Neoclassical Models of Endogenous Growth: The Effects of Fiscal Policy, Innovation and Fluctuations*

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1 Introduction

Despite its role as the centerpiece of modern growth theory, the Solow model is decidedly silent on some of its basic questions: Why is average growth in per capita income so much higher now than it was 200 years ago? Why is per capita income so much higher in the member countries of the OECD than in the less developed countries (LDC) of the world? The standard implementation of the Solow model really has no answers for these questions except, perhaps for differences, across time and across countries in the production possibility set. This is typically summarized by differences in Total Factor Productivity (TFP). The fundamental reasons for why TFP might be different in different countries, or in different time periods is left open for speculation. If these differences are supposed to be due to differences in innovations, it is not made clear why access to these innovations should be different, nor is it noted that these innovations themselves are economic decisions– they have costs and benefits, and are made by optimizing, private agents.

This basic weakness in the Solow model (and its followers) was the driving force behind the development of the class of endogenous growth models. This literature has been wide and varied, with the models developed ranging from perfectly competitive, convex models to ones featuring a range of types of market failures (e.g., increasing returns, external effects, imperfectly competitive behavior by firms, etc.). But, a common feature that has been emphasized throughout is knowledge, or human capital, and its production and dissemination. In come cases, this has been

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directly treated in the modeling, in others it has been more tangential, an important consideration for quantitative development, but less so for qualitative work. That this focus is essential follows from the fact that the Solow model already accurately reflects the quantitative limits of using models with only physical capital. (That is, capital's share is determined by the data to put us in the Solow range, technologically.) Although they differ in their details, in the end, what this class of models points to as differences in development are differences in social institutions across time and countries. Thus, countries that have weaker systems of property rights, or higher wasteful taxation and spending policies, will tend to grow more slowly. Moreover, these differences in performance can be permanent if these institutions are unchanging. As a corollary, those countries who developed these growth enhancing institutions more recently (and some still have not), have levels of income that are lower than those in which they were adopted earlier, even if current growth rates show only small differences.

In this paper we limit ourselves to studying neoclassical models. By this we mean models with convex production sets, well behaved preferences and a market structure that is consistent with competitive behavior. Therefore, we do not review the large literature that adresses the role of externalities and non-competitive markets. As it turns out, most of the basic ideas behind this literature can be expressed in simple, convex models of aggregate variables without uncertainty. These are the models that are the first focus of this chapter. They have proven both highly flexible and easy to use. With them, we can give substance to statements like those above that property rights and other governmental institutions are key to long run growth rates in a society. Most of this branch of the literature is well known by now, and much of it appears on standard graduate macro reading lists. Accordingly, our discussion will be fairly brief.[1] One important, and as of yet unresolved issue, is the size of the growth effects of cross-country differences in fiscal policy. Thus, our review of the standard convex model is complemented with a discussion of the more recent findings about the quantitative effects of taxes (and government spending) on growth. Even though the theoretical effects of social institutions are well understood, this is less true of the recent work on perfectly competitive models of innovation, and so, comparatively more space is used to discuss that ongoing development. As a second focus, one issue that comes immediately to light in studying this class of models is the possibility that uncertainty per se might have an impact on long run performance. This points to the possibility that instability in property rights and institutions might change the incentives for investment. That is, how are the time paths of savings, consumption and investment affected by uncertainty in this class of models? How does this compare with how uncertainty affects decisions in the Solow model (i.e., Brock and Mirman

[1] Other authors have also presented comprehensive surveys of this literature (see Barro and Sala-i-Martin (1995), Jones and Manuelli (1997), and Aghion and Howitt (1998) for examples). Our aim is to complement those presentations, rather than repeat them, and hence, our focus is somewhat distinct.

(1972) vs. Cass (1965) and Koopmans (1965))?

Much less is known about the answers to these questions at the present time and that knowledge that does exist is much less widespread. For this reason, we present a fairly detailed discussion of the properties of stochastic, convex models of endogenous growth. To this end, we study models in which technologies and policies are subject to random shocks. We characterize the effects of differential amounts of uncertainty on average growth. We show that increased uncertainty can increase or decrease average growth depending on both the parameters of the model and the source of the uncertainty. A separate, but related topic, is the business cycle frequency properties of these models. This is left to future work.

In section 2, we lay out the basics of the class of neoclassical (i.e., convex) models of endogenous growth. We show how differences in social institutions across time and across countries can give rise to different performance, even over the very long run. We also lay out some of the interpretations of the model, including human capital investment and innovation and knowledge diffusion sectors, that lend richness to its interpretation. Section 3 discusses properties of the models when uncertainty is added, and shows how this can affect the long run growth rate of an economy.

2 Endogenous Growth: Infinite Lifetimes

Historically, the engine of growth as depicted in Solow's seminal work on the topic (1956) was the assumption of exogenous technical change. Thus, initially, growth models aimed at being consistent with growth facts, but gave up on the possibility of explaining them. Moreover, this approach has weaknesses in two distinct areas. First, it is difficult using the exogenous growth model to explain the observed long run differences in performance exhibited by different countries. Second, the productivity changes that are assumed exogenous in the Solow model are, in fact, the result of conscious decisions on the part of economic agents. If this is the case, it is then important to explore both the mechanism through which productivity changes as well as the factors that can give rise to the observed long run differences if we are to understand these phenomena. In this section we briefly review the basic optimal growth model as initially analyzed by Cass (1965) and Koopmans (1965). We then discuss the nature of the technologies consistent with endogenous growth and the role of fiscal policy in influencing the growth rate. We conclude with an analysis of the role of innovation in the context of convex models of equilibrium growth.

2.1 Growth and the Solow Model

In the simplest time invariant version of the Solow model, it can be shown that the per capita stock of capital converges to a unique value independent of initial conditions. It is then necessary to assume some exogenous source of productivity growth in order to account for long run growth. In Solow (1956), it is assumed that labor productivity grows continually and exogenously. In response, the capital stock (assumed homogeneous over time) is continually increased allowing for a continual expansion in the level of output and consumption. The literature on endogenous growth has concentrated on replacing this assumed exogenous productivity growth by an endogenous process. If this change in productivity of labor is thought to arise from the invention of techniques consciously developed, the literature on endogenous growth can then be thought of as explicitly modeling the decisions to create this technological improvement (see Shell (1967) and (1973)). For this to go beyond a reinterpretation of the Solow treatment, it must be that the technology for discovering and developing these new technologies does not have itself a source of exogenous technological change. Because of this, these models all feature technologies that are time stationary.

The consumer problem in the simple growth model is given by

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t(c_t + x_t) \leq W_0 + \sum_{t=0}^{\infty} p_t r_t k_t, \tag{1a}$$

$$k_{t+1} = (1 - \delta_k)k_t + x_t, \tag{1b}$$

where c_t is the level of consumption, x_t is investment, k_t is the capital stock, p_t is the price of consumption (relative to time 0), and r_t is the rental price of capital, all in period t, and W_0 is the present value of wealth net of capital income. The first order condition for (an interior) solution to this problem is just

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta_k + r_{t+1}]. \tag{2}$$

If, as is standard in the literature, the instantaneous utility function, $u(c_t)$, is assumed strictly concave, growth —defined as a situation in which $c_{t+1} > c_t$— requires

$$\beta[1 - \delta_k + r_{t+1}] > 1. \tag{3}$$

Condition (3) is fairly general, and must hold *independently of the details of the production side* of the economy. Thus, if the economy is going to display long run growth, the rate of return on savings must be sufficiently high.

What determines the economy's rate of return? In the standard Solow growth model —and in many convex models— firms can be viewed as solving a static problem. More precisely, each firm maximizes profits given by

$$\Pi_t = \max_{k,n} c + x - r_t k - w_t n,$$

subject to

$$c + x \leq F(k, n),$$

where F is a concave production function that displays constant returns to scale.

Since in equilibrium the household offers inelastically one unit of labor, the rental rate of capital must satisfy

$$r_t = f(k_t), \tag{4}$$

where $f(k) = F(k, 1)$, and k is capital per worker.

It is now straightforward to analyze growth in the Solow model. The equilibrium version of (2) is just

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta_k + f(k_{t+1})]. \tag{5}$$

If the *productivity of capital* is sufficiently low as the stock of capital per worker increases, then there is no long run growth. To see this, note that if $\lim_{k\to\infty} f'(k) = \underline{r}$, with $1 - \delta_k + \underline{r} < 1$, there exists a finite k^* such that $1 - \delta_k + f(k^*) = 1$. It is standard to show that the unique competitive equilibrium for this economy (as well as the symmetric optimal allocation) is such that the sequence of capital stocks $\{k_t\}$ converges to k^*. Given this, consumption is also bounded. (Actually, it converges to $f(k^*) - \delta_k k^*$.)

Can exogenous technological change 'solve' the problem. The answer depends on the nature of the questions that the model is designed to answer. If one is content to generate equilibrium growth, then the answer is a clear yes. If, on the other hand, the objective is to understand how policies and institutions affect growth, then the answer is negative.

To see this assume that technological progress is labor augmenting. Specifically, assume that, at time t, the amount of effective labor is $z_t = z(1+\gamma)^t$. In order to guarantee existence of a balanced growth path we assume that the utility function is isoelastic (see Jones and Manuelli (1990) for details), and given by $u(c) = c^{1-\theta}/(1-\theta)$. Let a ˆ over a variable denote its value relative to effective labor. Thus, $\hat{c}_t \equiv c_t/(z(1+\gamma)^t)$. In this case, the balanced growth version of (2) is

$$(1+\gamma)u'(\hat{c}_t) = \hat{\beta} u'(\hat{c}_{t+1})[1 - \delta_k + f'(\hat{k}_{t+1})]$$

where $\hat{\beta} = \beta(1+\gamma)^{1-\theta}$.[2]

Standard arguments show that the equilibrium of this economy converges to a steady state $(\hat{c}, \hat{k})$. Thus, this implies that, asymptotically, consumption is given by $c_t = (1+\gamma)^t z\hat{c}$. Thus, even though there is equilibrium growth, the growth rate is completely determined by the exogenous increase in labor augmenting productivity.

[2]Existence of a solution requires that $\beta(1+\gamma)^{1-\theta} < 1$, which we assume.

2.2 A One Sector Model of Equilibrium Growth

As we argued before, the critical assumption that results in the economy not growing is that the marginal product of capital is low. The modern growth literature has emphasized the analysis of economies in which the marginal product of capital remains (sufficiently) bounded away from zero. This induces positive long-run growth in equilibrium. As we will show, how fast output grows in these models depends on a variety of factors (e.g., parameters of preferences). Because of this, these models have the property that the rate of growth is determined by the agents in the model.

Throughout, there will be one common theme. This mirrors the point emphasized above, that for growth to occur, the interest rate (either implicit in a planning problem or explicit in an equilibrium condition) must be kept from being driven too low. This follows immediately from the discussion above.

In terms of key features of the environment that are necessary to obtain endogenous growth there is one that stands out: it is necessary that the marginal product of *some* augmentable input be bounded strictly away from zero in the production of some augmentable input which can be used to produce consumption.

Since we are dealing with convex economies, the arguments in Debreu (1954) apply to the environments that we study. Thus, in the absence of distortionary government policies, equilibrium and optimal allocations coincide. Thus, for ease of exposition, we will limit ourselves to analyzing planner's problems.

The planner's problem in the basic one sector growth model is given by

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to

$$\begin{aligned} c_t + x_t &\leq F(k_t, n_t), \\ k_{t+1} &\leq (1-\delta_k)k_t + x_t, \end{aligned}$$

where c_t is per capita consumption, k_t is the per capita stock of capital, x_t is the (nonnegative) flow of investment, and n_t is employment at time t. Since we assume that leisure does not yields utility, the optimal (and equilibrium) level of n_t equals the endowment, which we normalize to 1. The Euler equation for this problem is just (5) given that. as before, we set $f(k) = F(k, 1)$. It follows that if $\lim_{k\to\infty} \beta[1-\delta_k+f(k)] > 1$, then $\limsup_t c_t = \infty$. Thus, there is equilibrium growth. This result does not depend on the assumption of just one capital stock. More precisely, in the case of multiple capital stocks, the feasibility constraint is just

$$\begin{aligned} c_t + \sum_{i=1}^{I} x_{it} &\leq f(k_{1t}, ..., k_{It}), \\ k_{it+1} &\leq (1-\delta_{ik})k_{it} + x_{it}. \end{aligned}$$

In this case, the natural analogue of the assumption that the marginal product of capital is bounded is just that there is a homogeneous of degree one function —a linear function— that is a lower bound for the actual production function. However, it turns out that all that is required is that there exist a ray that has bounded marginal products. Formally, this corresponds to

Condition 1 (G) *Assume that* $f(k_1, ..., k_I) \geq h(k_1, ..., k_I)$, *where* h *is concave, homogeneous of degree one and* C^1 *for all* $(k_1, ..., k_I) \in \mathbb{R}^I_+$. *Moreover, assume that there exists a vector* $\hat{k} = (\hat{k}_1, ..., \hat{k}_I)$, $\hat{k} \neq 0$, *such that if* $\hat{k}_i > 0$,

$$\beta[1 - \delta_k + h_i(\hat{k})] > 1, \qquad i = 1, ..., I$$

The basic result is the following (see Jones and Manuelli (1990))

Proposition 2 *Assume that Condition G is satisfied. Then, any optimal solution* $\{c_t^*\}$ *is such that* $\limsup_t c_t^* = \infty$.

As Jones and Manuelli (1990) show, the planner's solution can be supported as a competitive equilibrium. An extension to multiple goods is presented by Kaganovich (1998) and it is based on similar insights. It is clear that Condition G does not rule out decreasing returns to scale. This, in turn implies that this class of models is consistent with a version of the notion of conditional convergence: relatively poor countries are predicted to grow faster than richer countries, with the consequent closing of the income gap. Put it differently, theory suggests that, with a finite amount of data, it is difficult to distinguish an endogenous growth model from a Cass-Koopmans exogenous growth model. The main difference lies in the tail behavior of the relevant variables (output or consumption), and not in the balanced (or unbalanced) nature of the equilibrium path.

2.3 Fiscal Policy and Growth

In this section we describe the effects of taxes and government spending on the long run growth rate. Consider the problem faced by a representative agent

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

subject to

$$(1+\tau^c)c_t + (1+\tau^x)p_t x_{kt} + (1+\tau^h)q_t \leq w_t(1-\tau^n)(n_{ct}h_t + n_{kt}h_t) + (1-\tau^k)r_t k_t + T_t + \Pi_t,$$

where τ^j represent tax rates, c_t is consumption, x_{kt} is investment in physical capital, q_t are market goods used in the production of human capital, $n_{it}h_t$ is effective labor

—the product of human capital and hours— allocated to sector i, k_t is the stock of capital, T_t is a government transfer, and Π_t are net profits.

Accumulation of human capital at the household level satisfies

$$h_{t+1} \leq (1 - \delta_h)h_t + F^h(q_t, n_{ht}h_t),$$

where F^h is homogeneous of degree one, concave and increasing in each argument.

The economy has two sectors: producers of capital and consumption goods. Output of the capital goods industries satisfies

$$x_t \leq F^k(k_{kt}, n_{kt}h_t),$$

where F^k is homogeneous of degree one and concave.

Feasibility in the consumption goods industry is given by

$$c_t \leq F^c(k_{ct}, n_{ct}h_t),$$

where F^c is increasing and concave. It is not necessary to assume that this production function displays constant returns to scale.

