theta^i, \xi^j \), and \( \eta^j \) are essentially bounded, so is \( \hat{\theta}^i \). If \( \sigma_{qj}(s^0) > 0 \), then \( \hat{\theta}^i \) generates a consumption plan that is strictly greater than \( c^i \) at date 0 and equal to \( c^i \) in all future events. This would contradict optimality of \( c^i \). Therefore \( \sigma_{qj}(s^0) = 0 \). \( \square \)

We now establish the equivalence between budget sets in complete security markets with essentially bounded portfolio strategies and in Arrow-Debreu markets. Let \( B_q(q; y^i, \omega^i) \) denote the set of budget feasible consumption plans in sequential markets at prices \( q \) with essentially bounded portfolio strategies. We have \( c \in B_q(q; y^i, \omega^i) \), if \( c \in C_+ \) and there exists a portfolio strategy \( \theta \) such that (3) holds, with \( \Theta^i \) defined by (15).

**Lemma A.2** Let \( P \) and \( q \) be such that

\[
\pi_q(s^t) = p(s^t) \quad \forall s^t. \tag{48}
\]

If \( P \) is countably additive, \( q \) is positive and with zero price bubbles, security markets are complete at \( q \), and if there exists an essentially bounded portfolio strategy \( \eta \) such that

\[
-\frac{1}{\pi_q(s^t)} \sum_{s^r \in \mathcal{E}^+(s^t)} \pi_q(s^r) y^i(s^r) s^t \geq q(s^t) \eta(s^t) \quad \forall s^t, \tag{49}
\]

then

\[
B_q(q; y^i, \omega^i) = B_{AD}(P; \omega^i). \tag{50}
\]

**Proof:** Let \( c \in B_q(q; y^i, \omega^i) \). As in the proof of Lemma A.1, budget constraint (3) implies (34). Taking limits in (34) as \( \tau \) goes to infinity, we obtain

\[
\sum_{s^t \in \mathcal{E}} \pi_q(s^t)c(s^t) + \lim_{\tau \to \infty} \sum_{s^r \in \mathcal{F}_\tau} \pi_q(s^r)q(s^r) \theta(s^r) \leq \sum_{s^t \in \mathcal{E}} \pi_q(s^t)y^i(s^t) + q(s^0) \omega^i. \tag{51}
\]

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Note that (49) for \( s^0 \) implies that
\[
\sum_{s^l \in \mathcal{E}} \pi_q(s^l)y^i(s^l) \leq y^i(s^0) - q(s^0)\eta(s^0),
\] (52)
which shows that \( \sum_{s^l \in \mathcal{E}} \pi_q(s^l)y^i(s^l) \) is finite. We claim that
\[
\liminf_{\tau \to \infty} \sum_{s^r \in \mathcal{F}_{\tau}} \pi_q(s^r)q(s^r)\theta(s^r) \geq 0.
\] (53)
Since \( \theta \) is essentially bounded, it follows that
\[
\sum_{s^r \in \mathcal{F}_{\tau}} \pi_q(s^r)q(s^r)\theta(s^r) \geq \sum_{s^r \in \mathcal{F}_{\tau}} \pi_q(s^r)q(s^r)b(s^r) \quad \forall \tau,
\] (54)
for some bounded portfolio strategy \( b \). Since \( \sigma_{qd}(s^0) = 0 \), (8) implies that the limit, as \( \tau \) goes to infinity, of the right-hand side of (54) is positive.

Inequalities (53) and (51) imply (37), that is
\[
\sum_{s^l \in \mathcal{E}} \pi_q(s^l)c(s^l) \leq \sum_{s^l \in \mathcal{E}} \pi_q(s^l)y^i(s^l) + q(s^0)\alpha^i,
\] (55)
Using the same arguments as in the proof of Lemma A.1 we can rewrite (55) as \( P(c) \leq P(\omega^i) \). Thus \( c \in B_{AD}(P; \omega^i) \).

Suppose now that \( c \in B_{AD}(P; \omega^i) \). In the proof of Lemma A.1 we constructed a portfolio strategy that generates \( c \) at security prices \( q \) and satisfies the wealth constraint. That is,
\[
q(s^i)\theta(s^i) \geq \frac{1}{\pi_q(s^i)} \sum_{s^r \in \mathcal{E}^{r \neq i}} \pi_q(s^r)y^i(s^r) \quad \forall s^i.
\] (56)
Using (49) we obtain
\[
q(s^i)\theta(s^i) \geq q(s^i)\eta(s^i) \quad \forall s^i,
\] (57)
which implies that \( \theta \) is essentially bounded. Consequently, \( c \in B_q(q; y^i, \alpha^i) \). □

**Proof of Theorem 6.2:** Lemma A.2 implies that consumption plan \( c^i \) is optimal for each \( i \) in sequential markets under the constraint of essentially bounded portfolio strategies. The proof that portfolio strategies that generate the optimal consumption plans clear security markets is the same as in Theorem 5.1. □

**Proof of Theorem 6.3:** It follows immediately from Lemma A.2. □
References