It is illustrative to consider several special cases. Throughout, we assume that the utility function is of the form that is consistent with the existence of a balanced growth path. Specifically, we assume that $u(c, \ell) = (cv(\ell))^{1-\theta}/(1-\theta)$. Moreover, since our emphasis is on the role of taxes and tax-like wedges between marginal rates of substitution and transformation, we assume that lump sum transfers, T_t, are adjusted to satisfy the government budget constraint.

Case I: One Sector Model with Capital Taxation We assume that the consumer supplies one unit of labor inelastically. In this case $F^c = F^k = Ak + \hat{F}(k)$, where $\hat{F}(k)$ is strictly concave and $\lim_{k\to\infty} \hat{F}'(k) = 0$. For now we ignore human capital and set $F^h \equiv 0$. It follows that the balanced growth rate satisfies

$$\gamma^\theta = \beta[1 - \delta_k + \frac{1 - \tau^k}{1 + \tau^x}A].$$

Thus, in this setting, increases in the effective tax on capital, $(1 - \tau^k)/(1 + \tau^x)$ unambiguously decrease the equilibrium tax rate. Thus, unlike exogenous growth models, government policies affect the growth rate. Moreover, this simple example illustrates the size of the impact of changes in tax rates on the long run growth rate depend on the intertemporal elasticity of substitution $1/\theta$. More precisely the elasticity of the growth rate with respect to τ^k is given by

$$\frac{\partial \gamma}{\partial \tau^k}\frac{\tau^k}{\gamma} = -\frac{1}{\theta}\frac{\frac{\tau^k}{1+\tau^x}A}{1 - \delta_k + \frac{1-\tau^k}{1+\tau^x}A}.$$

It follows that, other things constant, high values of the intertemporal elasticity of substitution result in large changes in predicted growth rates in response to changes in tax rates. Thus, even an example as simple as this one illustrates that the quantitative predictions of this class of models will heavily depend on the values of the relevant preference (and technology) parameters.

Case II: Physical and Human Capital: Identical Technologies In this section we assume that $F^c = F^k$, and $F^h = q$. This implies that all three goods —investment, consumption and human capital— are produced using the same technology and, in particular, the same physical to human capital ratio. As in the previous section, τ^k and τ^x do not play independent roles. Thus, to simplify notation, we will set $\tau^x = 0$. However, the reader should keep in mind that increases in the tax rate on capital income are equivalent to increases in the tax rate on purchases of capital goods.

In this case, the balanced growth conditions are

$$\gamma^\theta = \beta[1 - \delta_k + (1 - \tau^k)F_k(\kappa, n)] \tag{6a}$$

$$\frac{c}{h}\frac{v'(1-n)}{v(1-n)} = \frac{1-\tau^n}{1+\tau^c}F_n(\kappa, n) \tag{6b}$$

$$(1-\tau^k)F_k(\kappa, n) - \delta_k = \frac{1-\tau^n}{1+\tau^h}F_n(\kappa, n)n - \delta_h \tag{6c}$$

$$\frac{c}{h} + (\gamma + \delta_k - 1) = F(\kappa, n). \tag{6d}$$

There are several interesting points. First, increases in the tax rate on consumption goods (i.e. sales or value added taxes) are equivalent to increases in the tax rate on labor income. Second, the relevant tax rate to evaluate the return on human capital is $(1-\tau^n)/(1+\tau^h)$. Thus, it is possible that increases in τ^n —as observed in the U.S. between the pre World War II and the post WWII periods— if matched by decreases in τ^h (corresponding, for example, to expansion in the quantity and quality of free public education) have no effect on the physical capital - human capital ratio, κ. Third, it is possible to show that increases in τ^k, τ^n, τ^h or τ^c result in lower growth rates. Last, without making additional assumptions about preferences and technology, it is not possible to sign the impact of changes in tax rates on other endogenous variables.

Case III: Physical and Human Capital: Different Factor Intensities In this case, we assume that only human capital is used in the production of human capital. Thus, $F^h = A_h n_h h$. This is the technology proposed by Uzawa (1964) and popularized in this class of models by Lucas (1988). For simplicity, we only consider capital and labor taxes. The relevant steady state conditions are (6a), (6b), and (6d). However, (6c) becomes

$$\gamma^\theta = \beta[1 - \delta_h + A_h n] \tag{7}$$

In this version of the model, changes in labor income taxes, reduce growth through their impact on hours worked (relative to leisure). However, if total work time is inelastically supplied, i.e. $v(\ell) \equiv 1$, the growth rate is pinned down by

$$\gamma^{\theta} = \beta[1 - \delta_h + A_h].$$

Thus, in this setting (which corresponds to Lucas (1988) model without the externality, and to Lucas (1990)), taxes have no effect on growth. Increases in the tax rate on capital income simply change physical capital - human capital ratio and they leave the after tax rate of return unchanged. The reason for this extreme for of neutrality is that even though taxes on labor income reduce the returns from education, they also reduce the cost of using time to accumulate human capital (the value of time decreases with increases in taxes), and the two changes are identical. Thus, the cost-benefit ratio of investing in education is independent of the tax code.

2.3.1 Quantitative Analysis of the Effects of Taxes

Since the development of endogenous growth theory there have been several studies of the implications of substituting lump-sum taxes for a variety of distortionary taxes. Jones, Manuelli and Rossi (1993), analyze the optimal choice of distortionary taxes in several models of endogenous growth. In the case that physical and human capital are produced using the same technology and labor supply is inelastic, they find that for parameterizations that make the predictions of the model consistent with observations from the U.S., the potential growth effects of drastically reducing (eliminating in most cases) all forms of distortionary taxation is quite high. For their preferred parameterization the increase in growth rates is about 3%. They study a version of the model in which $F^c = F^k \neq F^h$., and the functions F^k andF^h are both of the Cobb-Douglas variety, but differ in the average productivity of capital. Jones, Manuelli and Rossi estimate the capital share parameter to be equal 0.36 in the consumption sector, and to be somewhere in the 0.40-0.50 range in the human capital production sector.[3] They also allow labor supply to be elastic. Their findings suggest that switching to an optimal tax code result in increases in yearly growth rates of somewhere between 1.5% and 2.0% per year. These are substantial effects.

The third experiment that they consider involves the endogenous determination (by the planner) of the level of government consumption. In this case, they revert back to the one sector version of the model, and they explore not only the consequences of changing the intertemporal elasticity of substitution, but they allow for varying elasticity of substitution between capital and human capital. For their preferred characterization, they also find growth effects of about 2% per year. Moreover, as in the other experiments, the predictions are quite sensitive to the details of the model

[3] Jones, Manuelli and Rossi (1993) calibrate this share. Since they study the sensitivity of their results to changes in other parameters (e.g. the intertemporal elasticity of substitution), the market goods share is not constant across experiments.

—in particular, to the choice of the intertemporal elasticity of substitution, and the degree of substitutability between capital and human capital.

Stokey and Rebelo (1995) undertake a thorough review of several models that estimate the growth impact of tax reform. They argue that in the U.S. tax rates in the post WWII period are significantly higher than in the pre WWII era. This conclusion is based on the increase in the revenue from income taxes as a fraction of GDP in the early 1940s. To reconcile the models with this evidence, they conclude that the human capital share in the production of human capital must be large, and that this sector must be lightly taxed. This description is close to the Case III above and, as argued before, it results in no growth effects[4]. Thus, in agreement with Lucas (1990) —and using a very similar specification of the human capital production technology— they conclude that changes in tax rates cannot have large growth effects.

This conclusion depends on several assumptions. First, that the U.S. evidence shows an increase in the general level of taxes after WWII. Second, that even if there is a tax increase, the additional revenue is used to finance lump-sum transfers. Third, that the balanced growth path is a good description of the pre and post WWII economy.

Measuring changes in the relevant marginal tax rates is a difficult task. Barro and Sahasakul (1986) using tax records compute average marginal tax rates for the U.S. economy. Their estimates, consistent with the Stokey and Rebelo assumption, show an increase in the 1940s. Even though the evidence about changes in the tax rate consistently points to an increase, the implications of this result for the model are not obvious. Consider, first, the uses of tax revenue. If, for example, additional income tax revenues (at the local level) are used to finance local publicly provided goods (e.g. education), then Tiebout-like arguments suggest that the 'tax effect' of a tax increase is zero. In the U.S. a substantial increase in government spending corresponds to increases in expenditures on education and, hence, the possibility of individuals sorting themselves to buy the 'right' bundle of publicly provided private goods cannot be ignored. A second quantitatively important public spending program in the post WWII era is Social Security. To the extent that benefits are dependent on contributions, the statutory tax rate on labor income used to finance social security overstates the true tax rate.[5] In this case, tax payments purchase the right to an annuity whose value is dependent on the payment. Finally, in a model with multiple tax rates an increase in a single tax does not imply, necessarily, a decrease in the growth rate. For the U.S. the evidence on the time path of capital income taxes is mixed. In a recent study, Mulligan (2003) argues that the tax rate on capital income has steadily fallen in the last 50 years. Similarly, Prescott and McGrattan (2003) and (2004) find that a decrease in the tax rate on corporate income —one form of

[4]The results are continuous in the parameters. Thus, for market goods share close to zero, as Stokey and Rebelo prefer, the growth effects are small.

[5]In a pay-as-you-go system, even if the share of total payments that an individual receives is sensitive to his contributions, the same effect obtains.

capital income— is instrumental in explaining the increase in the value of corporate capital relative to GDP. Overall, we find that the quantitative evidence on the time path of the relevant tax rates to be difficult to ascertain. More work is needed, with an emphasis on matching model and data.

The next section considers the effects of endogenous government spending and transitional effects.

2.3.2 Productive Government Spending

A Simple Balanced Growth Result In this section we study a simple one sector model that provides a role for productive government spending. Our discussion follows the ideas in Barro (1990). Assume that firm $i's$ technology is given by

$$y_{it} \leq Ak_{it}^{\alpha}h_{it}^{\eta}G_t^{1-\alpha-\eta},$$

where k_{it} and h_{it} are the amounts of physical and human capital used by the firm, and G_t is a measure of productive public goods that firms take as given. The government budget constraint is balanced in every period, and it satisfies

$$G_t = \tau^k r_t K_t + \tau^h w_t H_t,$$

where τ^k and τ^h are the tax rates on capital and income, and r_t and w_t are rental prices. For simplicity we assume that the instantaneous utility function is given by

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}.$$

We also assume that the technologies to produce market goods and human capital are identical. In this case, it is immediate to show that the equilibrium is fully described by

$$\begin{aligned} \delta_h - \delta_k &= A^{1/(\alpha+\eta)}(\alpha\tau^k + \eta\tau^h])^{(1-\alpha-\eta)/(\alpha+\eta)}[\eta(1-\tau^h)\kappa^{\alpha/(\alpha+\eta)} - \alpha(1-\tau^k)\kappa^{-\eta/(\alpha+\eta)}], \\ \gamma^{\theta} &= \beta[1 - \delta_k + \alpha(1-\tau^k)A^{1/(\alpha+\eta)}(\alpha\tau^k + \eta\tau^h)^{(1-\alpha-\eta)/(\alpha+\eta)}\kappa^{-\eta/(\alpha+\eta)}], \end{aligned}$$

where κ is the physical capital - human capital ratio.

Some tedious algebra shows that the growth rate is not a monotonic function of the tax rates. In general, there is no growth when taxes are either too low (not enough public goods are provided) or too high (the private returns to capital accumulation are too low). For intermediate values of the tax rates, growth is positive (if A is sufficiently high). Thus, in general, increases in tax rates need not result in lower growth if they are accompanied by changes in government spending. Thus, a variant of the model with endogenous government spending (or endogenous taxation and optimally chosen government spending) has potential to reconcile positive growth effects associated with the removal of distortions with the U.S. evidence.

What does the U.S. evidence show? In the U.S. there is a substantial increase in the ratio of government spending to GDP in the post WWII period on the order of 15%. Even ignoring defense related expenditures, the size of the federal government relative to output is close to 5% in the pre WWII period, and it increases steadily in the post war to reach about 20% of income. Of course, not all forms of government spending are productive, but if the trend in the productive component follows the trend in overall spending, ignoring changes in government spending result in biased estimates of the effects of distortions.

The Barro model is silent about the reasons why the desired ratio of (productive) government spending to GDP would increase. For this, it is necessary to have a model of the collective decision making mechanisms which is clearly beyond the scope of this chapter.

Progressive Taxes and Transition Effects Our discussion of the assumptions that suffice for sustained growth clearly shows that homogeneity of degree one is not necessary. In both theoretical and applied work it is common to appeal to linearity in order to ignore transitional dynamics (see, Bond, Wang and Yip (1996)) and Ladron de Guevara, Ortigueira and Santos (1997)) for analysis of the dynamics of endogenous growth models). However, when taking the model to the data, the assumption that the economy is on the balanced growth path may not be appropriate.

In this section we describe the results of Li and Sarte (2001). The basic insight from their model that is relevant for our discussion is that in the presence of heterogeneity in individual preferences and nonlinearities in the tax code, shocks to the tax regime (they consider an increase in the degree of progressivity of the tax code) that ultimately result in a decrease in the growth rate can have basically no effects for several decades.

The basic model that they consider is one in which goods are produced according to the following technology

$$Y_t \leq AK_t^{\alpha} L_t^{1-\alpha} G_t^{1-\alpha},$$

where K_t is capital at time t, L_t is the flow of labor, and G_t is a measure of productive public goods. All individuals have isoelastic preferences given by $u(c) = (c^{1-\theta} - 1)/(1-\theta)$, but they differ in their discount factors, β_i. Li and Sarte assume that each type has mass $1/N$, where N is population. The tax code is nonlinear. Given aggregate income Y, and individual income y_i, the *tax rate* is given by a function $\tau(z)$, where z is the ratio of individual to average income. In this application, Lin and Sarte assume that

$$\tau(\frac{y_i}{Y}) = \zeta(\frac{y_i}{Y})^{\phi}.$$

Note that the case of proportional taxes —the case discussed so far— corresponds to $\phi = 0$. In this setting, higher values of ϕ are interpreted as corresponding to more progressive tax codes. Individual income is defined as the sum of capital and labor income. Government spending is financed with revenue from income taxes. Li and

Sarte show that the equilibrium is the solution to the following system of equations

$$
\begin{aligned}
\gamma^{\theta} &= \beta_i\{1-\delta+[1-(1+\phi)\zeta(\frac{y_i}{Y})^{\phi}]\alpha A^{1/\alpha}(\frac{G}{Y})^{(1-\alpha)/\alpha}, \qquad i=1,2,...I, \\
\frac{G}{Y} &= \sum_{i=1}^{I}\zeta(\frac{y_i}{Y})^{1+\phi}\frac{1}{N}, \\
1 &= \sum_{i=1}^{I}\frac{y_i}{Y}\frac{1}{N}.
\end{aligned}
$$

In this model, changes in the progressivity of the tax code affect the rate of return —this is the standard effect— as well as the distribution of income. It is this last effect that generates the slow adjustment. It is possible to show that an increase in ϕ decreases long run growth, γ.

Li and Sarte explore the dynamic effects of a one time increase in ϕ that result in a decrease in the growth rate of 1.5%. On impact, output growth increases because since the distribution of income does not adjust immediately, government revenues increase and this, in turn, increase output. As the low discount factor individuals adjust their relative income (an increase in progressivity affects them more than proportionally), government revenue (and spending decrease. For parameter values that are designed to mimic the U.S. economy, Li and Sarte find that the half-life of the adjustment is over 40 years. Thus, any test for breaks in the growth rate as suggested by models in which convergence is immediate would conclude (incorrectly) that the tax increase has no effects on growth.

It is difficult to evaluate how appropriate the Li and Sarte model is to study the impact of tax reform in the U.S. economy. However, it casts doubt on the approach by Stokey and Rebelo which ignores transitional dynamics. Models that rely on changes in tax rates that, in turn, affect the distribution of income, are consistent with the view that the effects of those changes are not monotone, and that the full impact may not be felt for decades.

2.4 Innovation in the Neoclassical Model

One of the things that seems unsatisfactory to many economists in the presentation up to this point is the starkness with which the technological side of the model is described. As we argued above, the key in improving over the Solow model is to explicitly consider decisions made by private agents about investments they make that cause technology to improve. This both endogeneizes the growth process envisaged by Solow and breaks away from another key assumption of the exogenous growth literature, that technological change happens without any resource cost. But, much of the detail that one thinks about as being an important part of the innovation process seems to be missing from the simple convex models of growth described above. The idea that innovation is carried out by specialized researchers who pass on their newfound knowledge to production line workers is just one example of this. Indeed, one question is whether or not that kind of structure is consistent with convex models of growth at all.

Because of this, in this section we describe a variant of the models presented in the last section that is more directed at identifying innovation as a special activity. The purpose of this exercise is not to fully exhaust the possibilities, but rather to show the reader that more is possible with the class of convex models than one might first think. In particular, since the model we will analyze is convex, standard price taking behavior is consistent with equilibrium behavior. In this sense, the example we will present is similar to the ideas developed by Boldrin and Levine (2002).

There are many models of innovation that do not have convex technology sets (e.g., see the surveys in Barro and Sala-i-Martin (1995) and Aghion and Howitt (1998)). In this setting, standard price taking behavior is either not consistent with equilibrium in those settings or they must include external effects. Because of this, all policy experiments in those models mix two conceptually distinct aspects of policy, the desire to correct for monopoly power and/or external effects and the distortionary effects of 'wedges' (e.g. taxes). This, in turn, implies that the answers to questions about the effects of alternative policies on both the incentives to innovate and overall welfare depends on the details of the specifications of external effects (e.g., do other researchers learn new innovations for free after one month, or one decade) and/or market power (e.g., is there only one researcher at the frontier and so a monopoly analysis is in order, or are there two, or many). Thus, one thing that a convex model of innovation has to add is answers to some of these questions which are less dependent on those details.

2.4.1 Notation

We will follow the notation above as closely as is possible. We assume that there are two types of labor supply available, researchers and workers. Each individual of each type has his own level of knowledge. We will assume that there is a continuum of identical households each with some researcher time and some worker time to supply to the market. These are given by L_1 researcher hours per household, and L_2 worker hours per household, where $L = L_1 + L_2$ is total labor supply within the household. We will assume that L_1 and L_2 are fixed, with no ability to move hours between them. (In this sense, it might be easier to think of the household as being made up of L_1 researchers and L_2 workers.)

Each household has the level of knowledge H_t that they can use with researcher hours during period t. Thus, if households are symmetric, H_t symbolizes the absolute frontier of what 'society' knows at date t. Similarly, the level of knowledge for the average worker hour at date t is denoted by h_t. This will represent the average knowledge of those workers that work in the final goods sector below.

Final consumption at date t is denoted by c_t. We abstract from physical capital to simplify.

Production Functions We will assume that:

$$\begin{aligned} H_{t+1} &= H_t + A_H L_{1t}^H H_t, \\ h_{t+1} &= (1-\delta_h) h_t + A_h (L_{1t}^h H_t)^\alpha (L_{2t}^h h_t)^{1-\alpha}, \\ c_t &= A_c L_{2t}^c h_t, \\ L_{1t}^H + L_{1t}^h &= L_1, \\ L_{2t}^h + L_{2t}^c &= L_2. \end{aligned}$$

This formulation is equivalent to one in which quality adjusted labor of the form $Z = LH$, (resp. $Z = Lh$) is employed in each activity.

The idea here is that $I_H = A_H L_{1t}^H H_t$ is new research and development or innovation, increasing H corresponds to learning more at society's highest level. Note that we have assumed that there is no depreciation– the level of frontier knowledge does not go backward (positive depreciation is easily included). If no innovation is done, $L_{1t}^H = 0$ for all t, $H_{t+1} = H_t$ for all t, that is, the frontier is static. In this case, h_t would also be bounded, and hence the level of output would be bounded above. In this sense, innovation is necessary for growth to occur in this model.

Similarly, we think of the $I_h = A_h (L_{1t}^h H_t)^\alpha (L_{2t}^h h_t)^{1-\alpha}$ technology as Education and/or Worker Training. This is where family members at the frontier spend part of their time educating the workers from the family on the use of new techniques. The more time researchers spend in I_h, the less time they have to spend in I_H, and hence workers are better prepared and more productive, but the frontier moves out more slowly. Note that increasing $L_{2t}^h h_t$, holding $L_{1t}^h H_t$ constant increases total output of worker productive knowledge (new h) but lowers the average product of frontier knowledge workers in educating (bigger classes give more total new training, but less output per student). Symmetrically, the more time that production line workers spend learning new knowledge, the less time they have available for production of current consumption goods.

Preferences We assume that each researcher/worker supplies one unit of labor to the market inelastically and that each has preferences of the form:

$$U(c) = \sum_t \beta^t u(c_t),$$

where $u(c) = c^{1-\sigma}/(1-\sigma)$. This model, although different in detail, shares one common critical feature with those above: linearity in the reproducible factors.

2.4.2 Balanced Growth Properties of the Model

Like many Ak, style models, this one has the feature that it converges to a Balanced Growth Path. Indeed if the initial levels of relative human capitals (H_0/h_0) are right the economy is on the BGP in every period. Standard techniques can be used to characterize this BGP. After some algebra, we find:

$$\gamma^{\sigma} = \beta \left[1 + A_H L_1\right]$$

This equation gives γ as a function of the basic parameters σ, β, A_H, and L_1. By construction, the comparative statics of growth rates with respect to the deep parameters of the model are identical to what one would find in an Ak model. The one difference is the inclusion of the endowment of researcher hours, L_1, note that γ is increasing in L_1. That is, if one country had a higher proportion of researchers in its population, output would grow faster.

Since γ does not depend on the other parameters of the model, it can be shown that the only way income taxes affect growth rates here is through their effect on the R&D sector. That is, if we have a linear income tax (at rate τ) either on income generated in all sectors, or on income generated only in the H sector the growth rate will fall to:

$$\gamma_{\tau}^{\sigma} = \beta[1 + (1 - \tau)A_H L_1]$$

In particular, if income from the h and/or c sectors are taxed, but that from the H sector is not, there are no effects on growth. This is reminiscent of the Lucas (1988) model and the 2-sector model in Rebelo (1991).

The amount of time spent on R&D on the BGP is given by:

$$\begin{aligned} L_1^H &= \frac{\left[\left(\beta\left[1 + A_H^*\right]\right)^{1/\sigma} - 1\right]}{A_H^*} L_1 = \frac{\left[\left(\beta\left[1 + A_H L_1\right]\right)^{1/\sigma} - 1\right]}{A_H L_1} L_1 \\ &= \left[\left(\beta\left[1 + A_H L_1\right]\right)^{1/\sigma} - 1\right] / A_H \end{aligned}$$

Thus, if we compare two countries with different discount factors, but identical in other respects, the one with the higher β will devote a higher fraction of its researcher time to innovation and a lower percentage to teaching. This causes worker productivity in the consumption sector to be lower at first (and consumption as well), but growing faster and hence, eventually overtaking the low β country.

As a second point, note that increases in A_h do not change γ (and neither do changes in A_c). Thus, in this case, L_1^H is not affected and so the time series of H_t will be identical. This implies that h_t must be higher. Thus, wages of both researchers and workers will be higher. This is similar in spirit to the result in Boldrin and Levine (2002) that improvements in the copying technology raises the value of being an innovator. Since the only 'copying' being done here is the passing on of new knowledge to final goods workers, this is analogous in this setting.

These are simple comparative statics exercises which are meant only to show that much intuition about the process of innovation, and its comparative statics properties with respect to incentives can be illustrated in this class of models.

There are many interesting extensions of the analysis that one could imagine. These include heterogeneity among households (e.g., some researcher households, some worker households), the inclusion of uncertainty about the results of researcher time (and the questions that this raises about ex post hold up problems when one researcher is the 'first' discoverer), the training of researchers by other researchers when they have different H_t 's, the inclusion of more than one good or process, what types of Industrial Organization are possible through decentralizations of the allocation as a competitive equilibrium (e.g., firms specializing in R&D vs. each firm having an R&D division), etc.

But, the reader can see that much of the analysis will go through unchanged. Notable exceptions are when there are assumed to be external effects in the learning process. The simplest example of this here would be to assume that $h = H$ no matter what L_2^h is. In this case, unless this is completely internalized within a firm (i.e., there are no spillovers across firms) the Planner's problem will not be implementable as a competitive equilibrium.

2.4.3 Adding a Non-Convexity

Most models of innovation differ from the one outlined above in that they assume that there is a non-convexity in the innovation technology. There are two ways to include this in the specification above, and the differences between them highlight a key question about innovation.

These are:

$$H_{t+1} = H_t + A_H L_{1t}^H H_t if L_{1t}^H \geq L^*,\ H_{t+1} = H_t\ if\ L_{1t}^H < L^* \tag{8}$$

and

$$H_{t+1} = H_t + A_H(L_{1t}^H - L^*)H_t\ if\ L_{1t}^H \geq L^*,\ H_{t+1} = H_t\ if\ L_{1t}^H < L^*. \tag{9}$$

Although these two specifications look similar, they differ in one key aspect. The technology in (8) is convex anytime R&D is 'active' (i.e., $L_{1t}^H \geq L^*$). The technology in (9) has constant marginal costs when R&D is active, but features a set-up cost as well (given by L^* denominated in labor units). Technology (9) is the specification that is most commonly employed in the R&D literature while that in (8) is similar in spirit to that used in Boldrin and Levine. Because of this difference, all of the analysis outlined above can be used if the technology is that given in (8) so long as in the solution to the planner's problem we have that $NL_{1t}^H \geq L^*$ for all t where N is the number of households. That is, the allocation can be supported as a competitive equilibrium with price taking behavior, etc. There are some restrictions on the implicit Industrial Organization in the equilibrium however. For example, if $NL_{1t}^H = L^*$ for all t it follows that there can be at most one R&D firm in any equilibrium (or one firm with an R&D division). This, were it true, would cause serious concern for the price-taking assumption in the decentralization.

One interesting implication of this model is that whether or not the solution to the planning problem above (without the non-convexity) describes the competitive allocation depends on the size of the country, N. Thus, large countries would, in equilibrium, conduct R&D while small countries would not. Adding in a fourth sector in which researchers in large countries could train researchers in small countries would be a natural extension in which R&D was done in large countries, these researchers train high H workers from small countries (e.g., in Engineering schools), those newly trained 'researchers' return to their home countries where they subsequently train production line workers, etc.

This description of equilibrium cannot be true for (9), however. Price taking behavior in this setting implies that prices for the rental of new knowledge equal their marginal cost of production. This implies that there is no way to recover the set up cost of researchers spending L^* hours. Thus, there can be no competitive equilibrium. It follows that there must be some monopoly rent generated somewhere to decentralize any allocation. Typically this will be accompanied by inefficiencies and incorrect incentives to conduct R&D.

3 Fluctuations and Growth

3.1 Introduction

In this section we describe the existing results on the effects of 'volatility,' both in technologies and policies, on the long-run growth rate. We start with a brief summary of the empirical research in this area, and we then describe some simple theoretical models that are useful in understanding the empirical results. We end with the description of some recent work based on the theoretical models but aimed at evaluating their ability to *quantitatively* match the growth observations. As before, we ignore models based on aggregate non-convexities, and with non-competitive market structures.

3.2 Empirical Evidence

A relatively small (but growing) empirical literature has tried to shed light on the relationship between 'instability' and growth. This literature has concentrated on estimating reduced form models that try to capture, with varying degrees of sophistication, how 'volatility' (defined in a variety of different ways) affects long-run growth.

Kormendi and Meguire (1985) is probably the earliest study in this literature. They consider a sample of 47 countries with data covering the 1950-1977 period. Their methodology is to run a cross-country growth regression with the average (over the sample period) growth rate as the dependent variable, and a number of control variables, including the standard deviation of the growth rate (one measure of instability), as well as the standard deviations of policy variables such as the inflation rate and the money supply. Kormendi and Meguire find that the coefficient of the volatility measure (the standard deviation of the growth rate) is *positive* and significant. Thus, a simple interpretation of their results is that more volatile countries —as measured by the standard deviation of their growth rates— grow at a higher rate.

Grier and Tullock (1989) use panel data techniques on a sample of 113 countries covering a period from 1951 to 1980. Their findings on the effect of volatility on growth are in line with those of Kormendi and Meguire. They find that the standard deviation of the growth rate is *positively,* and significantly, associated with mean growth rates.

Ramey and Ramey (1995) first report the results of regressing mean growth on its standard deviation on a sample of 92 countries as well as a subsample of 25 OECD countries, covering (approximately) the 1950-1985 period. They find that for the full sample the estimated effect of volatility is negative and significant, while for the OECD subsample the point estimate is positive, but insignificant. In order to allow for the variance of the innovations to the growth rate to be jointly estimated with the effects of volatility, Ramey and Ramey posit the following statistical model

$$\gamma_{it} = \beta X_{it} + \lambda \sigma_i + u_{it} \tag{10}$$

where X_{it} is a vector of variables that affect the growth rate and

$$u_{it} = \sigma_i \epsilon_{it}, \qquad \epsilon_{it} \sim N(0,1). \tag{11}$$

The model is estimated using maximum likelihood. The control variables used were the (average) investment share of GDP (Average I/Y), average population growth rate (Average γ_{Pop}), initial human capital (measured as secondary enrollment rate, H_0), and the initial level of per capita GDP (Y_0). They study separately the full sample (consisting of 92 countries) as well as a subsample of 25 OECD (more developed) economies. Their results are reproduced in columns (1) and (3) of Table I.

Table I: Growth and Volatility I				
Variables	(1) (92−*Country*) N = 2,184	(2) (92−*Country*) N = 2,184	(3) (*OECD*) N = 888	(4) (*OECD*) N = 888
Constant	0.07 (3.72)	0.08 (3.73)	0.16 (5.73)	0.16 (4.48)
σ_i	−0.21 (−2.61)	−0.109 (−1.22)	−0.39 (−1.92)	−0.401 (−1.93)
Average I/Y	0.13 (7.63)	0.12 (6.99)	0.07 (2.76)	0.071 (2.67)
Average γ_{Pop}	−0.06 (−0.38)	−0.115 (−0.755)	0.21 (0.70)	0.230 (0.748)
H_0	0.0008 (1.18)	0.0007 (1.03)	0.0001 (2.00)	0.0001 (1.954)
Y_0	−0.009 (−3.61)	−0.009 −3.53)	−0.017 (−5.70)	−0.017 (−4.7445)
$\sigma_{\ln(I/Y)}$	-	−0.023 (1.81)		0.007 (0.22)

Note: t-statistics in parentheses
Source: Columns (1) and (3) Ramey and Ramey (1995)
Columns (2) and (4), Barlevy (2002)

For both sets of countries, Ramey and Ramey find that the standard deviation of the growth rate is *negatively* related to the average growth rate. However, for the OECD subsample, the coefficient is less precisely estimated (even though the point estimate is larger in absolute value). Ramey and Ramey also consider more 'flexible' specifications that try to capture differences across countries in the appropriate forecasting equations. Considering the most parsimonious version of their model, the estimated effect of volatility on growth is still positive. However, the strength of the estimated relationship is reversed: for the OECD subsample the point estimate is four times the size of the estimate for the full sample and highly significant.

In more recent work, Barlevy (2002) reestimates the Ramey and Ramey model with one change: he adds the standard deviation of the logarithm of the investment-output ratio ($\sigma_{\ln(I/Y)}$) as one of the explanatory variables. Barlevy hypothesizes that this variable can capture non-linearities in the investment function. His results, using the same basic data as Ramey and Ramey are in columns (2) and (4) of Table I.[6] For the full 92-country sample, the introduction of this measure of investment volatility halves the size of the coefficient of σ_i, and it is no longer significant at conventional levels. The coefficient on $\sigma_{\ln(I/Y)}$ is *negative* and significant (at 5%). For the OECD sample, the addition of $\sigma_{\ln(I/Y)}$ does not affect much the estimate of the effect of σ_i on growth. However, Barlevy points out that this is finding is not robust, since eliminating two outliers, Greece and Japan where high volatility of the investment share seems to be due to transitional dynamics, implies that neither the volatility of the growth rate nor $\sigma_{\ln(I/Y)}$ are significant.[7]

[6] We thank Gadi Barlevy for providing us the estimated coefficient for the control variables.
[7] The point estimates are negative but insignificant.

One possible explanation for the differences in the estimates of the effects of volatility on growth found in Kormendi and Meguire, Grier and Tullock and Ramey and Ramey, is —as pointed out by Ramey and Ramey and Barlevi— that Kormendi and Meguire and Grier and Tullock include among their explanatory variables the standard deviations of policy variables that could be proxying for $\sigma_{\ln(I/Y)}$.

Kroft and Lloyd-Ellis (2002) also start from the basic statistical model of Ramey and Ramey but offer a different way of decomposing volatility. They hypothesize that uncertainty can be split into two orthogonal components: uncertainty about changes in regime (e.g. expansion-contraction) and fluctuations within a given regime. To this end, they generalize the empirical specification of the Ramey and Ramey statistical model to account for this. They assume that

$$\gamma_{ist} = \beta X_{it} + \lambda_w \sigma_{iw} + \lambda_b \sigma_{ib} + \upsilon_{ist}, \tag{12a}$$

$$\upsilon_{ist} = \sigma_{iw}\epsilon_{it} + \mu_{is}, \qquad \epsilon_{it} \sim N(0,1), \tag{12b}$$

$$\mu_{is} = \begin{cases} \mu_{ie} & \text{with probability } p_i = \frac{T_{ie}}{T} \\ \mu_{ir} & \text{with probability } 1 - p_i \end{cases} \tag{12c}$$

Kroft and Lloyd-Ellis interpret the standard deviation of the random variable μ_{is}, σ_{ib} —which they assumed observed by the economic agents but unobserved by the econometrician— as a measure of variability *between* regimes, while σ_{iw} is viewed as the *within-regime* variability. Kroft and Lloyd-Ellis estimate their model by maximum likelihood using the same sample as Ramey and Ramey. The results are in Table II

Table II: Growth and Volatility II

Independent Variable	92-Country Sample (2,208 observations)	OECD Sample (888 observations)
Constant	0.00132 (0.022)	0.095 (1.89)
Within-phase volatility (σ_{iw})	2.63 (4.69)	0.90 (1.44)
Between-phase volatility (σ_{ib})	−2.65 (−6.35)	−1.11 (−2.33)
Average investment share of GDP	−0.01 (−0.26)	−0.004 (−0.073)
Average population growth rate	0.58 (1.24)	0.28 (0.62)
Initial human capital	0.001 (0.66)	−0.00001 (−0.096)
Initial per capita GDP	0.002 (0.25)	−0.0008 (−1.30)

Note: t-statistics in parentheses.
Source: Kroft and Lloyd-Ellis (2002).

The major finding is that the 'source' of volatility matters: Increases in σ_{iw} — the within phase standard deviation— have a positive impact on growth for the full sample. For the OECD, the coefficient estimate is still positive but about one third of the size. The effect of the between-phase volatility, σ_{ib}, is negative in both cases.

However, the effects are stronger for the full sample. It is not easy to interpret the phases identified by Kroft and Lloyd-Ellis in terms of a switching model because their estimation procedure assumes that the econometrician can identify whether a particular period corresponds to either a recession or an expansion.[8]. Kroft and Lloyd-Ellis also use the same controls as Ramey and Ramey. However, they find that, when the two variances are allowed to differ, none of the control variables is significant. It is not clear why this is the case. One possibility is that the 'phases' that they identify are correlated with the control variables (this seems like a likely situation in the case of investment). Another possibility is that the control variables, in the Ramey and Ramey formulation, capture the non-linearity associated with the regime shift and that, once the shifts are taken into account, the control variables have no explanatory power. In any case, this illustrates a point that we will come back to: the fragility of the "growth" regressions suggest that better theoretical models are necessary to more provide restrictions that will allow to identify the parameters of interest.

The results of Ramey and Ramey and Kroft and Lloyd-Ellis are consistent with the existence of nonlinearities in the relationship between measures of instability and growth. Fatás (2001) estimates a number of different specifications of the relationship between instability and growth. His approach is to run standard cross country regressions. His data set is taken from the most recent version of the Heston-Summers sample and includes 98 countries with information covering the period 1950-1998. His estimates (see Table III) support the view that the effect of volatility on growth is nonlinear. Using Fatás' basic estimate —shown in column (1) of Table III— the pure effect of volatility is *negative* with a coefficient of -2.772 indicating that a one standard deviation increase in volatility reduces the growth rate by over 2.5%. However, the interaction term, corresponding to the variable Volatility * GDP is positive and equal to 0.340. According to these estimates, the net effect of σ_i on γ_i for the *richest* countries in the data is *positive* and greater than 0.3. For the less developed countries the estimate of the effect of volatility is *negative*. Columns (2) and (3) use other measures of non-linearity (initial per capita GDP and M3/Y, a measure of financial development), with similar outcomes: In all cases there is a significant effect, and increases in volatility are less detrimental to growth —and could even have a positive effect— the more developed a country is according to the proxy variables.

[8] Kraft and Lloyd-Ellis estimate the probabilites p_i as the fraction of the time that an economy spends in the recession "phase," defined as periods of negative output growth. Thus, not only is the process assumed to be *i.i.d.* but the transition probabilites are not jointly estimated with the parameters.

Table III: Growth and Volatility III			
Independent Variable	(1)	(2)	(3)
Volatility (σ_i)	−2.772	−1.700	−0.270
	(0.282)	(0.645)	(0.091)
GDP per capita (1960)	−2.229	−1.856	−0.953
	(0.235)	(0.422)	(0.220)
Human capital (1960)	0.037	0.040	0.026
	(0.015)	(0.018)	(0.017)
Average investment share of GDP	0.083	0.143	0.120
	(0.013)	(0.021)	(0.024)
Average population growth rate	−0.624	−0.562	−0.465
	(0.153)	(0.205)	(0.465)
Volatility * GDP	0.340	–	-
	(0.036)		
Volatility * GDP (1960)	–	0.212	-
		(0.082)	
Volatility * M3/Y	-	-	0.004
			(0.001)
R^2	0.77	0.58	0.57

Note: Sample 1950-1998. Robust standard errors in parentheses
Source: Fatás (2001

Martin and Rogers (2000) also study the relationship between the standard deviation of the growth rate and its mean, in a cross section of countries and regions. They study two samples —European regions and industrialized countries— and in both cases they find a *negative* relationship between σ_γ and γ. However, when they consider a sample of developing countries the point estimates are positive, but in general insignificant.

It is not easy to explain the differences between Ramey and Ramey, Fatás and Martin and Rogers. The period used to compute the growth rates (1962-1985 for Ramey and Ramey, 1950-1998 for Fatás and 1960 to 1988 for Martin and Rogers), and the set of less developed countries included (68 in Ramey and Ramey's study, and 72 in Martin and Rogers') are fairly similar. The two studies differ on their definition of the growth rate (simple averages in the Ramey and Ramey and Fatás papers, and estimated exponential trend in Martin and Rogers), and in the variables that are used as controls. However, it is somewhat disturbing that what appear, in the absence of a theory, as ex-ante minor differences in definitions can result in substantial differences in the estimates.

Siegler (2001) studies the connection between volatility in inflation and growth rates and mean growth for the pre 1929 period. Specifically, he uses panel data methods for a sample of 12 (presently developed) countries over the 1870-1929 period. He finds that volatility and growth are negatively correlated, and this finding is robust to the inclusion of standard growth regression type of controls.

Dawson and Stephenson (1997) estimate a model similar to (10) and (11) applied to U.S. states. They use the average (over the 1970-1988 period) growth rate of gross state product per worker for U.S. states as their growth variable, and its standard deviation as a measure of volatility. In addition, they include in their cross-sectional regression the standard (in growth regressions) control variables (investment rate, initial level of gross state product per worker, labor force growth rate, and initial human capital). Dawson and Stephenson find that volatility has *no impact* on the growth rate, once the other effects are included. Unfortunately, they do not report the 'raw' correlation between mean growth and its standard deviation. Thus, it is not possible to determine if the lack of significance is due to the use of controls, or is a more robust feature of U.S. states growth performance.

Mendoza (1997) differs from the previous studies in terms of his definition of instability. Instead of the standard deviation of the growth rate, which, in general, is endogenous, he identifies instability with the standard deviation of a country's terms of trade. He estimates a linear model using a cross section of countries and finds a *negative* relationship between instability and growth. His sample is limited to only 40 developed and developing countries, and it only covers the period 1971-1991.

A fair summary of the existing results is that there is no sharp characterization of the relationship between fluctuations and growth. Variation across studies in samples or specifications yield fairly different results. Moreover, the findings do not seem robust to details of how the statistical model is specified.

Are the empirical findings of the channel through which uncertainty affects growth more robust? Unfortunately, the answer is negative. Ramey and Ramey find that volatility —measured as the standard deviation of the growth rate— does not affect the investment-output ratio. More recently, Aizenman and Marion (1999) find that volatility is negatively correlated with investment, when investment is disaggregated between public and private. Fatás estimates a non-linear model of the effect of volatility on investment. He finds that increases in volatility decrease investment in poor countries, but that the opposite is true in high income countries. Thus his findings are consistent with the view that changes in volatility affect mean growth rates through (at least partially) their impact upon investment decisions.

How should these empirical results be interpreted? Even though it is tempting to take one's preferred point estimate as a measure of the impact of fluctuations (or business cycles) on growth there are two problems with this approach. First, the empirical estimates are not robust to the choice of specification of the reduced form. Second, and more important in our opinion, is that from the point of view of policy design, the relevant measures of volatility is the —in general unobserved— volatility in *policies and technologies*. In most models, the growth rate (and its standard deviation) are endogenous variables and, as usual, the point estimate of one endogenous variable on another is at best difficult to interpret. One way of contributing to the interpretation of the empirical results is to study what simple theoretical models predict for the estimated relationships. In the next section we present a number of very simple models to illustrate the possible effects of volatility in fundamentals on mean growth. In the process, we find that it is very difficult to interpret the empirical findings. To put it simply, there are theoretical models that —depending on the sample— do not restrict the sign of the estimated coefficient of the standard deviation of the growth rate on its mean. Moreover, the sign and the magnitude of the coefficient is completely uninformative to determine the effect of volatility on growth.

3.3 Theoretical Models

The analysis of the effect of uncertainty on growth can be traced to the early work of Phelps (1962), and Levhari and Srinivasan (1969) who studied versions of the stochastic consumption-saving problem that are similar to the linear technology versions of endogenous growth models. More recently, Leland (1974), studies a stochastic Ak model, and he shows that the impact of increased uncertainty on the consumption/output ratio depends on the size of the coefficient of risk aversion.

Even in deterministic versions of models that allow for the possibility of endogenous growth, existence of equilibria (and even optimal allocations) requires strong assumptions on the fundamentals of the economy (see Jones and Manuelli (1990) for a discussion). At this point, there are no general results on existence of equilibrium in stochastic versions of those models. In special cases, most authors provide conditions under which an equilibrium exists (see Levhari and Srinivasan (1969), Mendoza (1997), Jones et. al (2003a) and (2003b) for various versions). A recent, more general result is contained in de Hek and Roy (2001). These authors consider fairly general utility and production functions, but limit themselves to *i.i.d.* shocks. It is clear that more work is needed.

In what follows, we will describe a general linear model and we will use it to illustrate the predictions of the theory for the relationship between mean growth rates and their variability. To simplify the presentation we switch to a continuous time setting. In order to obtain closed-form solutions we specialize the model in terms of specifying preferences and technology. Moreover, we will limit ourselves to *i.i.d.* shocks. Generalizations of these assumptions are discussed in the section that presents quantitative results.

3.4 A Simple Linear Endogenous Growth Model

We begin by presenting a stochastic analog of a standard Ak model with a 'twist.' Specifically, we consider the case in which there are multiple linear technologies, all producing the same good. In order to obtain closed-form results we specify that the utility of the representative household is given by

$$U = E\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\theta}}{1-\theta}dt\,|F_0\right]. \tag{13}$$

We assume that each economy has two types of technologies to produce consumption (alternatively, the model can be interpreted as a two sector model with goods that are perfect substitutes). Output for each technology satisfies

$$
\begin{aligned}
dk_t &= ((A-\delta_k)k_t - c_{1t})dt + \sigma_k k_t dW_t + \eta_k k_t dZ_t^k, && (14a)\\
db_t &= ((r-\delta_b)k_t - c_{2t})dt + \sigma_b b_t dW_t + \eta_b b_t dZ_t^b, && (14b)
\end{aligned}
$$

where (W_t, Z_t^k, Z_t^b) is a vector of three independent standard Brownian motion processes, and k_t and b_t are two different stocks of capital. This specification assumes that each sector is subject to an aggregate shock, W_t, as well as sector (or technology) specific shocks, Z_t^j.

To simplify the algebra, we assume that capital can be costlessly reallocated across technologies, and we denote total capital by $x_t \equiv k_t + b_t$. Setting (without loss of generality) $k_t = \alpha_t x_t$ (and, consequently $b_t = (1-\alpha_t)x_t$) it follows that total capital evolves according to

$$
\begin{aligned}
dx_t = {} & [(\alpha_t(A-\delta_k) + (1-\alpha_t)(r-\delta_b))x_t - c_t]dt + && (15)\\
& [(\alpha_t\sigma_k + (1-\alpha_t)\sigma_b)dW_t + \alpha_t\eta_k dZ_t^k + (1-\alpha_t)\eta_b dZ_t^b]x_t.
\end{aligned}
$$

Given the equivalence between equilibrium and optimal allocations in this convex economy, we study the solution to the problem faced by a planner who maximizes the utility of the representative agent subject to the feasibility constraint. Formally, the planner solves

$$
\max E\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\theta}}{1-\theta}dt \,|F_0\right],
$$

subject to (15).

Let the value of this problem be $V(x)$. Then, it is standard to show that the solution to the planner's problem satisfies the Hamilton-Jacobi-Bellman equation

$$
\rho V(x) = \max_{c,\alpha}\left[\frac{c^{1-\theta}}{1-\theta} + V'(x)(\mu(\alpha)x - c) + \frac{V''(x)x^2}{2}\sigma^2(\alpha)\right],
$$

where

$$
\begin{aligned}
\mu(\alpha) &= r + \alpha(A-r) - (\alpha\delta_k + (1-\alpha)\delta_b), && (16a)\\
\sigma^2(\alpha) &= (\alpha\sigma_k + (1-\alpha)\sigma_b)^2 + \alpha^2\eta_k^2 + (1-\alpha)^2\eta_b^2. && (16b)
\end{aligned}
$$

It can be verified that the solution is given by $V(x) = v\frac{x^{1-\theta}}{1-\theta}$, where

$$
v = \left[\frac{\rho - (1-\theta)[\mu(\alpha^*) - \delta(\alpha^*) - \theta\frac{\sigma^2(\alpha^*)}{2}]}{\theta}\right]^{-\theta} \tag{17}
$$

and $\delta(\alpha) = \alpha\delta_k + (1-\alpha)\delta_b$.

The optimal decision rules are

$$\alpha^* = \frac{\frac{A-\delta_k-(r-\delta_b)}{\theta} - \sigma_b(\sigma_k - \sigma_b) + \eta_b^2}{(\sigma_k - \sigma_b)^2 + \eta_b^2 + \eta_k^2}, \tag{18a}$$

$$c = \frac{\rho - (1-\theta)[\mu(\alpha^*) - \delta(\alpha^*) - \theta\frac{\sigma^2(\alpha^*)}{2}]}{\theta} x. \tag{18b}$$

For the solution to be well defined it is necessary that $\rho - (1-\theta)[\mu(\alpha^*) - \delta(\alpha^*) - \theta\frac{\sigma^2(\alpha^*)}{2}] > 0$, which we assume. (In each case we make enough assumptions to guarantee that this holds.)[9]

It follows that the equilibrium stochastic differential equation satisfied by aggregate wealth is given by

$$dx_t = [\frac{\mu(\alpha^*) - (\delta(\alpha^*) + \rho)}{\theta} - (1-\theta)\frac{\sigma^2(\alpha^*)}{2}]x_t dt + [(\alpha^*(\sigma_k - \sigma_b) + \sigma_b)dW_t + \alpha_k^*\eta_k dZ_t^k + (1-\alpha^*)\eta_b dZ_t^b]x_t, \tag{19}$$

and the instantaneous growth rate of the economy, γ, and its variance, σ_γ^2, satisfy

$$\gamma = \frac{\mu(\alpha^*) - (\delta(\alpha^*) + \rho)}{\theta} - (1-\theta)\frac{\sigma^2(\alpha^*)}{2}, \tag{20a}$$

$$\sigma_\gamma^2 = (\alpha^*(\sigma_k - \sigma_b) + \sigma_b)^2 + \alpha^{*2}\eta_k^2 + (1-\alpha^*)^2\eta_b^2. \tag{20b}$$

One is tempted to interpret (19) as the theoretical analog of (10) by defining the stochastic growth rate as

$$\gamma_t = \frac{dx_t}{x_t}.$$

Given this definition, the discrete time —with period length equal to one— version of the stochastic process followed by the growth rate is

$$\gamma_t = \frac{\mu(\alpha^*) - (\delta(\alpha^*) + \rho)}{\theta} - (1-\theta)\frac{\sigma_\gamma^2}{2} + \varepsilon_t, \tag{21a}$$

$$\varepsilon_t = (\alpha^*(\sigma_k - \sigma_b) + \sigma_b)dW_t + \alpha^*\eta_k dZ_t^k + (1-\alpha^*)\eta_b dZ_t^b. \tag{21b}$$

This simple model driven by i.i.d. shocks has a stark implication: the growth rate is i.i.d. and is independent of other endogenous (or exogenous) variables, except through the joint dependence on the error term. Using panel data, it is relatively easy

[9] In endogenous growth models existence of an equilibrium is not always guaranteed. The main problem is that with unbounded instantaneous utility and production sets, utility can be infinite. For a discussion of some conditions that guaranteee existence see Jones and Manuelli (1990) and Alvarez and Stokey (1998). The key issue is that the return function is unbounded above when $0 < \theta < 1$, and unbounded below if $\theta > 1$. In this setting, it can be shown that $c > 0$ is equivalent to ensuring boundedness.

to reject this implication. This, however, is not an intrinsic weakness of this class of models. The theoretical setting *can* be generalized to include serially correlated shocks and a non-linear structure, which could account for "convergence" effects, and would provide a role for lagged dependent variables. However, generalizing the theoretical model comes at the cost of not being able to discuss the impact of different factors on the growth rate, except numerically.

What is the (simple) class of model that we study useful for? We view the class of theoretical models that we present as more appropriate to discuss the implications of the theory for cross section regressions since, in this case, the constant $\frac{\mu(\alpha^*)-(\delta(\alpha^*)+\rho)}{\theta} - (1-\theta)\frac{\sigma_\gamma^2}{2}$ can be correlated with other variables like the investment-output ratio.

Even though there is a formal similarity between (21) and (10)-(11), the theoretical model suggests that the simple approach that ignores that the same factors that affect σ_γ, also influence the true value of β in (10) can result in incorrect inference. Alternatively, the "deep parameters" are not the means and the standard deviation of the growth rates. They are the means and standard deviations of the driving stochastic processes. In terms of those parameters, the "true" model is non-linear.

Whether the model in (21) implies a positive or negative relationship between fluctuations and growth depends on the sources of shocks. At this general level it is difficult to illustrate this point, but we will come back to it in the context of specific examples.

It is not obvious how to define the investment ratio in this model. The change in cumulative investment in k, X_k, is given by,

$$dX_{kt} = \delta_k k_t dt + dk_t,$$

while the change in total output can be defined as[10]

$$dY_t = \mu(\alpha^*)x_t dt + \sigma_\gamma dM_t,$$

where M_t is a standard Brownian motion defined so that

$$\sigma_\gamma dM_t = (\alpha^*(\sigma_k - \sigma_b) + \sigma_b)dW_t + \alpha^* \eta_k dZ_t^k + (1-\alpha^*)\eta_b dZ_t^b.$$

In order to avoid technical problems, we consider a discrete time approximation in which the capital stocks change only at the beginning of the period. The investment-output ratio (for physical capital) is given by

$$z_t = \frac{\gamma + \delta_k + \sigma_\gamma \varepsilon_t}{\mu(\alpha^*) + \sigma_\gamma \varepsilon_t},$$

[10]This is not the only possible way of defining output. It assumes that the economy two sectors (or technologies). However, another interpretation of this basic framework considers b_t as bonds, and k_t as the only real stock of capital. We will be precise about the notion of output in each application.

where ε_t is the same noise that appears in (21). Since the previous expression is non-linear, we approximate it by a second order Taylor expansion to obtain

$$z_t = \frac{\gamma + \delta_k}{\mu(\alpha^*)} + \frac{\sigma_\gamma[\mu(\alpha^*) - (\gamma + \delta_k)]}{\mu(\alpha^*)^2}\varepsilon_t - \frac{\sigma_\gamma^2[\mu(\alpha^*) - (\gamma + \delta)]}{\mu(\alpha^*)^3}\varepsilon_t^2. \tag{22}$$

The mean investment ratio, which we denote z, is given by

$$z = \frac{\gamma + \delta_k}{\mu(\alpha^*)}[1 + \frac{\sigma_\gamma^2}{\mu(\alpha^*)^2}] - \frac{\sigma_\gamma^2}{\mu(\alpha^*)^2}. \tag{23}$$

Given this approximation, the model implies that the covariance between the growth rate and the investment-output ratio is

$$cov(\gamma_t, z_t) = \frac{\sigma_\gamma^2[\mu(\alpha^*) - (\gamma + \delta_k)]}{\mu(\alpha^*)^2}, \tag{24}$$

while the standard deviation of z_t is

$$\sigma_z = \frac{\sigma_\gamma[\mu(\alpha^*) - (\gamma + \delta_k)]}{\mu(\alpha^*)^2}\frac{(1 + \mu(\alpha^*)^2)^{1/2}}{\mu(\alpha^*)}. \tag{25}$$

Simple algebra shows that, given that the existence condition (17) is satisfied, $cov(\gamma_t, z_t) > 0$. Thus, in a simple regression, the investment ratio has to appear to affect positively growth. At this general level it is more difficult to determine if high σ_z economies are also high γ economies. The problem is that there are a number of factors that jointly affect γ and σ_z. In order to be more precise, it is necessary to be specific about the sources of heterogeneity across countries. We will be able to discuss the sign of this relationship in specific contexts.

We now use this 'general' model to discuss —in a variety of special cases— the connection between the variability of the growth rate of output and its mean

3.4.1 Case 1: An Ak Model

Probably the simplest model to illustrate the role played by differences in the variability of the exogenous shocks across countries is the simple Ak model. Even though it is a special case of the model described in the previous section, it is useful to describe the technology in a slightly different way. Let the feasibility constraint for this economy be given by

$$\int_0^t \hat{A}k_s ds + \int_0^t \sigma_y \hat{A}k_s dW_s \geq \int_0^t c_s ds + \int_0^t dX_{ks}.$$

The left hand side of this condition is the accumulated flow of output until time t, and the right hand side is the accumulated uses of output, consumption and investment. The law of motion of capital is

$$dk_t = -\delta_k k_t dt + dX_{kt},$$

where δ_k is the depreciation rate. Expressing the economy's feasibility constraint in flow form, and substituting in the law of motion for physical capital, the resource constraint satisfies

$$dk_t = [(\hat{A} - \delta_k)k_t - c_t]dt + \sigma_y \hat{A} k_t dW_t. \tag{26}$$

The planner's problem —which coincides with the competitive equilibrium in this economy— is to maximize (13) subject to (26). This problem resembles the more general model we introduced in the previous section if we set $\eta_b = \sigma_b = \eta_k = 0$, and

$$\begin{aligned} A &= \hat{A} - \delta_k, \\ \sigma_k &= \sigma_y \hat{A}. \end{aligned}$$

In addition, we need to make sure that the "b" technology is not used in equilibrium. A simple way of guaranteeing this is to view $r - \delta_b$ as endogenous, and to choose it so that, in equilibrium, $\alpha^* = 1$; that is, all of the investment is in physical capital. It is immediate to verify that this requires

$$r - \delta_b = \hat{A} - \delta_k - \theta \sigma_y^2 \hat{A}^2.$$

In this case is it follows that $x_t = k_t$ and the formulas in (20) imply that the mean growth rate and the variance of the growth rate satisfy

$$\begin{aligned} \gamma &= \frac{\hat{A} - (\rho + \delta_k)}{\theta} - (1 - \theta)\frac{\sigma_\gamma^2}{2}, \\ \sigma_\gamma^2 &= \sigma_y^2 \hat{A}^2. \end{aligned}$$

This result, first derived by Phelps (1962) and Levhari and Srinivasan (1969), shows that, in general, the sign of the relationship between the variance of the technology shocks, σ_y^2, and the growth rate is ambiguous:

- If preferences display less curvature than the logarithmic utility function, i.e. $0 < \theta < 1$, increases in σ_y are associated with decreases in the mean growth rate, γ.
- If $\theta > 1$, increases in σ_y are associated with increases in the mean growth rate, γ.
- In the case in which the utility function is the log (this corresponds to $\theta = 1$) there is no connection between fluctuations and growth.

The basic reason for the ambiguity of the theoretical result is that the total effect of a change in the variance of the exogenous shocks on the saving rate —and ultimately on the growth rate— can be decomposed in two effects that work in different directions:

- An increase in the variance of the technology makes acquiring future consumption less desirable, as the only way to purchase this good is to invest. Thus, an increase in variance of the technology shocks has a *substitution effect* that increases the demand for current (relative to future) consumption. This translates into a lower saving and growth rates.

- On the other hand, an increase in the variability of the exogenous shocks induces also an *income effect*. Intuitively, for concave utility functions, the fluctuations of the marginal utility decrease with the level of consumption. Thus, the (negative) effect of fluctuations is smaller when consumption is high. This income effect increases savings, as this is the only way to have a 'high' level of consumption (i.e. to spend more time on the relatively flat region of the marginal utility function).

The formula we derived shows that the relative strength of the substitution and income effects depends on the degree of curvature of the utility function: if preferences have less curvature than the logarithmic function, the substitution effect dominates and increases in the variance of the exogenous shocks reduce growth. If the utility of the representative agent displays more curvature than the logarithmic function, the income effect dominates and the relationship between fluctuations and growth is positive.

In this simple economy, the variance of the technology shock, σ_y^2, and the variance of the growth rate of output, σ_γ^2, coincide up to scale factor $\hat{A}$.[11] If one views the differences across countries as due to differences in σ_y^2,[12] the theoretical model implies that the true regression equation is very similar to the one estimated in the empirical studies. The only difference is that the theory implies that it is σ_γ^2, and not σ_γ, that enters the right hand side of (10). If we use this model to interpret the results of Ramey and Ramey (1995), one must conclude that the negative relationship between mean growth and its standard deviation is evidence that preferences have less curvature than the logarithmic utility, i.e. $0 < \theta < 1$. On the other hand, the Kormendi and Meguire (1985) findings suggest that $\theta > 1$.

In this simple example, the mean investment ratio —the appropriate version of (23)— is

$$z = \frac{\gamma + \delta_k}{\hat{A}}[1 + \sigma_y^2] - \sigma_y^2$$

As was pointed out in the previous section, the covariance between the investment-ratio and the growth rate is positive. In this example, the appropriate version of (25) is

$$\sigma_z = \sigma_y \left(\frac{\rho - (1-\theta)(\hat{A} - \delta_k - \frac{\theta}{2}\sigma_y^2 \hat{A}^2}{\theta} \right) \left(1 + \hat{A}^2\right)^{1/2}.$$

[11] In general, this is not the case.

[12] This is not necessary. In addition to differences in preferences —which we will ignore in this chapter— countries can differ in terms of $(\hat{A}, \delta_k)$ as well.

In this case, the increases in σ_y are associated with increases (decreases) in σ_z if $\theta <\ (>)1$. Thus, if $\theta < 1$, the higher the (unobserved) variance of the technology shocks (σ_y^2), the higher the (measured) variances of both the growth rate, σ_γ^2, and the investment rate, σ_z^2, and the lower the mean growth rate. Moreover, in this stochastically singular setting the standard deviation of the growth rate and the investment rate are related (although not linearly). Thus, this simple model is consistent with the findings of Barlevy (2002) that the coefficient of σ_z is estimated to be negative, and that its introduction reduces the significance of σ_γ.

This simple model cannot explain the apparent non-linearity in the relationship between mean and standard deviation of the growth rate process which, according to Fatás (2001), is such that the effect of σ_γ on γ is less negative (and can be positive) for high income countries. In order to account for this fact it is necessary to increase the degree of heterogeneity, and to consider non-linear models.

Finally, the model can be reinterpreted as a multi-country model in which markets are incomplete and the distribution of the domestic shocks —the productivity shocks— is common across all countries.[13] More precisely, consider a market structure in which all countries can trade in a perfectly safe international bond market. In this case —which of course implies that mean growth rates are the same across countries— there is an equilibrium in which all countries choose to hold no international bonds, and the world interest rate is

$$r^* = \hat{A} - \delta_k - \theta\sigma_{y_i}^2\hat{A}^2.$$

If there is a common shock that decreases the variability of every country's technology shocks, this has a positive effect on the "world" interest rate, r^*, and an ambiguous impact on the world growth rate.

3.4.2 Case 2 : A Two Sector (Technology) Model

In the previous model, the variance of the growth rate is exogenous and equal to the variance of the technology shock. This is due, in part, to the assumption that the economy does not have another asset that can be used to diversify risk. In this section we present a very simple two-technology (or two sector) version of the model in which the variance of the growth rate is *endogenously* determined by the portfolio decisions of the representative agent. The main result is that, depending on the source of heterogeneity across countries, the relationship between σ_γ and γ need not be monotone. In particular, and depending on the source of heterogeneity across countries, the model is consistent with increases in σ_γ initially associated with increases in γ, and then, for large values of σ_γ, with decreases in the mean growth rate.

[13] It is possible to allow countries to share the same realization of the stochastic process. Even in this case, the demand for bonds is zero at the conjectured interest rate.

To keep the model simple, we assume that the second technology is not subject to shocks, and we ignore depreciation. Thus, formally, we assume that $\eta_b = \sigma_b = \eta_k = 0$. However, unlike the previous case, the "safe" rate of return r satisfies

$$A - \theta\sigma_k^2 < r < A.$$

This restriction implies that $\alpha^* \varepsilon (0, 1)$, and guarantees that both technologies will be used to produce consumption. Since this model is a special case of the results summarized in (20) (we set the depreciation rates equal to zero for simplicity), it follows that the equilibrium mean growth rate and its variance are given by

$$\gamma = \frac{r-\rho}{\theta} + \left(\frac{A-r}{\theta\sigma_k}\right)^2 \frac{1+\theta}{2}, \tag{27a}$$

$$\sigma_\gamma^2 = \left(\frac{A-r}{\theta\sigma_k}\right)^2. \tag{27b}$$

How can we use the model to interpret the cross country evidence on variability and growth? A necessary first step is to determine which variables can potentially vary across countries. In the context of this example, a natural candidate is the vector (A, r, σ_k). Before we proceed, it is useful to describe the connection between γ and σ_γ implied by the model. The relationship is —taking a discrete time approximation—

$$\begin{aligned} \gamma_t &= \frac{r-\rho}{\theta} + \sigma_\gamma^2 \frac{1+\theta}{2} + \varepsilon_t, \\ \varepsilon_t &= \sigma_\gamma \omega_t, \qquad \omega_t \sim N(0,1). \end{aligned}$$

It follows that if the source of cross-country differences are differences in (A, σ_k) the model implies that —independently of the degree of curvature of preferences— the relationship between σ_γ^2 and γ is positive. To see why increases in σ_k result in such a positive association between the two endogenous variables σ_γ and γ, note that, as σ_k rises, the economy shifts more resources to the safe technology (α^* decreases) and this, in turn, results in a decrease in the variance of the growth rate (which is a weighted average of the variances of the two technologies). Since the 'risky' technology has higher mean return than the 'safe' technology, the mean growth rate decreases. The reader can verify that changes in A have a similar effects.

If the source of cross-country heterogeneity is due to differences in r, the implications of the model are more complex. Consider the impact of a decrease in r. From (27b) it follows that σ_γ^2 increases and this tends to increase γ. However, as (27a) shows, this also decreases the growth rate, as it lowers the non-stochastic return. The total effect depends on the combined impact. A simple calculation shows that

$$\frac{\partial\gamma}{\partial r} \lesseqqgtr 0 \qquad \Leftrightarrow \qquad r \lesseqqgtr \hat{r},$$

where

$$\hat{r} = A - \frac{\theta \sigma_k^2}{1+\theta}.$$

To better understand the implications of the model consider a "high" value of r; in particular, assume that $r > \hat{r}$. A decrease in r reduces σ_γ and, given that $r > \hat{r}$, it results in an increase in γ. Thus, for low σ_γ (high r) countries, the model implies a positive relationship between γ and σ_γ. If $r < \hat{r}$, decreases in the return to the safe technology still increase σ_γ, but, in this region, the growth rate decreases. Thus, in (σ_γ, γ) space the model implies that, due to variations in r, the relationship between σ_γ and γ has an inverted U-shape.

Can this model explain some of the non-linearities in the data? In the absence of further restrictions on the cross-sectional joint distribution of (A, r, σ_k) the model can accommodate arbitrary patterns of association between σ_γ and γ. If one restricts the source of variation to changes in the return r the model implies that, for high variance countries, variability and growth move in the same direction, while for low variance countries the converse is true. If one could associate low variance countries with relatively rich countries, the implications of the model would be consistent with the type of non-linearity identified by Fatás (2001).

3.4.3 Case 3: Aggregate vs. Sectoral Shocks

The simple Ak model that we discussed in the previous section is driven by a single, aggregate, shock. In this section we consider a two sector (or two technology) economy to show that the degree of sectoral correlation of the exogenous shocks can affect the mean growth rate. To capture the ideas in as simple as possible a model, we specialize the specification in (14) by considering the case

$$\begin{aligned} \sigma_k &= \sigma_b = \sigma > 0, \\ \eta_b &= 0, \ \eta_k = \eta, \\ \delta_k &= \delta_b = 0. \end{aligned}$$

Note that, in this setting, there is an aggregate shock, W_t, which affects both sectors (technologies) while the A sector is also subject to a specific shock, Z_t^k. Using the formulas derived in (18) and (20) it follows that the relevant equilibrium quantities are

$$\begin{aligned} \alpha^* &= \frac{A-r}{\theta \eta^2}, \\ \gamma &= \frac{r-\rho}{\theta} - (1-\theta)\frac{\sigma^2}{2} + \left(\frac{A-r}{\theta\eta}\right)^2 \frac{1+\theta}{2}, \\ \sigma_\gamma^2 &= \sigma^2 + \left(\frac{A-r}{\theta\eta}\right)^2. \end{aligned}$$

As before, it is useful to think of countries as indexed by (A, r, σ, η). Since changes in each of these parameters has a different impact, we analyze them separately.

- `An increase in` σ. The increase in the standard deviation of the economy-wide shock affects both sectors equally, and it does not induce any 'portfolio' or sectoral reallocation of capital. The share of capital allocated to each sector (technology) is independent of σ. Since increases in σ increase σ_γ (in the absence of a portfolio reallocation, this is similar to the one sector case), the total effect of an increase in σ is to decrease the growth rate if $0 < \theta < 1$, and to increase it if $\theta > 1$.

- `A decrease in` r. The effect of a change in r parallels the discussion in the previous section. It is immediate to verify that a decrease in r results in an increase in σ_γ. However, the impact on γ is not monotonic. For high values of r, decreases in r are associated with increases in γ, while for low values the direction is reversed. Putting together these two pieces of information, it follows that the predicted relationship between σ_γ and γ is an inverted U-shape, with a unique value of σ_γ (a unique value of r) that maximizes the growth rate.

- `An increase in` η. This change increases the 'riskiness' of the A technology and results in a portfolio reallocation as the representative agent decreases the share of capital in the high return sector (technology). The change implies that σ_γ and γ decrease. Thus, differences in η induce a positive correlation between mean and standard deviation of the growth rate.

- What is the impact of differences in the degree of correlation between sectoral shocks. Note that the correlation between the two sectoral shocks is

$$\nu = \frac{\sigma}{(\sigma^2 + \eta^2)^{1/2}}.$$

  In order to isolate the impact of a change in correlation, let's consider changes in (σ, η) such that the variance of the growth rate is unchanged. Thus, we restrict (σ, η) to satisfy

$$\sigma_\gamma^2 = \sigma^2 + \left(\frac{A - r}{\theta\eta}\right)^2,$$

  for a given (fixed) σ_γ. It follows that the correlation between the two shocks and the growth rate are

$$\begin{aligned} \nu &= \left(1 + (\frac{A - r}{\theta})^2 \frac{1}{\sigma^2(\sigma_\gamma^2 - \sigma^2)}\right)^{-1}, \\ \gamma &= \frac{r - \rho}{\theta} - \sigma^2 + \frac{1 + \theta}{2}\sigma_\gamma^2. \end{aligned}$$

Thus, lower correlation between sectors (in this case this corresponds to higher σ) unambiguously lower mean growth. If countries differ in this correlation then the implied relationship between σ_γ and γ need not be a function; it can be a correspondence. Put it differently, the model is consistent with different values of γ associated to the same σ_γ.

3.5 Physical and Human Capital

In this section we study models in which individuals invest in human and physical capital. We consider a model in which the rate of utilization of human capital is constant. Even though the model is quite simple it is rich enough to be consistent with **any** estimated relationship between σ_γ and γ.

We assume that output can be used to produce consumption and investment, and that market goods are used to produce human capital. This is equivalent to assuming that the production function for human capital is identical to the production function of general output. The feasibility constraints are

$$\begin{aligned} dk_t &= ([F(k_t, h_t) - \delta_k k_t - x_t - c_t)\, dt + \sigma_y F(k_t, h_t) dW_t, \\ dh_t &= -\delta_h h_t + x_t dt + \sigma_h h_t dW_t + \eta h_t dZ_t, \end{aligned}$$

where (W_t, Z_t) is a vector of independent standard Brownian motion variables, and F is a homogeneous of degree one, concave, function. As in the previous sections, let $x_t = k_t + h_t$ denote total (human and non-human) wealth. With this notation, the two feasibility constraints collapse to

$$\begin{aligned} dx_t &= ([F(\alpha_t, 1-\alpha_t) - (\delta_k \alpha_t + \delta_h (1-\alpha_t))] x_t - c_t)\, dt + \quad (28) \\ \sigma_y F(\alpha_t, 1-\alpha_t) x_t dW_t + & \quad \sigma_h (1-\alpha_t) x_t dW_t + \eta (1-\alpha_t) x_t dZ_t. \end{aligned}$$

As in previous sections, the competitive equilibrium allocation coincides with the solution to the planner's problem. The planner maximizes (13) subject to (28). The Hamilton-Jacobi-Bellman equation corresponding to this problem is

$$\rho V(x) = \max_{c,\alpha} \left[\frac{c^{1-\theta}}{1-\theta} + V'(x)[(F(\alpha, 1-\alpha) - \delta(\alpha)) x_t - c_t] + \frac{V''(x) x^2}{2} \sigma^2(\alpha) \right],$$

where

$$\begin{aligned} \delta(\alpha) &= \delta_k \alpha + \delta_h (1-\alpha), \\ \sigma^2(\alpha) &= \sigma_y^2 F(\alpha, 1-\alpha)^2 + \sigma_h^2 (1-\alpha)^2 + \eta^2 (1-\alpha)^2 + \sigma_y \sigma_h F(\alpha, 1-\alpha)(1-\alpha). \end{aligned}$$

It can be verified that a function of the form $V(x) = v \frac{x^{1-\theta}}{1-\theta}$ solves the Hamilton-Jacobi-Bellman equation. The solution also requires that

$$\rho = \theta v^{-1/\theta} + (1-\theta)\{F(\alpha, 1-\alpha) - \delta(\alpha) - \frac{\theta}{2}\sigma^2(\alpha)\},$$

where α is given by

$$\alpha = \arg\max(1-\theta)\{F(\alpha, 1-\alpha) - \delta(\alpha) - \frac{\theta}{2}\sigma^2(\alpha)\}.$$

For any homogeneous of degree one function F, the solution is a constant α. Moreover, α does not depend on v. Existence requires $v > 0$, and this is just a condition on the exogenous parameter that we assume holds.[14]

The growth rate and its variance are given by

$$\begin{aligned}
\gamma &= F(\alpha, 1-\alpha) - \delta(\alpha) - v^{-1/\theta}, \\
\sigma_\gamma^2 &= \sigma_y^2 F(\alpha, 1-\alpha)^2 + \sigma_h^2(1-\alpha)^2 + \eta^2(1-\alpha)^2 + \sigma_y\sigma_h F(\alpha, 1-\alpha)(1-\alpha)
\end{aligned}$$

It follows that, for the class of economies for which the planner problem has a solution (i.e. economies for which $v > 0$, and $\gamma > 0$), the conjectured form of $V(x)$ solves the HJB equation, for any homogeneous of degree one function F. However, in order to make some progress describing the implications of the theory, it will prove convenient to specialize the technology and assume that F is a Cobb-Douglas function given by

$$F(x, y) = Ax^\omega y^{1-\omega}, \qquad 0 < \omega < 1.$$

The next step is to characterize the optimal share of wealth invested in physical capital, α, and how changes in country-specific parameters affect the mean and standard deviation of the growth rate. It turns out that the qualitative nature of the solution depends on the details of the driving stochastic process. To simplify the algebra, we assume that the human capital technology is deterministic (i.e. $\sigma_h = \eta = 0$), and that both stocks of capital (physical and human) depreciate at the same rate ($\delta_k = \delta_h$). As indicated above, we assume that the production function is Cobb-Douglas. The first order condition for the optimal choice of α is simply

$$\phi(\alpha)\hat{F}(\alpha)[1 - \theta\sigma_y^2\hat{F}(\alpha)] = 0,$$

where

$$\begin{aligned}
\hat{F}(\alpha) &\equiv A\alpha^\omega(1-\alpha)^{1-\omega}, \\
\phi(\alpha) &= \frac{\omega}{\alpha} - \frac{1-\omega}{1-\alpha}.
\end{aligned}$$

The second order condition requires that

$$-\omega(1-\omega)[\alpha^{-2} + (1-\alpha)^{-2}]\hat{F}(\alpha)[1 - \theta\sigma_y^2\hat{F}(\alpha)] - \theta\sigma_y^2\hat{F}(\alpha)^2\phi(\alpha) < 0.$$

Since $\hat{F}(\alpha) > 0$ in the relevant range, the solution is either $\phi(\alpha) = 0$, which corresponds to $\alpha^* = \omega$, or $\hat{F}(\alpha^*) = 1/\theta\sigma_y^2$. The latter, of course, does not result in a unique α^*[15].

[14] This is just the stochastic analog of the existence problem in endogenous growth models.

[15] In the case of the Cobb-Douglas production function there are two values of α that satisfy $\hat{F}(\alpha^*) = 1/\theta\sigma_y^2$

The nature of the solution depends on the size of σ_y^2. There are two cases characterized by

- *Case A*: $\sigma_y^2 \leq \frac{1}{\theta \hat{F}(\omega)}$.
- *Case B*: $\sigma_y^2 > \frac{1}{\theta \hat{F}(\omega)}$.

In *Case A*, the low variance σ_y^2 case, the maximizer is given by $\alpha^* = \omega$, since $1 - \theta\sigma_y^2\hat{F}(\alpha) > 0$ for all feasible α. The second order condition is satisfied.

In *Case B*, there are two solutions to the first order condition. They correspond to the values of α, denoted α^- and α^+ that solve $\hat{F}(\alpha^*) = 1/\theta\sigma_y^2$. By convention, let's consider $\alpha^- < \omega < \alpha^+$. It can be verified that in both cases the second order condition is satisfied.[16] The implications of the model for the expected growth rate and its standard deviation in the two cases are

$$
\begin{aligned}
\gamma_A &= \frac{\hat{F}(\omega) - (\rho+\delta)}{\theta} - \frac{1-\theta}{2}\sigma_y^2\hat{F}(\omega)^2, \\
\sigma_{\gamma_A} &= \sigma_y\hat{F}(\omega), \\
\gamma_B &= \frac{1}{\theta}[\frac{1+\theta}{2}\frac{1}{\theta\sigma_y^2} - (\rho+\delta)], \\
\sigma_{\gamma_B} &= \frac{1}{\theta\sigma_y}.
\end{aligned}
$$

It follows that for large σ_y^2, that is in *Case B*, the model predicts a positive relationship between mean growth and the standard deviation of the growth rate, while for small values of σ_y^2, *Case A*, the sign of the relationship depends on the magnitude of θ.

Much more interesting from a theoretical point of view is the fact that the model is consistent with two countries with different σ_y^2 to have *exactly the same* σ_γ. To see this, note that for any σ_γ in the range of feasible values —corresponding to the set $[0, \left(\frac{\hat{F}(\omega)}{\theta}\right)^{1/2}]$ in this example— there are two values of σ_y, one less than $\left(\frac{1}{\theta\hat{F}(\omega)}\right)^{1/2}$, and the other greater than this threshold that result in the same σ_γ. The relationship between σ_γ and γ is a correspondence. Figure 1 displays such a relationship in the small risk aversion case, $0 < \theta < 1$.

If the only source of cross-country heterogeneity are differences in the variability of the technology shocks, σ_y, the model implies that all data points should be in one of the two branches of the mapping depicted in Figure 1. By arbitrarily choosing the location of these points, the estimated relationship between σ_γ and γ can have any sign, and the estimated value says very little about the deep parameters of the model

[16] The reader can check that, in this case, the solution $\alpha^* = \omega$ does not satisfy the second order condition.

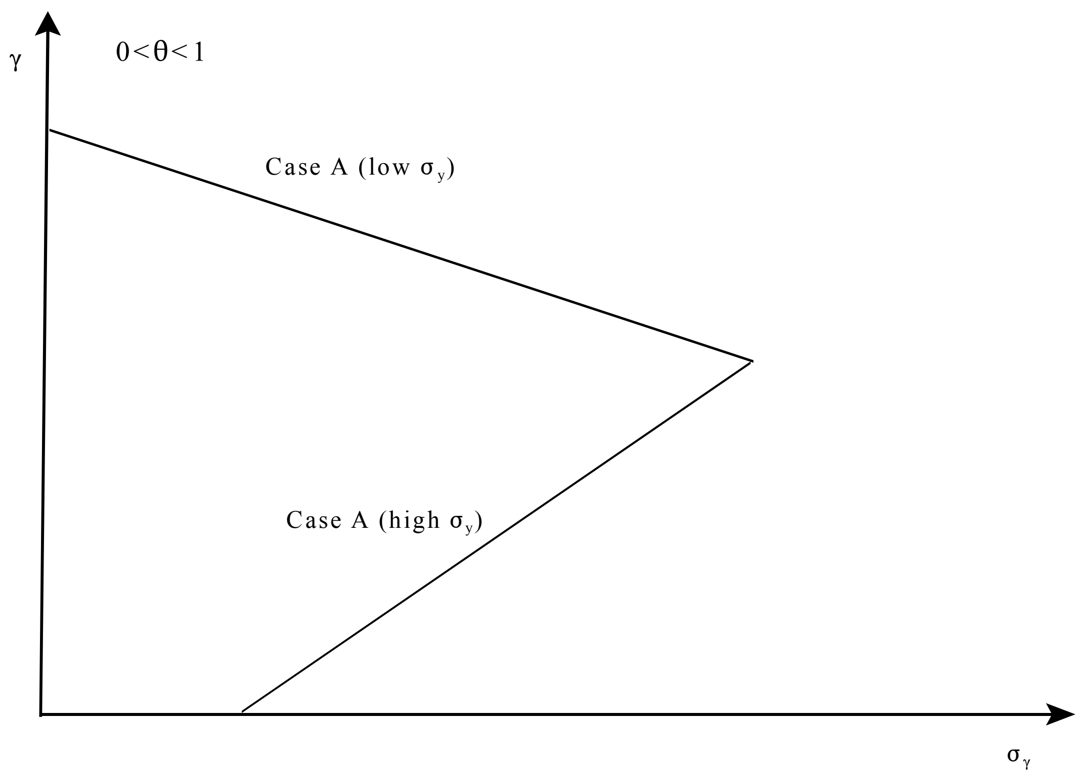

Figure 1: Figure 1: The mapping between σ_γ and γ. $[0 < \theta < 1]$

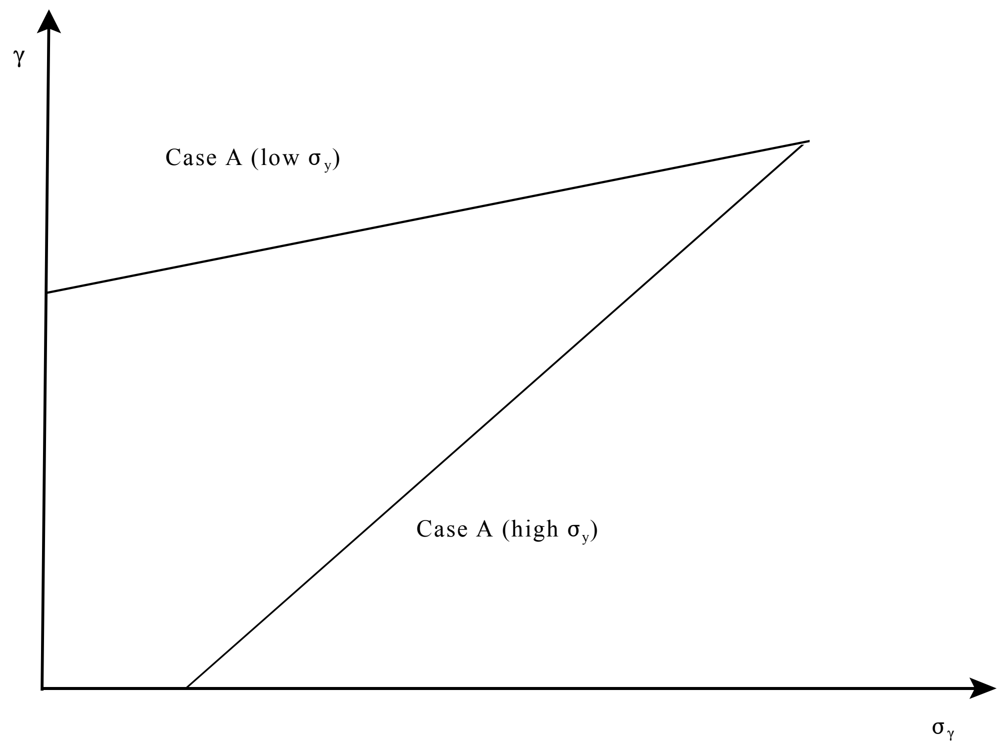

Figure 2: Figure 2: The mapping between σ_γ and γ. $[\theta > 1]$

or, more importantly, about the effects of reducing the variability of shocks on the average growth rate.

Does the nature of the result depend on the assumption $0 < \theta < 1$? For $\theta > 1$ the relationship between σ_γ and γ is also a correspondence, and hence that the model —in the absence of additional assumptions— does not pin down the sign of the correlation between σ_γ and γ.[17] In the case of $\theta > 1$, the size of θ matters only to determine which branch is steeper. I both *cases A* and *B* the relationship between the standard deviation of the growth rate and the mean growth rate is upward sloping. However, the low σ_y^2-branch is flatter (and lies above) the high σ_y^2-branch (see Figure 2).

Since we have studied a very simple version of this class of models, it does not seem useful to determine the relevance of each branch by asigning values to the parameters. In ongoing work, we are studying more general versions of this setup. However, even this simple example suggests that some caution must be exercised when interpreting the empirical work relating the variability of the growth rate and its mean. Unless

[17] At this point, we have not explored what are the consequences of adding the investment output ratio to the (theoretical) regression. However, to do this is a complete manner it seems necessary to model measurement errors, as the model is stochastically singular. We leave this for future work.

one can rule out some of these cases, theory gives ambiguous answers to the question that motivated much of the literature, i.e. Do more stable countries grow faster? Moreover, the theoretical developments suggest that progress will require to estimate structural models rather than reduced form equations.

3.6 The Opportunity Cost View

So far the models we discussed emphasize the idea that increases in the variability of the driving shocks can have positive or negative effects upon the growth rate depending on the relative importance of income and substitution effects. An alternative view is that recessions are "good times' to invest in human capital because labor —viewed as the single most important input in the production of human capital— has a low opportunity cost. In this section we present a model that captures these ideas. The model implies that the time allocated to the formation of human capital is independent of the cycle.[18] It also implies that shocks to the goods production technology have no impact on growth, but that the variability of the shock process in the human capital technology decreases growth.

As before, we concentrate on a representative agent with preferences described by (13). The goods production technology is given by

$$c_t + x_t \leq z_t A k_t^{\alpha} (n_t h_t)^{1-\alpha},$$

where n_t is the fraction of the time allocated to goods production, k_t is the stock of physical capital, and h_t is the stock of human capital. The variable z_t denotes a stationary process. To simplify the theoretical presentation we assume that capital depreciates fully. Thus, goods consumption is limited by

$$c_t \leq z_t A k_t^{\alpha} (n_t h_t)^{1-\alpha} - k_t.^{19}$$

Human capital is produced using only labor in order to capture the idea that the opportunity cost of investing in human capital is market production. The technology is summarized by

$$dh_t = [1 - \delta + B(1 - n_t)]h_t dt + \sigma_h [1 - \delta + B(1 - n_t)]h_t dW_t,$$

[18] The empirical relationship between investment in human capital and the cycle is mixed. Dellas and Sakellaris (1997) using CPS data for all individuals aged 18 to 22 find that college enrollment id procyclical. Christian (2002) also using the CPS but restricting the sample to 18-19 years olds (sa as to be able to control for family variables) finds no cyclical effects. Sakellaris and Spilimbergo (2000) study U.S. college enrollment of foreign nationals and conclude that, among those individuals coming from risch countries enrollment is countercyclical, while among students from less developed countries it is countercyclical. Moreover, college enrollment is only a partial measure of investment in human capital. Training (inside and outside business firms) is another (difficult to measure) component of increases in skill acquisition.

[19] This restriction makes it possible to derive the theoretical implications of the model in a simple setting.

where, as before, W_t is a standard Brownian motion.[20]

Given that the problem is convex[21] the competitive allocation solves the planner's problem. It is clear that, given $n_t h_t$, physical capital will be chosen to maximize net output. This implies that consumption is

$$c_t = A^* \hat{z}_t n_t h_t,$$

where $A^* = (A\alpha)^{1/(1-\alpha)}(\alpha^{-1} - 1)$ and $\hat{z}_t = z_t^{1/(1-\alpha)}$. We guess that the relevant state variable is the vector $(\hat{z}_t, h_t)$, and that the value function is of the form

$$V(\hat{z}_t, h_t) = v\frac{(\hat{z}_t h_t)^{1-\theta}}{1-\theta}.$$

Given this guess, the relevant Hamilton-Jacobi-Bellman equation is

$$\rho v \frac{(\hat{z}h)^{1-\theta}}{1-\theta} = \max_x \left\{ \frac{[\frac{A^*}{B}(\mu - x)\hat{z}h]^{1-\theta}}{1-\theta} + v(\hat{z}h)^{1-\theta}x - v(\hat{z}h)^{1-\theta}\theta\frac{\sigma_h^2}{2}x^2, \right\}$$

where $\mu \equiv 1-\delta+B$, and $x = 1-\delta+B(1-n)$. It follows that choosing x is equivalent to choosing n. The solution to the optimization problem is given by the solution to the following quadratic equation

$$x^2 = \frac{2(1+\mu\sigma_h^2)}{(1+\theta)\sigma_h^2}x + \frac{2(\rho-\mu)}{\theta(1+\theta)\sigma_h^2}.$$

In order to guarantee that utility remains bounded even in the case $\sigma_h = 0$ is is necessary to assume that $\rho - \mu > 0$. Simple algebra shows that the positive root of the previous equation is such that increases in σ_h decrease x. It follows that the stochastic process for h_t is given by

$$dh_t = xh_t dt + \sigma_h h_t dW_t$$

We now discuss the implications of the model for the growth rate of consumption (or output). Even though our results do not depend on the particular form of the z_t process, it is convenient to consider the case in which z_t is a geometric Brownian motion that is possibly correlated with the shock to the human capital. Specifically, we assume that

$$dz_t = z_t(\sigma_w dW_t + \sigma_m dM_t),$$

where M_t is a standard Brownian motion that is uncorrelated with W_t. Ito's lemma implies that

$$d\hat{z}_t = \frac{\alpha}{(1-\alpha)^2}\frac{\sigma_w^2 + \sigma_m^2}{2}\hat{z}_t dt + \frac{\alpha}{(1-\alpha)}\hat{z}_t(\sigma_w dW_t + \sigma_m dM_t).$$

[20] A special case of this model in which utility is assumed logarithmic, and the goods production function is not subject to shocks is analyzed in De Kek (1999).

[21] Even though our choice of notation somewhat obscures this, the convexity of the technology is apparent by defining $h_{mt} = n_t h_t$ and $h_{st} = (1-n_t)h_t$, and adding the constraint $h_{mt} + h_{st} \leq h_t$.

In equilibrium, consumption (and net output) is given by

$$c_t = \frac{A^*}{B}(\mu - x)\hat{z}_t h_t.$$

Applying Ito's lemma to this expression, we obtain that the growth rate of consumption

$$\frac{dc_t}{c_t} = \frac{dh_t}{h_t} + \frac{d\hat{z}_t}{\hat{z}_t} + \frac{\alpha x}{(1-\alpha)}\sigma_h \sigma_w dt,$$

or, taking a discrete time approximation,

$$\begin{aligned}
\gamma_t &= x(1 + \frac{\alpha}{(1-\alpha)}\sigma_h \sigma_w) + \frac{\alpha}{(1-\alpha)^2}\frac{\sigma_w^2 + \sigma_m^2}{2} + \\
&\quad [(\frac{\alpha}{(1-\alpha)}\sigma_w + \sigma_h x)\tilde{W}_t + \frac{\alpha}{(1-\alpha)}\sigma_m \tilde{M}_t], \qquad (29a)\\
\gamma_t &= \gamma + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma_\gamma^2), \qquad (29b)\\
\sigma_\gamma^2 &= (\frac{\alpha}{(1-\alpha)}\sigma_w + \sigma_h x)^2 + (\frac{\alpha}{(1-\alpha)}\sigma_m)^2 \qquad (29c)
\end{aligned}$$

Equation (29a) completely summarizes the implications of the model for the data. There are several interesting results. To simplify the notation, we will refer to W_t as the aggregate shocks and to M_t as the idiosyncratic component of the productivity shock in the goods sector.

- The share of the time allocated to human capital formation —the engine of growth in this economy— is independent of the variability of the technology shock in the goods sector, as measured by (σ_w, σ_m).

- High (σ_w, σ_m) economies are also high growth economies. Thus, if cross-country differences in σ_γ are mostly due to differences in (σ_w, σ_m), the model implies a positive correlation between the standard deviation of the growth rate and mean growth.

- It can be shown that increases in σ_h result in *decreases* in $\sigma_h x$. Thus, if countries differ in this dimension the model also implies a positive relationship between σ_γ and γ.

- In the model, investment in physical capital (as a fraction of output) is α, independently of the distribution of the shocks. Thus, there is no sense that a regression that shows that variability does not affect the rate of investment provides evidence against the role of shocks in development.

- This lack of (measured) effect on both physical and human capital investment should not be interpreted as evidence against the proposition that incentives for human or physical capital accumulation matter for growth. It is easy enough to include a tax/subsidy to the production of human capital —consider a policy that affects B— and it follows that this policy affects growth.

3.7 More on Government Spending, Taxation, and Growth

In this section we consider a simple Ak model in which a government uses distortionary taxes to finance an exogenously given stochastic process for government spending. Our analysis follows Eaton (1981).[22]

The representative household maximizes utility —given by (13)— by choosing consumption and saving in either capital or bonds. However, given that tax policy is exogenously fixed, it is not the case that the rate of return on bonds is risk free. On the contrary, since the government issues bonds to make up for any difference between revenue and spending it is necessary to let the return on bonds to be stochastic.

The representative household problem is

$$\max U = E\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\theta}}{1-\theta}dt\,|F_0\right]. \tag{30}$$

subject to

$$\begin{aligned} dk_t &= (r_k k_t - c_{1t})dt + \sigma_k k_t dW_t, & (31a)\\ db_t &= (r_b b_t - c_{2t})dt + \sigma_b b_t dW_t, & (31b)\\ c_t &= c_{1t} + c_{2t}, & (31c) \end{aligned}$$

where k_t is interpreted as capital and b_t as bonds. As before, it is possible to simplify the analysis by using wealth as the state variable. Let $x_t \equiv k_t + b_t$. With this notation, the single budget constraint is given by,

$$dx_t = [(\alpha_t r_k + (1-\alpha_t)r_b)x_t - c_t]dt + (\alpha_t\sigma_k + (1-\alpha_t)\sigma_b)x_t dW_t.$$

Since this problem is a special case of the "general" two risky assets model, it follows that the optimal solution is characterized by

$$\begin{aligned} \alpha &= \frac{\frac{r_k - r_b}{\theta} - \sigma_b(\sigma_k - \sigma_b)}{(\sigma_k - \sigma_b)^2}, & (32a)\\ c_t &= \underbrace{\frac{\rho - (1-\theta)[\alpha r_k + (1-\alpha)r_b - \theta\frac{(\alpha\sigma_k + (1-\alpha)\sigma_b)^2}{2}]}{\theta}}_{c} x_t. & (32b) \end{aligned}$$

The set of feasible allocations is the set of stochastic process that satisfy

$$\begin{aligned} dk_t &= (Ak_t - c_t)dt + \sigma A k_t dW_t - dG_t,\\ dG_t &= gAk_t dt + g'\sigma A k_t dW_t. \end{aligned}$$

Thus, the government consumes a fraction g of the non-stochastic component of output, and a fraction g' of the stochastic component. Taxes are levied on the

[22] For extensions of this model, see Turnovski (1995)

deterministic and stochastic components of output at (possibly) different rates. The stochastic process for tax revenue is assumed to satisfy

$$dT_t = \tau A k_t dt + \tau' \sigma A k_t dW_t.$$

In equilibrium, the parameters that determine the rate of return on capital (r_k, σ_k) are given by

$$\begin{aligned} r_k &= (1-\tau)A, \\ \sigma_k &= (1-\tau')\sigma A. \end{aligned}$$

The government budget constraint requires that the excess of spending over tax revenue be financed through bond issues. Thus,

$$B_t + dG_t - dT_t = p_t dB_t,$$

where $p_t B_t = b_t$ is the value of bonds issued. The stock of capital evolves according to

$$dk_t = ((1-g)A - \frac{c_t}{k_t})k_t dt + \sigma(1-g')Ak_t dW_t.$$

Note that

$$\frac{c_t}{k_t} = c\frac{x_t}{k_t} = c(1 + \frac{1-\alpha}{\alpha}) = \frac{c}{\alpha}.$$

Since, in equilibrium, it must be the case that, in all states of nature, the growth rate of private wealth and the growth rate of the capital stock are the same[23], it is necessary that

$$\alpha r_k + (1-\alpha) r_b - c = (1-g)A - \frac{c}{\alpha}, \quad (33a)$$

$$\alpha \sigma_k + (1-\alpha)\sigma_b = \sigma(1-g')A. \quad (33b)$$

The system formed by the four equations described in (32) and (33) provides the solution to the endogenous variables that need to be determined: c, α, r_b, σ_b. It is convenient to define the excess rate of return of capital, and the excess instant variability of capital as

$$\begin{aligned} \Delta_r &= r_k - r_b, \\ \Delta_\sigma &= \sigma_k - \sigma_b. \end{aligned}$$

Some simple but tedious algebra shows that

$$\begin{aligned} \alpha &= \frac{\sigma_\gamma - \sigma_k + \Delta_\sigma}{\Delta_\sigma}, \\ \Delta_r &= \theta \sigma_\gamma \Delta_\sigma \end{aligned}$$

[23]This, of course, depends on the fact that the solution to the individual agent problem is such that bonds and capital are held in fixed proportions.

Substituting in the remaining equations, and recalling that the instantaneous mean and standard deviation of the growth rate process is given by

$$\begin{aligned} \gamma &= \alpha r_k + (1-\alpha) r_b - c, \\ \sigma_\gamma &= \sigma(1-g')A, \end{aligned}$$

it follows that,

$$\gamma = \frac{(1-\tau)A-\rho}{\theta} - \left(\frac{1-\theta-\tau'+\theta g'}{1-g'}\right)\sigma_\gamma^2 \tag{34a}$$

$$\sigma_\gamma = \sigma(1-g')A. \tag{34b}$$

Equation (34a) summarizes the impact of both technology and fiscal shocks on the expected growth rate. Consider first the impact of variations in the tax regime on the relationship between the variability of the growth rate, σ_γ, and the average growth rate. If technology shocks, σ, are the main source of differences across countries in the standard deviation of the growth rate, then high variability countries are predicted to be low mean countries if $1-\theta-\tau'+\theta g' > 0$; that is, if a country has a relatively low tax rate on the stochastic component of income. This, would be the case if the base of the income tax allowed averaging over several periods. On the other hand, countries with relatively high tax rates on the random component of income display a positive relationship between the mean and the variance of their growth rates.

As in more standard models, high capital income tax countries (high τ countries) have lower average growth. Differences across countries in the average size of the government, g in this notation, have no impact on growth. Finally, cross country differences in the fraction of the random component of income consumed by the government, g', induce a positive correlation between γ and σ_γ. This is driven by the negative impact that increases in g' have on mean growth, and the equally negative effect that those changes have on σ_γ. Thus, high g' countries display low average growth rates, which do not fluctuate much.[24]

3.8 Quantitative Effects

In this section we summarize some of the quantitative implications various models for the relationship between variability and growth. Unlike the theoretical models described above, the quantitative exercises concentrate on the role of technology shocks in models with constant –relative to output– government spending.

[24] The impact of some variables in the previous analysis differs from the results in Eaton (1981) since our specification of the fiscal policy allows the demand for bonds (as a fraction of wealth) to be endogenous, and driven by changes in the tax code. Eaton assumes that the share of bonds, $1-\alpha$ in our notation, is given, and some tax must adjust to guarantee that demand and supply of bonds are equal.

Mendoza (1997) studies an economy in which the planner solves the following problem

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta}$$

subject to

$$A_{t+1} \leq R_t(A_t - p_t c_t),$$

where A_t is a measure of assets and p_t is interpreted as the terms of trade. Since this equation can be rewritten as

$$k_{t+1} = r_{t+1}(k_t - c_t),$$

where $k_t = A_t/p_t$ is the stock of assets (capital) measured in units of consumption, and $r_{t+1} = R_t p_t/p_{t+1}$ is the random rate of return, it is clear that Mendoza's model is a stochastic version of an Ak model. The rate of return, r_t is assumed to be lognormally distributed with mean and variance given by

$$\begin{aligned} \mu_r &= e^{\mu+\sigma^2/2}, \\ \sigma_r^2 &= \mu_r^2 e^{\sigma^2-1}. \end{aligned}$$

It follows that the standard deviation of the growth rate is $\sigma_\gamma = \sigma$. Mendoza studies the effect of changing σ from 0 to 0.15, holding μ_r constant. To put the exercise in perspective, the average across countries of the standard deviations of the growth rate or per capita output in the Summers-Heston dataset is 0.06. Thus, the model is calibrated at a fairly high level of variability. The results depend substantially on the assumed value of θ. For $\theta = 1/2$, the non-stochastic growth rate is 3.3%. If $\sigma = 0.10$, it decreases to 2.5%, while it is 1.6% when $\sigma = 0.15$. For $\theta = 2.33$ (Mendoza's preferred specification), the growth rate increases from 0.7% to 0.9% in the given range. For other values of the coefficient of risk aversion, the impact of uncertainty is also small. In summary, unless preferences are such that the degree of intertemporal substitution is large, increases in rate of return uncertainty have a small impact on mean growth rates.

Jones et. al (2003a) analyze the following planner's problem:

$$\max E_t\{\sum_t \beta^t c_t^{1-\theta} v(\ell_t)/(1-\theta)\},$$

subject to,

$$\begin{aligned} c_t + x_{zt} + x_{ht} + x_{kt} &\leq F(k_t, z_t, s_t), \\ z_t &\leq M(n_{zt}, h_t, x_{zt}) \\ k_{t+1} &\leq (1-\delta_k)k_t + x_{kt}, \\ h_{t+1} &\leq (1-\delta_h)h_t + G(n_{ht}, h_t, x_{ht}) \\ \ell_t + n_{ht} + n_{zt} &\leq 1. \end{aligned}$$

For their quantitative exercise, they specify the following functional forms

$$\begin{aligned}
n_h &= x_z = 0, \quad n_z = n, \\
v(\ell) &= \ell^{\psi(1-\sigma)}, \\
F(k,z,s) &= sAk^{\alpha}z^{1-\alpha}, \\
G(h,x_h) &= x_h, \\
M(n,h) &= nh, \\
s_t &= \exp[\zeta_t - \frac{\sigma_{\varepsilon}^2}{2(1-\rho^2)}], \\
\zeta_{t+1} &= \rho\zeta_t + \varepsilon_{t+1}.
\end{aligned}$$

The model is calibrated to match the average growth, in a cross section of countries, and its standard deviation. Jones et. al (2003) consider the impact of changing the standard deviation of the shock, s_t, from 0 to 0.15. The impact on the growth rate depends on the curvature of the utility function. For preferences slightly less curved than the log, the model predicts an increase in the growth rate of 0.7% an an annual basis, while for $\theta = 1.5$, the effects is an increase in the growth rate of 0.25%. However, the model predicts that σ_γ —which is endogenous— is unusually high (of the order of 0.10) unless $\theta \geq 1.5$.

Thus, Jones et. al (2003a) obtain results that are quite different from those of Mendoza (1997). There are two important differences between the models: First, while Mendoza (1997) assumes that shocks are *i.i.d.*, Jones et. al. (2003a) set the first order correlation parameter, ρ, to 0.95. Second, while Mendoza assumes a constant labor supply, Jones et. al allow for the number of hours to vary with the shock.

In order to disentangle the effect of the components of the standard deviation of the technology shock, Jones et. al. vary σ_ε and ρ in a series of experiments, where $\sigma_s = \sigma_\varepsilon/(1-\rho^2)^{1/2}$, where σ_ε is the standard deviation of the innovations. They find that changes in σ_ε appear quantitatively more important than those and ρ. Moreover, they also find that the relative variability of hours worked is very sensitive to the precise value of θ. Economies with high θ, and lower σ_γ in their specification, also display substantially less variability in hours worked. Even though it is not possible to determine on the basis of these two results which is the critical feature that accounts for the differences between the results obtained by Mendoza (1997) and Jones et. al (2003a), it seems that the assumption of a flexible supply of hours —which determines the rate of utilization of human capital— is a leading candidate.[25]

In a series of papers, Krebs (2003) and (2004) explores the impact of changes in uncertainty in models where markets are incomplete. Building on the work of de Hek (1999), Krebs (2003) studies the impact of shocks to the depreciation rate of the

[25] In subsequent work, Jones et. al (2003a) analyze the business cycle properties of the same class of models, and they show that are capable of generating higher serial correlation in the growth rate of output than similar exogneous growth models.

capital stock. Even though he assumes that instantaneous utility is logarithmic, he finds that increases in the standard deviation of the shock, decrease growth rates. This result is driven by the "location" of the shocks, and it does not require market incompleteness.[26] Quantitatively, Krebs (2003) finds that an increase in the standard deviation of the growth rate from 0 to 0.15 (a fairly large value relative to the world average) reduces growth from 2.13% to 2.00%. If the variability is increased to 0.20. the growth rate drops to 1.5%. However, these values are substantially higher than those observed in international data.[27]

The theoretical analysis of the impact of different forms of uncertainty on the growth of the economy is still in its infancy. Simple existing suggest that the sign of the relationship between the variability of a country's growth rate and its average growth rate is ambiguous. Thus, theoretical models that restrict more moments can help in understanding the effect of fluctuations on growth and welfare. The few quantitative studies that we reviewed have produced conflicting results. It seems that the precise nature of the shocks, their serial correlation properties and the elasticity of hours with respect to shocks all play a prominent role in accounting for the variance in predicted outcomes. Much more work is needed to identify realistic and tractable models that will be capable of confronting both time series and cross country observations.

4 Concluding Comments

In this chapter we briefly presented the basic insights about the growth process that can be learned from studying standard convex models with perfectly functioning markets. We emphasized three aspects of those models. First, the impact of fiscal policy on growth. A summary of the current state of knowledge is that theoretical models have ambiguous implications about the effect of taxes on growth. The key feature is the importance of market goods in the production of human capital. If, as Lucas (1988) assumes, no market goods are needed to produce new human capital, the impact of income taxes on growth is small (or zero in some cases). If, on the other hand, market goods are necessary to produce human capital then taxes play a more important role, and they have a large impact on growth. It seems that the next step is to use detailed models of the process of human capital formation and to explore the implications that they have for the age-earnings profile to identify the parameters of the production function of human capital. A first step in that direction is in Manuelli and Seshadri (2004).

A second important issue that features prominently in the discussion of the relative

[26] de Hek (1999) shows theoretically that increases in the variance of the depreciation shock decrease average growth even if markets are complete, and the shocks are aggregate shocks.

[27] Krebs (2003) does not have an aggregate shock. His model predicts that aggregate growth is constant. He calibrates his model to match the standard deviation of the growth rate of individual income.

merits of convex and non-convex models is the role of innovation. The standard argument claims that innovation is a one-off investment (with low copying costs) and hence that this technology is inconsistent with price taking behavior. In this chapter we elaborate on the ideas discussed by Boldrin and Levine (2002) and show that it is possible to reconcile the existence of a non-convexity with competitive behavior.

The last major theme covered in this survey is the relationship between fluctuations and growth. An important question is whether technological or policy induced fluctuations affect the growth rate of an economy. This is relevant for the time series experience of a single country(e.g. the discussion about the role of post-war stabilization policies on the growth rate of the American economy), as well as the prescriptions of international agencies for national policies. We discuss the empirical evidence and find it conflicting. It is not easy to identify a clear pattern between fluctuations and growth. To shed light on why this might be the case, we discuss a series of theoretical models. We show that the relationship between the growth rate and its standard deviation has an ambiguous sign. We also describe more precisely how one might identify the parameters of preferences and technologies that determine the sign of the relationship. This is one area of research in which more theoretical and empirical work will have a high marginal value.

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